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A NUMERICAL STUDY OF GRAVITATIONAL STELLAR COLLAPSE*

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ABSTRACT

The dynamic gravitational collapse of stars of mass 1.25–10 $M_\odot$ is calculated in terms of general relativity. Neutrino flow is calculated in considerable detail to evaluate whether the energy flow of neutrinos can produce mass ejection. Most stars evidently eject no mass by the process of neutrino transport during their collapse even though they radiate about 10^{32} g of neutrinos. Only the low-mass star of 1.25 $M_\odot$ ejects material mass, in this case 1.1×10^{32} g. In this latter case neutrino flow is not important.

I. INTRODUCTION

Colgate and White (1966) have demonstrated that if neutrino energy were transported in a particular manner from the center of a star collapsing by iron decomposition, a large explosion of the outer regions of the star would result.

This paper investigates the effect of neutrino transport on the gravitational collapse of stars and, in particular, seeks to determine whether heat conduction by neutrinos can produce the ejection of material from a star. General relativity is used for the equations of motion of the material, and a Boltzmann transport equation in the appropriate metric is employed for the neutrino flow. Both electron and muon neutrinos are included in the calculations; however, reactions involving antineutrinos are assumed to be equal to reactions involving neutrinos. The interaction of neutrinos with matter is described by an opacity function which depends on the kind of neutrino, its energy, and the material temperature and density. The equation of state of material is derived under the assumption that the matter is in its most stable (nongravitational) state.

II. DETAILS OF THE MODEL

a) Equations of Motion

The equations of motion are derived from May and White (1966) and from Lindquist (1966). We observe the following metric:

$$ds^2 = a^2dt^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2) - b^2dm^2 , \tag{1}$$

where m is a radial mass coordinate. The energy tensor $T_\mu{}^\nu$ is taken to be

$$\begin{vmatrix} -P & 0 & 0 & -Gb/a \\ 0 & -P-H & 0 & 0 \\ 0 & 0 & -P-H & 0 \\ Ga/b & 0 & 0 & \rho(1+\epsilon) \end{vmatrix} , \tag{2}$$

where P is the total pressure, $H = \frac{1}{2}(\epsilon_r - 3P_r)$ with ϵ_r the neutrino energy density and P_r the neutrino pressure, G is the flux of neutrino energy, ρ the particle density, and ϵ the total energy per unit mass. Given this energy tensor, the equations of motion become

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$$M = 4\pi \int_0^R \left(\rho + \rho\epsilon + \frac{GU}{\Gamma}\right) R^2 dR \,, \tag{3}$$

$$\Gamma = \left(1 + U^2 - \frac{2M}{R}\right)^{1/2} = \frac{\partial R}{b\partial m} \,, \quad U = \frac{1}{a}\frac{\partial R}{\partial t} \,, \tag{4}$$

$$\frac{1}{a}\frac{\partial U}{\partial t} = \frac{-4\pi R^2 \Gamma \rho}{(\rho + \rho\epsilon + P)}\left[\frac{\partial P}{\partial m} + \frac{2H}{R}\frac{\partial R}{\partial m} + \frac{\partial}{\partial t}\left(\frac{Gb}{a}\right)\right] - \frac{1}{R^2}(M + 4\pi P R^3) \,, \tag{5}$$

$$a = \exp\left\{\int_R^{R_{\max}} \left[\frac{\partial P}{\partial m} + \frac{2H}{R}\frac{\partial R}{\partial m} + \frac{\partial}{\partial t}\left(\frac{Gb}{a}\right)\right]\frac{dm}{(\rho + \rho\epsilon + P)}\right\} \,, \tag{6}$$

$$\frac{\partial}{a\partial t}\log(\rho R^2) + \frac{\partial U/\partial m}{\partial R/\partial m} = 0 \,, \tag{7}$$

$$\frac{\partial \epsilon_m}{\partial t} + P_m \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \text{neutrino collision terms} = \frac{TdS}{dt} \,, \tag{8}$$

where M is the total mass to point R; U the material velocity; and ϵ_m and P_m the material energy and pressure, respectively.

The equation of neutrino transport is

$$\begin{aligned} \frac{\partial F}{a\partial r}(\mu, \nu, m, t) = {} & \frac{-\mu\Gamma}{aR^2}\frac{\partial}{\partial R}(aR^2 F) - \Gamma\left(\frac{1}{R} - \frac{\partial}{\partial R}\log a\right)\left\{\frac{\partial}{\partial \mu}[F(1 - \mu^2)]\right\} \\ & + \frac{F}{a\rho}\frac{\partial \rho}{\partial t} + R\frac{\partial}{\partial R}\left(\frac{U}{R}\right)\left\{\frac{\partial}{\partial \mu}[\mu(1 - \mu^2)F] + \mu^2 \nu \frac{\partial F}{\partial \nu}\right\} \\ & + \frac{\nu \partial F}{a \partial \nu}\left[\frac{\partial}{\partial t}\log\left(\frac{R}{a}\right)\right] + K\rho(B - F)\left[1 + \exp\left(-\frac{\nu}{aT}\right)\right] \,, \end{aligned} \tag{9}$$

$$\frac{\partial T}{a\partial t} = \frac{1}{C_\nu}\int\left[1 + \exp\left(-\frac{\nu}{aT}\right)\right]K(B - F)d\mu d\nu + \text{hydrodynamic terms} \,, \tag{10}$$

where μ is the cosine of the angle of the neutrino velocity with respect to the radius vector, ν/a is the neutrino energy, and F is the neutrino flux, so that $\epsilon_r = \int F/a \, d\mu d\nu$, $G = \int(\mu F/a)\, d\mu d\nu$, and $P_r = \int(\mu^2 F/a)\, d\mu d\nu$. The quantities T and C_ν are the material temperature and heat capacity, $K = K(T, \rho, \nu)$ is the opacity of the material to neutrinos, and $B = \nu^3/a^3[\exp(\nu/aT) + 1]$ is the blackbody source function for the neutrinos. (B is weighted doubly so as to represent both neutrinos and antineutrinos.) All interactions are treated as absorptions, though part of the opacity is due to elastic scattering. This treatment of elastic scattering overestimates the energy exchange for neutrino energies not near the electron mass ($\sim kT/c^2$).

b) Equations of State

We define the total pressure as

$$P = P_0(\rho) + P_1(\rho, T) + P_2(T) \,, \tag{11}$$

where P_0 is the pressure at zero temperature; and for this, three variations were used (see Fig. 1). For variation I, the Harrison-Wheeler equation was used (Harrison *et al.*

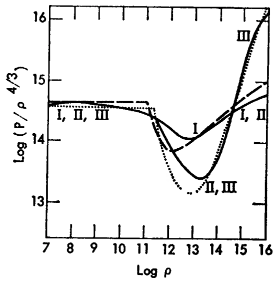

FIG. 1.—Pressure at zero temperature divided by $\rho^{4/3}$ versus density. *Variation I*, Harrison-Wheeler equations of state. *Variation II*, variation I equations except Langer equation for $3 \times 10^{11} < \rho < 3 \times 10^{14}$. *Variation III*, "hard core" pressure added to variation II (ρ is particle density in cm^{-3}). *Dotted line*, equation used by Colgate and White (1966); *dashed line*, equation used by Arnett (1966).

1965). For variation II, the Harrison-Wheeler equation was used except for the case $3.16 \times 10^{11} < \rho < 3.16 \times 10^{14}$, where we let

$$P_0 = 4.03 \times 10^{29} + 5.7 \times 10^{17}\rho + 2.06 \times 10^{10}\rho^3 ,$$

which is a fit to the work of Langer *et al.* (1969) (cgs units are used here and henceforth, except for temperature which is given in kilovolts). Variation III is the same as variation II except that a term $2.5 \times 10^5\, \rho^4/(\rho^2 + 10^{30})$ is added to P_0 to represent hard-core repulsion. Note that, since ρ is particle density, the velocity of sound approaches that of light at infinite ρ.

We define the thermal pressure P_1, which is due to electrons and nucleons, by the following expression:

$$P_1 = 1.7 \times 10^{13}\, \rho N T , \tag{12}$$

where

$$N = AZ(1 + 26Z) + 56(1 - AZ)(1 + Z)$$

and

$$Z = 1 - \exp \{-(T + 8888)/2.58\rho^{1/3}Z^{1/2}[A + 11(1 - A)]\} ,$$

$$A = 1 + [1 - (1 + 2\beta)^{1/2}]\, \beta^{-1} , \quad \beta = \rho \exp [(8888/T)]/(4.85 \times 10^5 T^{3/2}) .$$

In effect, Z represents the number of electrons per baryon and A the fraction of baryons in iron nuclei. This equation represents the thermal decomposition of ^{56}Fe to protons and/or neutrons.

In equation (11) we define P_2 as the pressure of particle pairs and electromagnetic radiation, given by

$$P_2 = 4.57 \times 10^{13} T^4\, [1 + \tfrac{7}{4} \exp (- 510/T) + 4.5 \exp (- 120000/T)] . \tag{13}$$

The associated energy is taken to be

$$\begin{aligned}\epsilon = \textstyle\int P_0\, d\rho/\rho^2 + (2.56 \times 10^{13} NT) + [1.28 \times 10^{19}(1 - A)] \\ + 1.37 \times 10^{14}\, T^4[1 + \tfrac{7}{4}(1 + 170/T) \exp (-510/T) \\ + 4.5(1 + 40000/T) \exp (-420000/T)]/\rho \text{ ergs g}^{-1} .\end{aligned} \tag{14}$$

The behavior of this equation of state is illustrated in Figure 2. It should be noted that this equation of state represents material in its most stable form. All nuclear reactions are taken as instantaneous.

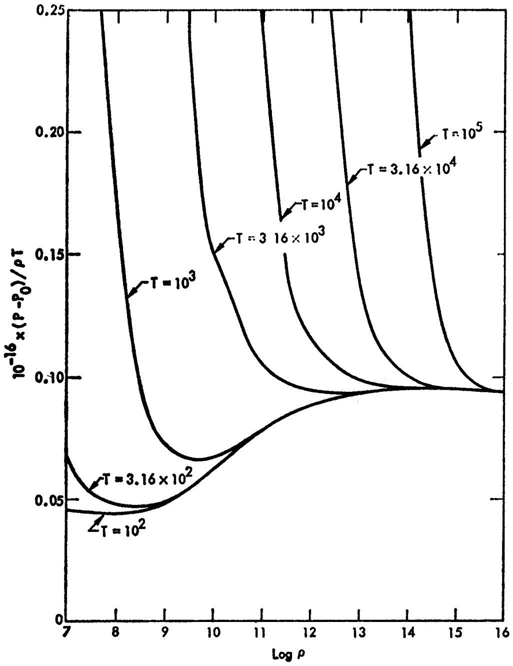

FIG. 2.—Pressure minus zero-temperature pressure divided by density and temperature versus density.

c) Opacity Formulae

The opacity of electron neutrinos is taken as the sum of K_p (plasma interaction), K_Δ (electron scattering), K_m (nuclear absorption), and K_ν (neutrino-antineutrino annihilation) as follows:

$$K_p = 2.0 \times 10^{-30}\rho Z^2\left[\frac{1}{\nu^2}(\nu^2 - 0.44Z^{2/3})^{1/2}\right], \tag{15}$$

$$K_\Delta = [6.3 \times 10^{-26}ZT/A + 2.48 \times 10^{-27}T^4/\rho]\nu \exp[-2.8(\rho Z)^{1/3}/\nu], \tag{16}$$

$$K_m = 2.4 \times 10^{-26}\rho\nu^2 \exp[-2.8(\rho Z])^{1/3}/\nu], \tag{17}$$

$$K_\nu = 1.57 \times 10^{-35}\left[\nu^3 + 170\left(\frac{R_1}{R + R_1}\right)^2 \int F\frac{d\nu' d\mu}{\nu'}\right] \exp\left[\frac{-2.8(\rho Z)^{1/3}}{\nu'}\right], \tag{18}$$

where ν is now the neutrino energy. The term $\exp[-2.8(\rho Z)^{1/3}/\nu]$ is a cutoff factor for the electron-degeneracy effect, and R_1 is the radius at one mean free path inward from the surface of the star. The factor $[R_1/(R + R_1)^2]$ is inserted into the K_ν formula to correct for nonisotropy of the neutrino velocity distribution. The ν^3 term in the neutrino-neutrino interaction is inserted to give the proper rate of neutrino production by electron-pair interaction. The nuclear absorption term is too high for low-energy neutrinos in a low-temperature region where nuclear structure is important. (See Fig. 3 for spectra at the surface of the star. Little energy is transported by these low-energy neutrinos.)

The opacity of muon neutrinos is taken as the sum of K_e, K_n (nuclear absorption), K_μ, and K_π (π-absorption) as follows:

$$K_e = 2.75 \times 10^{-28}(T^4/\rho + 2.53ZT/A)\nu \exp\left(\frac{-2.8 \times 10^9}{\nu T}\right)$$

$$\text{for} \quad (\nu_\mu + e^\pm \leftrightarrow \nu_e + \mu^\pm) \quad \text{and} \quad (\nu_\mu + \nu_e \leftrightarrow e^+ + \mu^-), \tag{19}$$

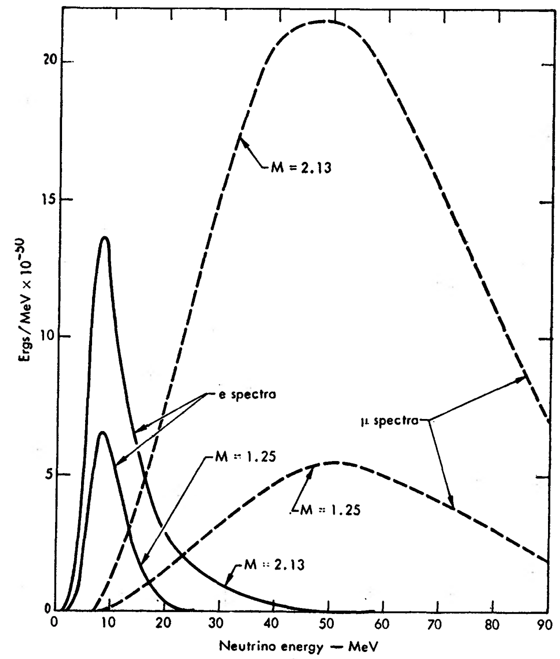

FIG. 3.—Spectra of emitted neutrinos for 1.25 and 2.13 $M_\odot$ stars. *Solid curves*, electron neutrinos; *dashed curves*, muon neutrinos.

$$K_n = 3.83 \times 10^{-26}\nu(\nu^2 - 1.12 \times 10^{10})^{1/2}\,, \tag{20}$$

$$K_\mu = \frac{1.66 \times 10^{-11}}{\rho}\left[T^{3/2}(1 + 1.5 \times 10^{-4}\nu) + 1.1 \times 10^{-16}T^4 \exp\left(\frac{-1.06 \times 10^5}{T}\right)\right]$$
$$\times \exp\left(\frac{-1.06 \times 10^5}{T}\right) \quad \text{for} \quad (\nu_\mu + \mu^\pm \leftrightarrow \nu_e + e^\pm)\,, \tag{21}$$

$$K_\pi = \begin{Bmatrix} 1.75 \times 10^{-9}T^{3/2}\nu/\Delta\nu \\ 1.8 \times 10^{-4}T^{1/2}\Delta\nu/\nu \end{Bmatrix} \exp\left(\frac{-1.06 \times 10^5}{T}\right), \tag{22}$$

where (case 1)λ is greater than R or (case 2)λ is less than R, where λ represents the mean free path for neutrino-muon capture. The quantity $\Delta\nu$ is the width of the neutrino energy group used in the calculation; it enters because the line width is much less than the group width. The mean free path for the process $\nu + \mu \rightarrow \pi$ is taken as $2.7 \times 10^4 \exp(1.12 \times 10^5/T)/T$.

The first choice is selected to give the correct rate of energy loss when the system is transparent. The second choice is selected when the system is opaque to the line. In this latter case, the line opacity is multiplied by the ratio of the width of the calculational group to the line width, since the radiation cannot be emitted directly by the line but must diffuse out.

It is assumed in the calculation of muon opacities that the muons, pions, and electron neutrinos are in statistical equilibrium. No account is taken of the degeneracy of muons. The aim of these opacity formulae is to give order-of-magnitude accuracy in the relevant regions of (ν, ρ, T)-space.

As an example of a typical derivation of these opacity formulae, consider the reaction $\nu_\mu + \mu^\pm \leftrightarrow \nu_e + e^\pm$. From Bahcall (1964), we take the cross-section to be

$$\sigma = \frac{\sigma_0}{24\pi}\left[k^2 + 2k\cdot P + \frac{k^2}{m^2}(4k\cdot P - k^2)\right]$$

for the reaction going to the right, where $\sigma_0 = 1.7 \times 10^{-44}$, k is the total four-momentum, P is the electron momentum, and m is the muon mass in units of electron mass. Neglecting m_e compared with all other energies and letting $k = q + \nu$, where q is the muon momentum and ν the muon-neutrino momentum,

$$\sigma \approx \frac{\sigma_0}{24\pi m_e^2}[3m^2 + 4\nu(m^2 + q^2)^{1/2} - 4\boldsymbol{\nu}\cdot \boldsymbol{q}]\,,$$

we approximate the number density of muons by

$$\frac{2}{(\hbar c)^3}\exp\left[-\frac{(m^2+q^2)^{1/2}}{T}\right];$$

then

$$K\rho = \frac{\sigma_0^2}{24\pi m_e^2(\hbar c)^3}\int dq^3[3m^2 + 4\nu(m^2+q^2)^{1/2} - 4\boldsymbol{\nu}\cdot\boldsymbol{q}]\exp\left[-\frac{(m^2+q^2)^{1/2}}{T}\right].$$

The integral is evaluated first for T, q, $\nu \ll m$, and then with q, ν, $T \gg m$. The high-energy $K\rho$ is multiplied by $\exp(-2m/T)$ and added to the low-energy $K\rho$ to give equation (21).

d) Numerical Techniques

The equations of motion (§ II*a*) are approximated by difference equations. The stars are described by fifty space zones (see Fig. 4 for typical mass zoning), four angle zones for the neutrino, fourteen energy zones for the electron-neutrino energy extending from 1 to 93 MeV in geometric steps, and thirteen energy zones for the muon-neutrino energy extending from 10 to 640 MeV. The zone energies are in the ratio $\sqrt{2}:1$.

The radiation difference equations are quite complicated, and hence it is difficult to ensure that they are error-free. Several comparison calculations have been made with other computer programs to test the radiation transport. Comparisons with laboratory explosion experiments have yielded good agreement for both optically thick and optically thin systems. The calculations presented here, however, contain more critical conditions for the interaction of radiation and hydrodynamics than any experiment on which they have been tested. When the star is radiating at its peak rate, the radiation pressure at 1 optical depth in from the outside is greater than the material pressure.

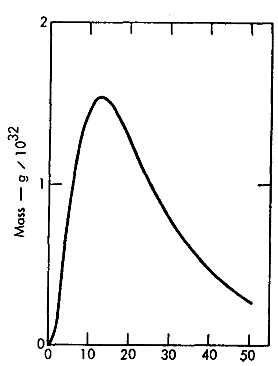

FIG. 4.—Mass of radial zones versus zone number from inside to outside for a total mass of 4.2×10^{33} g.

III. DISCUSSION

The initial configurations of the stars are established by selecting an isentropic configuration with a central density of about 10^7 g cm^{-3}. Until either the central γ-value $(\partial \log P/\partial \log \rho)_s$ is less than $\frac{4}{3}$ or the central temperature exceeds 500–700 keV, hydrodynamics is replaced by hydrostatics and the neutrino transport is replaced by equations of energy loss that assume stellar transparency (see the discussion by Chiu 1968). After the full calculation is turned on, collapse follows in 1–2 sec.

In Figure 5, the density and temperature are shown at the time when the full calculation is started. The 1.25 $M_\odot$ star ($M_\odot = 2.0 \times 10^{33}$ g) comes very close to the white-dwarf limit and hence has a very low initial temperature. This model has a central γ-value of $\frac{4}{3}$ at a density of about 2×10^8 g cm^{-3}. All the other masses that reach the temperature range 600–700 keV before γ become less than $\frac{4}{3}$. For the 4.3 $M_\odot$ star, γ is just under $\frac{4}{3}$ at starting time.

Generally, a star emits neutrinos slowly, cooling for about 1 sec, then entering a region where $\gamma < \frac{4}{3}$ describes the equation of state. Collapse of the star proceeds rapidly for some distance, being halted by $\gamma > \frac{4}{3}$. Increasingly rapid neutrino cooling sets in, which

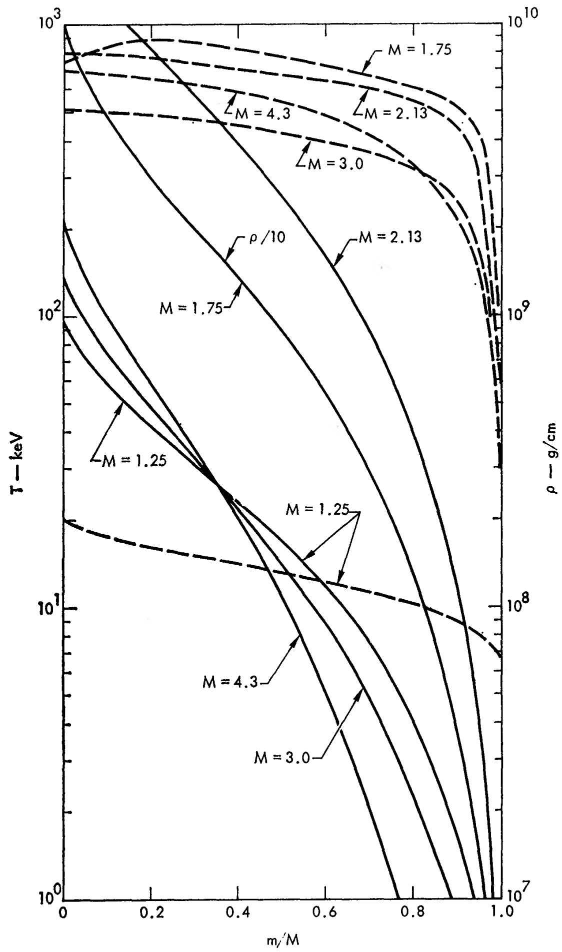

FIG. 5.—Initial density and temperature profiles for several star masses versus fraction of total mass. *Solid curves*, density; *dashed curves*, temperature. Labels are masses in units of solar mass.

lasts until final relativistic collapse of the star. Collapse starts at temperatures of about 100 MeV and densities near 10^{15} g cm^{-3}. In Figure 6, the rates of neutrino emission are given for several stellar masses. A star with the "hard core" addition to the zero-temperature pressure has a critical mass for gravitational collapse of about 2.1 $M_\odot$. For this equation of state and masses of less than 2.1 $M_\odot$, the rate of neutrino emission peaks out about as shown in Figure 6, slowly decreasing thereafter as the star cools. The emitted spectra are all fairly similar (see the sample spectra in Fig. 3). It should be noted that about 80–90 percent of the emitted neutrinos are μ-neutrinos, with the electron-neutrino spectra peaking at 8 keV and the μ-neutrino spectra peaking at 50 keV. The density-temperature trajectories of the central zones for the several masses are shown in Figure 7. The 1.25 $M_\odot$ star undergoes relatively greater shock heating after its initial collapse, hence at the higher densities is only slightly lower in entropy.

The 1.25 $M_\odot$ star is the only example from which material is ejected. In this case, 1.1×10^{32} g are ejected with a total energy of 2.8×10^{50} ergs (kinetic + internal − gravitational energy). From Figures 6 and 8, it can be seen that little energy is added to the ejected material during the period of high neutrino emission. At 1.2 sec, when the ejected material has acquired most of the energy it ever will, the star has emitted only 1.2×10^{51} ergs of neutrinos. Since 36×10^{51} ergs of neutrinos eventually are emitted, most neutrinos have little effect on the explosion.

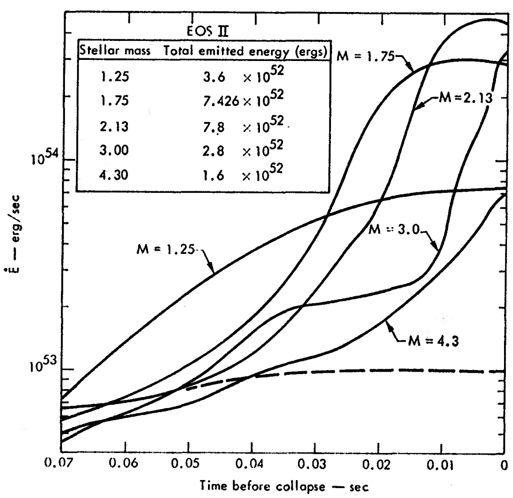

FIG. 6.—Neutrino luminosity as a function of time before final collapse, for several masses. Equation of state II was used for these examples. *Dashed curve*, electron-neutrino luminosity for the 2.13 $M_\odot$ star.

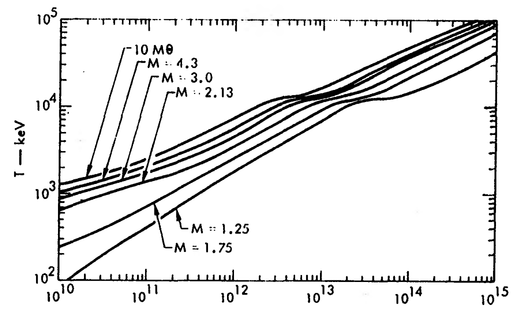

FIG. 7.—Central temperature versus central density for several stellar masses

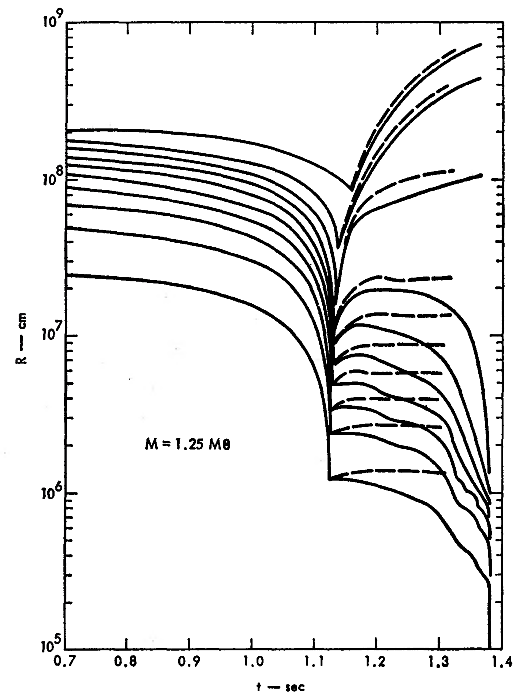

FIG. 8.—Radius of every fifth radial zone versus time for 1.25 $M_{\odot}$ star. Solid curves are with neutrino flow; dashed curves are without neutrino flow.

The calculation was repeated with no neutrino flow. The same amount of material was ejected, but the energy of the ejected material increased to 3.5×10^{50}. The neutrinos appear to act only as a slight damper on the shock wave. In Figure 8, the dashed lines represent results for the no-neutrino case.

Since little material is ejected, the only way to form a neutron star in the present model is to construct an equation of state in which the critical mass for the density of a neutron star is greater than the critical mass for a white dwarf (1.2 $M_{\odot}$ for the present equations). With equations of state III, stars of mass 1.75 and 2.1 $M_{\odot}$ settle quietly into a stable and relatively cold configuration (see Fig. 9 for radius changes versus time for the 2.1 $M_{\odot}$ model).

In Figure 10, several quantities are shown for the calculation of a 2.13 $M_{\odot}$ star just as its peak radiating time is reached. Particularly note the $\dot{T}$ curve. Very little neutrino heating occurs relative to cooling. From the curve P_r/P_m, it is seen that in the region of strong neutrino coupling the neutrino pressure is relatively small and that in the region of weak coupling the material pressure is relatively small. The difference between the total outward flux and the net outward flux is a measure of the degree of neutrino coupling. At this time the decoupling radius for electron neutrinos is about 7×10^6 cm, and that for muon neutronos is about 2.5×10^6 cm.

To test sensitivity to opacity, the model calculation for the 2.12 $M_{\odot}$ star was rerun with equation of state III, where K_e (electron neutrinos) and K_μ (muon neutrinos) were taken as follows:

$$K_e = 4 \times 10^{-26} \nu T \,, \quad K_\mu = 4 \times 10^{-26} \nu T \exp\left(\frac{-1.06 \times 10^5}{T}\right) . \tag{23}$$

These opacities agree roughly with the more complicated opacities (from 15 to 22) on the (ρ, T)-trajectory as given in Figure 7 for a mass of 2.1 $M_{\odot}$.

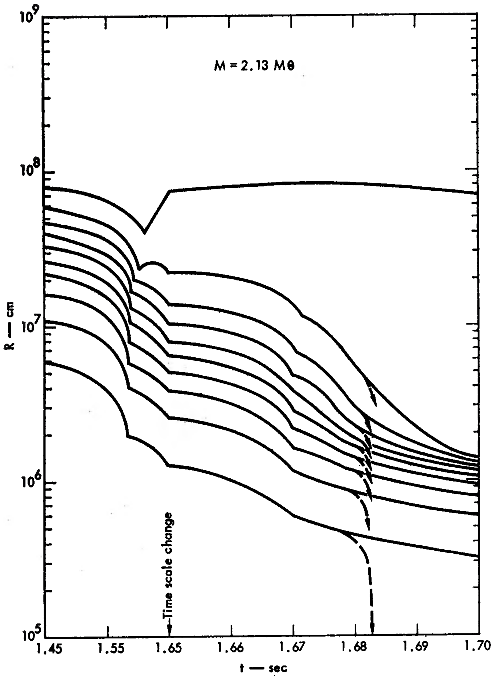

FIG. 9.—Radius of every fifth radial zone versus time. *Solid curves*, for equation of state III; *dashed curves*, for equation of state II. Note change of time scale at 1.65 sec.

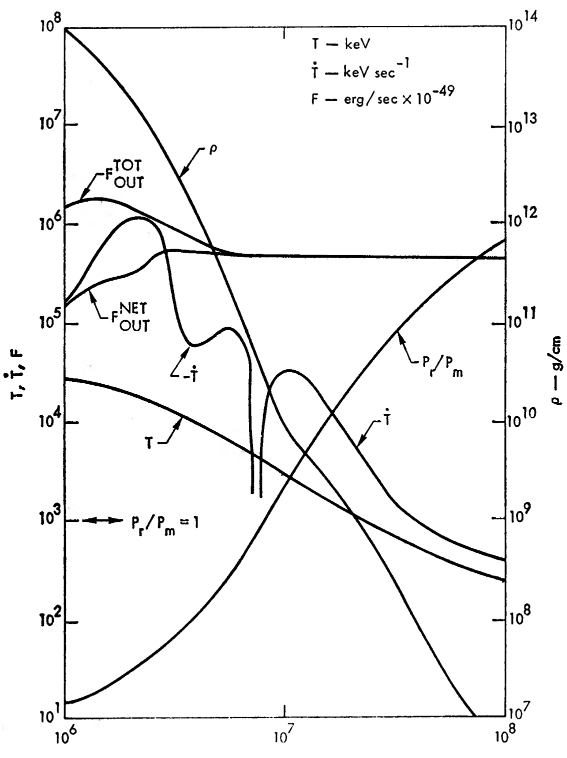

FIG. 10.—Density, temperature, rate of temperature change due to neutrino absorption and emission, ratio of neutrino pressure to material pressure, total outward neutrino flux, and net outward neutrino flux versus radius.

No significant differences were found between using equations of state I and II.

A low-density atmosphere of 5 $M_\odot$ extending out to 5×10^{10} cm was placed about a 2 $M_\odot$ core. The core mass was decreased by 5 percent in 0.03 sec, characteristic of the neutrino-emission time. This period is much shorter than that for hydrodynamic readjustment. An outward shock developed in the atmosphere, and material was ejected with an energy of about 10^{47} ergs. The adiabatic γ for the atmosphere was about 1.5. If γ were nearer $\frac{4}{3}$, this type of disruption could produce much more energy.

IV. CONCLUSIONS

The main conclusion of this study is that heat conduction by neutrinos does not blow off any material from a collapsing star. (In the calculation of the 1.25 $M_\odot$ star, neutrino flow degraded the explosion.) Little effect was found in these calculations for variations in equation of state and/or opacity. Previous authors (Colgate and White 1966; Colgate 1968; Arnett 1966, 1967, 1968; Schwartz 1967) have found explosions to result from neutrino flow. In order for a violent explosion to arise from neutrino flow, energy must be transported and deposited from the interior to the exterior in sufficient quantity to overcome the gravitational binding. In the present calculations, relatively little net heating in the exterior regions is observed (see Fig. 10). In the region where the neutrinos are about to decouple from the material, it is important to treat the the coupling properly. Arnett and Schwartz make the transition in a very abrupt manner from complete coupling of neutrinos and matter to a fully absorptive situation. In the calculation of a 2.1 $M_\odot$ star, the introduction of an artificial inconsistency (about 10 percent) in the ratio of energy deposition to energy flow produced an explosion comparable with that reported by the previous authors. Consider the situation in Figure 10. Gravitational energy of the material external to the cooling region ($R > 8 \times 10^6$) is about 2×10^{51} ergs in a mass ~ 0.1 $M_\odot$. The flux of neutrino energy passing through this region is 10^{53} ergs, hence the system is very sensitive to how the energy passes out through the last mean free path.

The high-mass stars (4.5–10 $M_\odot$) collapse supersonically and start to form an outward-moving shock from the core, but the core accumulates enough mass to be relativistically unstable before much of the neutrino energy can flow. Note that in Figure 6 the radiation rate $M = 4.3$ $M_\odot$ is rising rapidly at the end. The stars in the low-mass range $M < 4$ $M_\odot$ tend to have subsonic flow most of the time.

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