Analytical Lunar Ephemeris: Delaunay's Theory

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(Received 1 December 1970)

Delaunay's constants have been substituted into our analytical solution of the main problem of lunar theory. The results are compared with Delaunay's reduced formulas. Corrections are proposed to four terms in the mean motion of the perigee, three in the mean motion of the node, 45 in the reduced expression for the latitude, and 49 in that for the longitude.

I. RECONSTRUCTION OF DELAUNAY'S THEORY

THE solution to the main problem of lunar theory [Delaunay (1860, 1867), hereafter designated by TML] proceeds essentially in four steps.

(i) Canonical transformations from osculating elements (l,g,h,L,G,H) to normalized angles (l^*,g^*,h^*) and actions (L^*,G^*,H^*) are proposed to render the new angles and the Sun's mean anomaly l' ignorable in the transformed Hamiltonian $\mathfrak{M}^*$. This function is expressed implicitly in terms of the new actions (L^*,G^*,H^*) by means of the principal quantities a^*, e^*, and γ^* and of the secondary quantities n^*, m^*, α^* such that

$$n^{*2}a^{*3}=k^2(M+E),\quad m^*=n'/n^*,\quad \alpha^*=a^*/a'.$$

The algorithms used by Delaunay yield automatically the new actions L^*, G^*, H^* in terms of the quantities m^*, α^*, e^*, γ^*, e' (see TML, Chap. 6, pp. 235–236).

(ii) The normalizing phase variables are substituted in the longitude V, the latitude U, and the function $1/r$ (see TML, Chap. 7).

(iii) New functions $\bar{n}$, $\bar{e}$, $\bar{\gamma}$ of the actions L^*, G^*, H^* are defined implicitly by the following rules:

(a) $\bar{n}$ or the mean motion of the Moon is the coefficient of t in the transformed longitude V;

(b) $\bar{e}$ or the constant of eccentricity is so chosen that the coefficient of $\sin l^*$ in the transformed longitude V has the same form as the coefficient of $\sin l$ in the longitude developed in powers of the osculating eccentricity and the osculating quantity $\gamma=\sin I/2$.

(c) $\bar{\gamma}$ or the constant of inclination is so chosen that the coefficient of $\sin F^*$ in the transformed latitude U has the same form as the coefficient of $\sin F$ in the latitude developed in powers of the osculating eccentricity and of γ.

Secondary constants $\bar{m}$, $\bar{a}$, $\bar{\alpha}$ are introduced so that

$$\bar{n}^2\bar{a}^3=k^2(M+E),\quad \bar{m}=n'/\bar{n},\quad \bar{\alpha}=\bar{a}/a'.$$

By means of these definitions, the quantities n^*, e^*, γ^* are expressed in power series of the constants $\bar{m}$, $\bar{\alpha}$, $\bar{e}$, $\bar{\gamma}$, and e'. (See TML, Chap. 11, pp. 799 and 800.)

(iv) The expressions for n^*, e^*, γ^* in terms of $\bar{m}$, $\bar{\alpha}$, $\bar{e}$, $\bar{\gamma}$, and e' are substituted into the transformed coordinates V, U, and $1/r$. The results constitute Delaunay's reduced formulas (see TML, Chap. 11, pp. 803–924). These substitutions were also performed in the mean motions for the perigee and the node, first by Cayley (1871) and then by Delaunay (1872).

Delaunay proposed originally to normalize the Hamiltonian to order 8 so as to derive the mean motions of perigee and node to order 7, and to develop the angle coordinates V and U to order 7 whereas the function $1/r$ would be produced to order 5. In Delaunay's conventions, the constants $\bar{m}$, $\bar{e}$, and $\bar{\gamma}$ are of order 1 whereas $\bar{\alpha}$ is of order 2. As for the Sun's eccentricity e', the conventions are somewhat less rigid (see TML, Chap. 2, p. 33): e' and e'^2 are taken as quantities of order 1 and 2, respectively, but e'^3, e'^4, and e'^5 are taken as being of order 4, 5, and 6, and e'^6 as being of order 8.

These conventions have saved Delaunay a considerable amount of computations. But they are not always justified: Some terms containing a power e'^j for $j\geq 3$ appear already in the expansion of the perturbation function with substantially large coefficients. In our developments of the main problem, we have not adopted Delaunay's simplifications relative to e': For us e'^j is taken as being of order j.

We have normalized the Hamiltonian of the main problem in three steps. The monthly terms were eliminated first in one canonical transformation built on the model of a Lie transform (Deprit 1969), then the annual terms were all evacuated in one batch, and we completed the task by eliminating the long-period terms. We aimed at retaining in $\mathfrak{M}^*$ all secular terms of the type

$$m^{j_1}\alpha^{j_2}e^{j_3}\gamma^{j_4}e'^{j_5},\tag{1}$$

such that

$$k=j_1+2j_2\leq 15,\quad j_3+j_4+j_5\leq (19-k)/2.$$

In this way, we can reach the principal parts (i.e., not containing α, e, γ, e') in the mean motions down to power 15 of m (Deprit *et al.* 1970b).

By application of the formalism of Lie transforms, we have developed the longitude, the latitude, and the sine parallax in terms of the normalizing angles and actions. Then we constructed automatically Delaunay's reduced constants in terms of the normalizing actions and we reduced the expressions for the coordinates.

At this stage we needed to check our operations. We could have resorted to differential identities analogous to Adams' theorem (Desprit 1970), but it would have meant another major effort in programming and computing. Instead we decided for a comparison, term by

TABLE I. Corrections to Delaunay's solar terms in the mean motion of the perigee $d(h+g)/dt$.

m	α	e	γ	e'	Delaunay's numerator	ALE numerator
3			2	2	−349	−693/2
6		2			− 25 291 729	− 34 271 929
8					71 028 685 589	66 702 631 253
9					32 145 882 707 741	29 726 828 924 189

term, with Delaunay's published results. In this way our verifications could be useful to others. Delaunay's formulas are used in the theory of the outer satellites of Jupiter; they serve also to analyze the perturbations caused by the Earth on lunar orbiters (Lemekhova 1967). Recently they have been programmed to analyze laser data returned from the Moon (Barlier and Meyer, private communication).

II. CORRECTIONS TO DELAUNAY'S THEORY

Andoyer (1901) states that most of the terms of order 8 and 9 added by Delaunay in the Moon's longitude are erroneous. We have spot-checked this assertion and found it to be true. Therefore, we shall retain in Delaunay's formulas terms up to order 7 in the longitude and latitude and to order 5 in the sine parallax. Even at these low orders there are mistakes. Although he hardly entered the task of elaborating an analytical theory, Brown pointed to possible places where checks would be needed; this was done by comparing his numerical coefficients with the results of assigning a numerical value to the frequency ratio m in Delaunay's series (Brown 1892, 1893).

In the mean motions for the perigee (Table I) and for the node (Table II), the errors at order 5 and 7 may result from the faulty addition that we detected in the normalized Hamiltonian (Deprit *et al.* 1970a). The resulting adjustment in the motion of the node at order 5 is significant.

Delaunay's longitude contains 460 trigonometric, arguments numbered from 2 to 479 with gaps at 235, 238, 246, 248, 250–252, 313, 315, 317–319, 461, 464, 466, 468–470. Only 38 of them contain erroneous terms for a total of 49 corrections (see Table III). The order 7 in the variation (term sin $2D$, numbered 89 by Delaunay) is seriously in error. The new coefficients of m^6e' in sin $(4D-l')$ and sin $(6D+l')$ which are the terms 233 and 314 in Delaunay's list are identical to the corrections proposed by Andoyer (1902); they are very

TABLE II. Corrections to Delaunay's solar terms in the mean motion of the node dh/dt.

m	α	e	γ	e'	Delaunay's numerator	ALE numerator
3				2	23	33
3		2		2	−349	−693/2
5				2	73423	72873

substantial ones, the more so because they bear on the principal part of the terms.

Delaunay's latitude lists 423 trigonometric terms numbered from 1 to 436; the numbers 31, 289, 232–234, 277, 278, 295–297, 425–427 are missing from his list. We should mention two misprints: in term 152 (TML, Chap. 11, p. 883), instead of $m\gamma ee'^2$, read $m\gamma^3e'^2$, and in term 217 (TML, Chap. 11, p. 891), instead of sin $(2D-5l)$, read sin $(2D-5F)$. Thirty-five trigonometric arguments are in error for a total of 45 corrections (see Table IV).

We found no errors in the sine parallax as it is developed by Delaunay.

The lists of corrections given here are *not exhaustive*. In the longitude V, we have not yet examined the terms of order higher than 7 computed by Delaunay. Moreover, we do not present here the terms that should be added to Delaunay's formulas so that they contain all terms of characteristic

$$m^{j_1}\alpha^{j_2}e^{j_3}\gamma^{j_4}e'^{j_5} \quad \text{with} \quad j_1+2j_2+j_3+j_4+j_5\leq 7.$$

Printing these complements would be an unwieldy task. For instance, in the latitude, we found that 91 of Delaunay's trigonometric terms should be completed, and 33 trigonometric terms should be inserted, for a grand total of 164 additions. We hope that, in the near future, we shall have in the longitude, latitude, and sine parallax all terms with characteristics (1) satisfying either the conditions

$$k=j_1+2j_2\leq 15, \quad j_3+j_4+j_5\leq (19-k)/2$$

or the conditions

$$k=j_1+2j_2\leq 19, \quad j_3+j_4+j_5\leq (23-k)/2.$$

As far as the circumstances permit, they will be stored on tape in a format accessible to FORTRAN. But, at this point, we should already caution the users of Delaunay's original series against an essential omission at order 7 in Delaunay's work. In order to construct the longitude V and the latitude U, Delaunay started from their developments in powers of the osculating quantities e and γ carried only to order 6. However, the substitutions of the reduced constants brought in at some places terms of order 7 in e^* and γ^* that were similar in characteristic to some of the terms of order 7 neglected in the initial developments of V and U. For these terms, but for them only, Delaunay brought in these omitted contributions and combined them with the contributions yielded by the perturbations (see TML, Chap.

TABLE III. Corrections to Delaunay's solar terms in the longitude V.

No.	D	F	l	l'	m	α	e	γ	e'	Delaunay's numerator	ALE numerator	Delaunay's denominator	ALE denominator
8			1	−1	1		1	4	1	−351	−99		
12			1	1	1		1	4	1	351	99		
13			1	2	4		1		2	−1557809	−1702505		
58		2	−1		2		1	4		−6447	−6402		
58		2	−1		2		3	2		57025	56743		
58		2	−1		4		1	2		149363	74947		
68		2	−3		2		3	2		1231	8792		
89	2				3		2		2	−149497	−132025		
89	2				5				2	−61969	−91129		
95	2			2	3		2		2	−1782049	−1821475		
95	2			2	5				2	−28021	75659		
102	2		1	1	1	2	1		1	−945	−975		
111	2		3	−1	2		3		1	5705	5453		
111	2		3	−1	3		3		1	173819	170687		
118	2		−1		3		1		2	365281	363337		
118	2		−1		4		1		2	19912163	19901957		
124	2		−1	2	2		1	2	2	183	777	16	64
124	2		−1	2	3		1		2	−46561	−46399		
124	2		−1	2	4		1		2	−47974339	−73237483		
151	2	2		1	2		2	2	1	3785	3813		
151	2	2		1	3			2	1	613	617		
167	2	2	−2	−1	2		2	2	1	5255	5043		
169	2	2	−2	1	2		2	2	1	−6355	−6215		
184	2	−2		−1	4			2	1	−13733	−12837		
191	2	−2	1	−1	3		1	2	1	−81	−71	8	16
196	2	−2	2	−1	2		2	2	1	−6197	−7177		
198	2	−2	2	1	2		2	2	1	1881	2021		
205	2	−2	−1	−1	3		1	2	1	−40795	−39339		
221	2	−4		1	2			4	1	−9	9		
227	2	−4	−1		2		1	4		−45	159		
233	4			−1	6				1	54129983	52839679		
236	4			1	4		2		1	−60359	−145415		
237	4			2	4				2	161	201		
237	4			2	5				2	2429	14237	2560	7680
239	4		1		4		3			307749	320549		
240	4		1	−1	3		3		1	140105	136325		
255	4		−1	−2	4		1		2	2056689	2056609		
275	4	2	1		4		1	2		−467	−563		
314	6			1	6				1	2853	−3715		
369	1		−2		3	1	2			−680863	−797863		
372	1		−2	1	2	1	2		1	79689	216189		
399	1	−2		1		1	2	2	1	145	155	48	144
399	1	−2		1	2	1		2	1	25649	26795		
405	1	−2	2	1		1	2	2	1	−115	−395		
418	3		1		4	1	1			1712803	367957	30720	7680
419	3		1	−1	3	1	1		1	6095	25595	768	3072
421	3		1	1	3	1	1		1	−1847	−15181	128	1024
432	3		−2		3	1	2			−2937983	−2931935		
462	5				5	1				3911	1481		

7, p. 240, 241, and 414). To be consistent, Delaunay should have also thrown in the original terms of order 7 that were not modified by the perturbations. These are, for the longitude V,

$$-9893/960 \quad e^5\gamma^2 \sin(2F+5l),$$
$$137/12 \quad e^3\gamma^4 \sin(4F+3l),$$
$$-2 \quad e\gamma^6 \sin(6F+l),$$
$$-3 \quad e\gamma^6 \sin(6F-l),$$

and for the latitude,

$$-3/8 \quad \gamma^7 \sin F,$$
$$-7/8 \quad \gamma^7 \sin 3F,$$
$$1/4 \quad \gamma^7 \sin 5F,$$
$$-715/8008 \quad \gamma^7 \sin 7F,$$
$$-625/96 \quad \gamma e^6 \sin(F+4l),$$
$$117649/23040 \quad \gamma e^6 \sin(F+6l),$$
$$-2567/384 \quad \gamma^3 e^4 \sin(3F+4l),$$
$$75/32 \quad \gamma^5 e^2 \sin(5F+2l).$$

Delaunay's errors on the multipliers of $m^3e'^2$ and $m^5e'^2$ in the mean motion of the node have been detected first by Brown (1897). The first error being ascribed to a faulty addition in combining alike terms in Delaunay's normalized Hamiltonian (Deprit *et al.* 1970a), Brown proposes the correct coefficient for the term in $m^3e'^2$; however, he also suggests to change the multiplier of $m^5e'^2$ into 73413/1024, and this, according to our Table II, is still not correct. Adopting the good coefficient for $m^5e'^2$ reduces the gap in the characteristics of weight 2 between Delaunay and Brown.

We should also acknowledge that Brown (1897) gave

TABLE IV. Corrections to Delaunay's solar terms in the latitude U.

No.	D	F	l	l'	m	α	e	γ	e'	Delaunay's numerator	ALE numerator	Delaunay's denominator	ALE denominator
2		1		−1	5			1	1	−15533587	−15757075		
3		1		−2	4			1	2	−3005199	−2999367		
11		1	1	−1	4		1	1	1	3172499	3120659		
17		1	2		2		2	3		−1319	217		
31		1	−1		2		1	3		527	1103		
31		1	−1		2		3	1		−3903	−2751		
38		1	−2		2		2	3		−6939	−13743	128	256
38		1	−2		2		4	1		−9505	−9787		
38		1	−2		4		2	1		664607	587407		
67		3	−1		2		1	3		297	9		
72		3	−2		2		2	3		−283	−2219	32	256
87	2	1		1	5			1	1	1393231	1410727		
104	2	1	−1		3		1	3		−269275	−300379		
107	2	1	−1	−1	2		3	1	1	−23003	−23969		
107	2	1	−1	1	2		3	1	1	1269	1557		
113	2	1	−2	1	3		2	1	1	50767	51199		
114	2	1	−2	2	2		2	1	2	−17583	−17574		
115	2	1	−3		3		3	1		−5073	3567		
140	2	3	−3		1		3	3		105	195	16	64
143	2	−1			4			1	2	−249073	−97441		
144	2	−1		−1	3		2	1	1	150803	150355		
144	2	−1		−1	4			1	1	157133	158029		
144	2	−1		−1	5			1	1	3326245	3354277		
145	2	−1		−2	4			1	2	30398147	30424067		
148	2	−1		1	3			3	1	13421	5045		
149	2	−1		2	4			1	2	1666363	894689		
159	2	−1	2		2		4	1		11175	7335		
159	2	−1	2		2		2	3		3499	4459		
173	2	−1	−1		3		1	3		−20049	−21345		
173	2	−1	−1		3		3	1		−2421	−4365	64	128
174	2	−1	−1	−1	2		3	1	1	−5755	−5272		
181	2	−1	−2	−1	3		2	1	1	798241	373945		
183	2	−1	−2	1	3		2	1	1	1168081	1168297		
184	2	−1	−2	2	2		2	1	2	−114141	−114123		
197	2	−3		1	3			3	1	−7775	−3587		
199	2	−3	1		3		1	3		1363	931		
202	2	−3	1	1	2		1	3	1	−147	141		
208	2	−3	−1		1		1	3		−435	−135		
208	2	−3	−1		1		3	3		−3429	−3654		
208	2	−3	−1		3		1	3		−9381	−741		
222	4	1			2	2		1		−497	63		
225	4	1		1	3		2	1	1	−22065	−19125		
238	4	1	−1	1	4		1	1	1	−14157	−18477		
252	4	−1			2	2		1		−637	−77		
357	1	−1	1	1	2	1	1	1	1	78399	80691		

good reasons to admit that Delaunay's multiplier for e^2m^6 in the mean motion of the perigee was incorrect. But in this case he made no attempt at repairing the mistake.

III. CONCLUSIONS

All terms to order 7 have been reviewed in Delaunay's reduced formulas for the mean motions of perigee and node, the longitude, the latitude, and the sine parallax. Corrections have been made to the terms in error; these are relatively very few in number.

The examination of terms of order higher than 7 that have been added selectively by Delaunay is postponed until Andoyer's theory has been checked.

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