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THE EFFECT OF GRAVITATIONAL RADIATION ON THE SECULAR STABILITY OF THE MACLAURIN SPHEROID

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ABSTRACT

It is shown that the dissipation of energy by gravitational radiation induces (in the manner of viscosity) a secular instability of the Maclaurin spheroid at the point of bifurcation where the Jacobian sequence branches off. But the mode of oscillation that is made unstable by radiation reaction is not the same one that is made unstable by viscosity.

I. INTRODUCTION

It is known that the Maclaurin spheroid becomes *secularly unstable* (i.e., unstable in the presence of some dissipative mechanism such as viscosity) at the point of bifurcation where the sequence of the Jacobi ellipsoids branches off. (For an account of the theory of these classical spheroids and ellipsoids, see Chandrasekhar 1969; this book will be referred to hereafter as E.F.E.) Since energy is dissipated by gravitational radiation, the question arises whether the dissipation from this source can induce a secular instability of the Maclaurin spheroid in the manner viscosity does. This question cannot be answered without an explicit knowledge of the radiation-reaction terms in the equations of motion. This knowledge is now provided in a recent paper (Chandrasekhar and Esposito 1970) where the equations of motion governing a perfect fluid are derived to a sufficient approximation (in the framework of Einstein's field equations of general relativity) that terms representing the reaction of the fluid to the emission of gravitational radiation are explicitly present. With the aid of these equations we shall determine the effect of gravitational radiation on the modes of oscillation that eventually become dynamically unstable. It will appear that radiation reaction does induce a secular instability in the Maclaurin spheroid at the point of bifurcation (see Chandrasekhar 1970*a* where a preliminary account of this result is given).

II. THE RADIATION-REACTION TERMS AND THE NATURE OF THE OSCILLATIONS THAT ARE TO BE CONSIDERED

In a frame of reference in which the center of mass is at rest, the terms in the equations of motion which represent the radiation reaction are (Chandrasekhar and Esposito 1970, eqs. [58], [60], and [101])

$$\frac{1}{c}\underset{4}{T}^{\alpha j}{}_{;j} = \frac{1}{c^5}\left[-\rho Q_{00}{}^{(5)}\frac{dv_\alpha}{dt} - \tfrac{1}{2}\rho v_\alpha \frac{dQ_{00}{}^{(5)}}{dt} - \rho\frac{d}{dt}(v_\mu Q_{\mu\alpha}{}^{(5)}) - \tfrac{1}{2}\rho Q_{\mu\nu}{}^{(5)}\frac{\partial \mathfrak{B}_{\mu\nu}}{\partial x_\alpha} + \tfrac{1}{5}\rho x_\alpha G\frac{d^5 I_{\mu\mu}}{dt^5} - \tfrac{3}{5}\rho x_\mu G\frac{d^5 I_{\mu\alpha}}{dt^5}\right], \tag{1}$$

where

$$Q_{00}{}^{(5)} = \tfrac{4}{3}G\frac{d^3 I_{\mu\mu}}{dt^3} \quad \text{and} \quad Q_{\alpha\beta}{}^{(5)} = 2G\frac{d^3 I_{\alpha\beta}}{dt^3} - \tfrac{2}{3}G\delta_{\alpha\beta}\frac{d^3 I_{\mu\mu}}{dt^3}. \tag{2}$$

For axisymmetric configurations, such as the Maclaurin spheroids, $Q_{00}{}^{(5)}$ and $Q_{\alpha\beta}{}^{(5)}$ are both zero and the terms on the right-hand side of equation (1) vanish identically.

This result is consistent with the requirement that systems, which appear stationary to an inertial observer at infinity, cannot radiate gravitationally. Consequently, when considering the effect of radiation reaction on the normal modes of oscillation of such systems, it will suffice to evaluate $Q_{00}^{(5)}$ and $Q_{\alpha\beta}^{(5)}$ for the perturbed configuration.

If we describe the perturbed Maclaurin spheroid, in a frame of reference rotating with the angular velocity Ω of the equilibrium spheroid, by a Lagrangian displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$, then in the "virial method" (as described in E.F.E., chapters 2 and 5) the relevant perturbation equations are expressed in terms of the Cartesian tensor

$$V_{\alpha\beta} = \int_V \rho(\xi_\alpha x_\beta + \xi_\beta x_\alpha)dx \,. \tag{3}$$

Moreover, for the particular "toroidal" modes of oscillation in which we are interested, the only non-vanishing components of $\boldsymbol{V}$ are (cf. E.F.E., § *33b*, p. 83)

$$V_{11} - V_{22} \quad \text{and} \quad V_{12} \,; \tag{4}$$

the matrix $\boldsymbol{V}$ is accordingly traceless:

$$V_{\mu\mu} = 0 \,. \tag{5}$$

Before we can write down the modification of the equations (E.F.E., eqs. [46] and [47], p. 83) satisfied by the non-vanishing $V_{\alpha\beta}$'s, resulting from the allowance for radiation reaction, it is necessary to transform the terms in equation (1) to a uniformly rotating frame. To effect this transformation we must know in particular how the nth time derivative of the moment-of-inertia tensor transforms. In view of its special interest for problems involving gravitational radiation, we shall consider this transformation separately in § III below.

III. THE TRANSFORMATION OF d^nI/dt^n TO A UNIFORMLY ROTATING FRAME

To effect the required transformation, we shall follow a general method, due to Lebovitz, outlined in E.F.E., chapter 4 (§ 25).

Let $\boldsymbol{T}(t)$ denote the (orthogonal) linear transformation which relates the coordinates in the inertial and in the rotating frames:

$$x_{\text{rotating frame}} = \boldsymbol{T} x_{\text{inertial frame}} \,. \tag{6}$$

Then

$$\boldsymbol{\Omega}^* = \frac{d\boldsymbol{T}}{dt}\boldsymbol{T}^\dagger \tag{7}$$

(where $\boldsymbol{T}^\dagger$ is the transpose of $\boldsymbol{T}$) is an antisymmetric matrix whose vector-dual $\boldsymbol{\Omega}$, defined by

$$\Omega_i = \tfrac{1}{2}\epsilon_{ijk}\Omega^*_{jk} \,, \tag{8}$$

represents, in general, a time-dependent rotation. However, for our present purposes, it will suffice to restrict ourselves to the case when $\boldsymbol{\Omega}^*$ and $\boldsymbol{\Omega}$ are time independent and constants.

When $\boldsymbol{\Omega}^*$ is a constant matrix, it can be readily verified by induction that

$$\boldsymbol{T}\frac{d^n\boldsymbol{T}^\dagger}{dt^n} = (-\boldsymbol{\Omega}^*)^n \,. \tag{9}$$

Moreover, if the orientation of the axes in the rotating frame is so chosen that $\boldsymbol{\Omega}$ is along the x_3-axis, then we can write

$$\mathbf{\Omega}^* = \Omega \boldsymbol{\sigma} , \quad \text{where} \quad \boldsymbol{\sigma} = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} . \tag{10}$$

Turning to the transformation of $d^n I_{\alpha\beta}{}^{(i)}/dt^n$, where the superscript i distinguishes that the components of the moment-of-inertia tensor are referred to an inertial frame, we first observe that in a rotating frame this quantity must be replaced by

$$T_{\mu\alpha} T_{\nu\beta} \frac{d^n}{dt^n} (T^\dagger{}_{\alpha\xi} T^\dagger{}_{\beta\eta} I_{\xi\eta}{}^{(r)}) , \tag{11}$$

where the superscript r distinguishes that the moment-of-inertia tensor is now referred to the rotating frame. Expanding the expression (11) by Leibnitz's theorem, we obtain

$$\sum_{m=0}^{n} C^n{}_m \frac{d^{n-m} I_{\xi\eta}{}^{(r)}}{dt^{n-m}} T_{\mu\alpha} T_{\nu\beta} \frac{d^m}{dt^m} (T^\dagger{}_{\alpha\xi} T^\dagger{}_{\beta\eta}) , \tag{12}$$

while by equation (10),

$$\begin{aligned} T_{\mu\alpha} T_{\nu\beta} \frac{d^m}{dt^m} (T^\dagger{}_{\alpha\xi} T^\dagger{}_{\beta\eta}) &= \sum_{p=0}^{m} C^m{}_p \, T_{\mu\alpha} \frac{d^p T^\dagger{}_{\alpha\xi}}{dt^p} T_{\nu\beta} \frac{d^{m-p} T^\dagger{}_{\beta\eta}}{dt^{m-p}} \\ &= \sum_{p=0}^{m} C^m{}_p (-\mathbf{\Omega}^*)^p{}_{\mu\xi} (-\mathbf{\Omega}^*)^{m-p}{}_{\nu\eta} \\ &= \sum_{p=0}^{m} C^m{}_p (-1)^p (\mathbf{\Omega}^*)^p{}_{\mu\xi} (\mathbf{\Omega}^*)^{m-p}{}_{\eta\nu} , \end{aligned} \tag{13}$$

where in the last step we have used the antisymmetry of the matrix $\mathbf{\Omega}^*$. Inserting the result of the foregoing reduction in (12), we obtain

$$\frac{d^n \boldsymbol{I}^{(i)}}{dt^n} = \sum_{m=0}^{n} \sum_{p=0}^{m} C^n{}_m C^m{}_p (-1)^p \mathbf{\Omega}^{*p} \frac{d^{n-m} \boldsymbol{I}^{(r)}}{dt^{n-m}} \mathbf{\Omega}^{*m-p} . \tag{14}$$

If the moment-of-inertia tensor should be a constant in the rotating frame, so that $\boldsymbol{I}^{(r)}$ is independent of time, equation (14) gives

$$\frac{d^n \boldsymbol{I}^{(i)}}{dt^n} = \Omega^n \sum_{p=0}^{n} C^n{}_p (-1)^p \boldsymbol{\sigma}^p \boldsymbol{I}^{(r)} \boldsymbol{\sigma}^{n-p} . \tag{15}$$

If, in addition, the axis of rotation coincides with one of the principal axes of the moment-of-inertia tensor, then there is no loss of generality in supposing that $\boldsymbol{I}^{(r)}$ and $\boldsymbol{\sigma}$ are the following matrices of order 2:

$$\boldsymbol{I}^{(r)} = \begin{vmatrix} I_{11} & 0 \\ 0 & I_{22} \end{vmatrix} \quad \text{and} \quad \boldsymbol{\sigma} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} . \tag{16}$$

With this redefinition of $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^2 = -1$ and equation (15) can be simplified to give

$$\begin{aligned} \frac{d^{2n} \boldsymbol{I}^{(i)}}{dt^{2n}} &= (-1)^n 2^{2n-1} \Omega^{2n} (\boldsymbol{I}^{(r)} + \boldsymbol{\sigma} \boldsymbol{I}^{(r)} \boldsymbol{\sigma}) \\ &= (-1)^n 2^{2n-1} \Omega^{2n} (I_{11} - I_{22}) \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} , \end{aligned} \tag{17}$$

and

$$\frac{d^{2n+1}I^{(i)}}{dt^{2n+1}} = (-1)^n 2^{2n}\Omega^{2n+1}(I^{(r)}\boldsymbol{\epsilon} - \boldsymbol{\epsilon}I^{(r)}) = (-1)^n 2^{2n}\Omega^{2n+1}(I_{11} - I_{22})\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} . \tag{18}$$

We now turn to the reduction of equation (14) when the object considered is an axisymmetric configuration perturbed (in the rotating frame) by a Lagrangian displacement of the form

$$e^{\lambda t}\xi(x) , \tag{19}$$

where λ is a characteristic-value parameter to be determined. Then

$$\frac{d^{n-m}I^{(r)}}{dt^{n-m}} = e^{\lambda t}\lambda^{n-m}V , \tag{20}$$

where V is the matrix formed of the elements $V_{\alpha\beta}$ defined in equation (3). Inserting this last result in equation (14) and suppressing the common factor $e^{\lambda t}$, we have

$$\delta I^{(n)} = \sum_{m=0}^{n} C^n{}_m \lambda^{n-m} S_m , \tag{21}$$

where

$$S_m = \Omega^m \sum_{p=0}^{m} C^m{}_p (-1)^p \boldsymbol{\epsilon}^p V \boldsymbol{\epsilon}^{m-p} . \tag{22}$$

As we have stated in § II, we are interested in perturbations for which the only non-vanishing elements of V are V_{11}, V_{22}, and V_{12}. Under these circumstances we may suppose that V and $\boldsymbol{\epsilon}$ are the matrices

$$V = \begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} \quad \text{and} \quad \boldsymbol{\epsilon} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} . \tag{23}$$

As here redefined, $\boldsymbol{\epsilon}^2 = -1$; and this fact enables an explicit evaluation of the sum S_m. We find

$$S_{2m} = (-1)^m 2^{2m-1}\Omega^{2m}(V + \boldsymbol{\epsilon}V\boldsymbol{\epsilon}) \quad \text{and} \quad S_{2m+1} = (-1)^m 2^{2m}\Omega^{2m+1}(V\boldsymbol{\epsilon} - \boldsymbol{\epsilon}V) , \tag{24}$$

where it may be noted that

$$V + \boldsymbol{\epsilon}V\boldsymbol{\epsilon} = \begin{vmatrix} V_{11} - V_{22} & 2V_{12} \\ 2V_{12} & -V_{11} + V_{22} \end{vmatrix} \quad \text{and} \quad V\boldsymbol{\epsilon} - \boldsymbol{\epsilon}V = \begin{vmatrix} -2V_{12} & V_{11} - V_{22} \\ V_{11} - V_{22} & +2V_{12} \end{vmatrix} . \tag{25}$$

From equations (21), (22), and (24) we find, in particular,

$$\delta I^{(3)} = \lambda^3 V - 6\lambda\Omega^2(V + \boldsymbol{\epsilon}V\boldsymbol{\epsilon}) + \Omega(3\lambda^2 - 4\Omega^2)(V\boldsymbol{\epsilon} - \boldsymbol{\epsilon}V) , \tag{26}$$

$$\delta I^{(4)} = \lambda^4 V - 4\Omega^2(3\lambda^2 - 2\Omega^2)(V + \boldsymbol{\epsilon}V\boldsymbol{\epsilon}) + 4\lambda\Omega(\lambda^2 - 4\Omega^2)(V\boldsymbol{\epsilon} - \boldsymbol{\epsilon}V) , \tag{27}$$

and

$$\delta I^{(5)} = \lambda^5 V - 20\lambda\Omega^2(\lambda^2 - 2\Omega^2)(V + \boldsymbol{\epsilon}V\boldsymbol{\epsilon}) + \Omega(5\lambda^4 - 40\lambda^2\Omega^2 + 16\Omega^4)(V\boldsymbol{\epsilon} - \boldsymbol{\epsilon}V) . \tag{28}$$

IV. THE RADIATION-REACTION TERMS FOR THE TOROIDAL OSCILLATIONS OF THE MACLAURIN SPHEROIDS

We now turn to the evaluation of the radiation-reaction terms for the toroidal oscillations of the Maclaurin spheroid.

It follows from equations (21), (24), and (25), in agreement with the requirement that $V_{\mu\mu} = 0$ for these oscillations, that the terms in $Q_{00}^{(5)}$ and $d^5 I_{\mu\mu}/dt^5$ in the radiation-reaction terms (1) do not make any contributions. We are therefore left to consider only the terms

$$\frac{1}{c^5}\left(-\rho\frac{dv_\mu}{dt}Q_{\mu\alpha}^{(5)} - \rho v_\mu \frac{dQ_{\mu\alpha}^{(5)}}{dt} - \tfrac{3}{5}\rho x_\mu G\frac{d^5 I_{\mu\alpha}}{dt^5} - \tfrac{1}{2}\rho Q_{\mu\nu}^{(5)}\frac{\partial \mathfrak{B}_{\mu\nu}}{\partial x_\alpha}\right), \tag{29}$$

where, in accordance with equation (2) and the results of § III, we may now let

$$Q_{\alpha\beta}^{(5)} \to 2G\delta I_{\alpha\beta}^{(3)}, \tag{30}$$

with $\delta I^{(3)}$ having the value given in equation (26).

In evaluating the first and the second terms in (29) in the rotating frame, we may replace $\boldsymbol{v}$ and $d\boldsymbol{v}/dt$ by their values

$$\boldsymbol{v} = \boldsymbol{\Omega}\times\boldsymbol{x} \quad \text{and} \quad d\boldsymbol{v}/dt = -\Omega^2(x_1, x_2, 0), \tag{31}$$

appropriate for the equilibrium spheroid. Also, we know that (E.F.E., eq. [126] p. 57)

$$\mathfrak{B}_{\alpha\beta} = \pi G\rho\left[2B_{\alpha\beta}x_\alpha x_\beta + a_\alpha^2\delta_{\alpha\beta}\left(A_\alpha - \sum_{\mu=1}^{3}A_{\alpha\mu}x_\mu^2\right)\right] \tag{32}$$

(no summation over repeated indices except when explicitly indicated), where $A_{\alpha\ldots}$ and $B_{\alpha\ldots}$ are the "index symbols" as usually defined in this theory (E.F.E., p. 54).

Inserting from equations (30)–(32) in (29), we find that the 1- and the 2-components of the radiation-reaction terms, in the rotating frame, are

$$\begin{aligned}\frac{\rho G}{c^5}[\![&-\pi G\rho\{[4B_{11}\delta I_{11}^{(3)} - 2a_1^2A_{11}(\delta I_{11}^{(3)} + \delta I_{22}^{(3)})]x_1 + 4B_{11}\delta I_{12}^{(3)}x_2\} \\ &+2\Omega^2(\delta I_{11}^{(3)}x_1 + \delta I_{12}^{(3)}x_2) + 2\Omega(\delta I_{11}^{(4)}x_2 - \delta I_{12}^{(4)}x_1) \\ &-\tfrac{3}{5}(\delta I_{11}^{(5)}x_1 + \delta I_{12}^{(5)}x_2)]\!] \qquad \text{(1-component)}\end{aligned} \tag{33}$$

and

$$\begin{aligned}\frac{\rho G}{c^5}[\![&-\pi G\rho\{[4B_{11}\delta I_{22}^{(3)} - 2a_1^2A_{11}(\delta I_{11}^{(3)} + \delta I_{22}^{(3)})]x_2 + 4B_{11}\delta I_{12}^{(3)}x_2\} \\ &+2\Omega^2(\delta I_{12}^{(3)}x_1 + \delta I_{22}^{(3)}x_2) + 2\Omega(\delta I_{12}^{(4)}x_2 - \delta I_{22}^{(4)}x_1) \\ &-\tfrac{3}{5}(\delta I_{12}^{(5)}x_1 + \delta I_{22}^{(5)}x_2)]\!] \qquad \text{(2-component)}.\end{aligned} \tag{34}$$

a) The Contributions of the Radiation-Reaction Terms to the Second-Order Virial Equations

The required contributions to the second-order virial equations can be obtained by evaluating the x_1 and the x_2 moments of the terms (33) and (34). We thus obtain

(1,1)-component:

$$\begin{aligned}\frac{GI_{11}}{c^5}[2(\Omega^2 - 2\pi G\rho B_{11})\delta I_{11}^{(3)} &+ 2\pi G\rho a_1^2A_{11}(\delta I_{11}^{(3)} + \delta I_{22}^{(3)}) \\ &- 2\Omega\delta I_{12}^{(4)} - \tfrac{3}{5}\delta I_{11}^{(5)}];\end{aligned} \tag{35}$$

(1,2)-component:

$$\frac{GI_{11}}{c^5}[2(\Omega^2 - 2\pi G\rho B_{11})\delta I_{12}{}^{(3)} + 2\Omega\delta I_{11}{}^{(4)} - \tfrac{3}{5}\delta I_{12}{}^{(5)}]\,; \tag{36}$$

(2,1)-component:

$$\frac{GI_{11}}{c^5}[2(\Omega^2 - 2\pi G\rho B_{11})\delta I_{12}{}^{(3)} - 2\Omega I_{22}{}^{(4)} - \tfrac{3}{5}\delta I_{12}{}^{(5)}]\,; \tag{37}$$

(2,2)-component:

$$\frac{GI_{11}}{c^5}[2(\Omega^2 - 2\pi G\rho B_{11})\delta I_{22}{}^{(3)} + 2\pi G\rho a_1{}^2 A_{11}(\delta I_{11}{}^{(3)} + \delta I_{22}{}^{(3)}) + 2\Omega\delta I_{12}{}^{(4)} - \tfrac{3}{5}\delta I_{22}{}^{(5)}]\,. \tag{38}$$

V. THE CHARACTERISTIC EQUATION INCLUDING THE RADIATION-REACTION TERMS

In obtaining the characteristic equation for the toroidal modes of oscillation of the Maclaurin spheroid by the virial method (E.F.E., § 33*b*) one forms the sum of the (1,2)- and the (2,1)-components, and similarly the difference of the (1,1)- and the (2,2)-components, of the second-order virial equations. By making use of the expressions (35)–(38) given in § IV*a*, we find that the contributions of the radiation-reaction terms to these sum and difference equations are

$$\frac{GI_{11}}{c^5}[4(\Omega^2 - 2\pi G\rho B_{11})\delta I_{12}{}^{(3)} + 2\Omega(\delta I_{11}{}^{(4)} - \delta I_{22}{}^{(4)}) - \tfrac{6}{5}\delta I_{12}{}^{(5)}] \qquad \{[(1,2) + (2,1)]\text{-terms}\} \tag{39}$$

and

$$\frac{GI_{11}}{c^5}[2(\Omega^2 - 2\pi G\rho B_{11})(\delta I_{11}{}^{(3)} - \delta I_{22}{}^{(3)}) - 4\Omega\delta I_{12}{}^{(4)} - \tfrac{3}{5}(\delta I_{11}{}^{(5)} - \delta I_{22}{}^{(5)})] \qquad \{[(1,1) - (2,2)]\text{-terms}\}\,. \tag{40}$$

Now substituting for the various matrix elements of $\delta I^{(n)}$ in accordance with equations (25)–(28) and measuring λ and Ω in the unit $(\pi G\rho)^{1/2}$, we obtain

$$\begin{aligned}
&\frac{GI_{11}(\pi G\rho)^{5/2}}{c^5}\{4(\Omega^2 - 2B_{11})[\lambda(\lambda^2 - 12\Omega^2)V_{12} + \Omega(3\lambda^2 - 4\Omega^2)(V_{11} - V_{22})] \\
&\quad + 2\Omega[(\lambda^4 - 24\lambda^2\Omega^2 + 16\Omega^4)(V_{11} - V_{22}) - 16\lambda\Omega(\lambda^2 - 4\Omega^2)V_{12}] \\
&\quad - \tfrac{6}{5}[\lambda(\lambda^4 - 40\lambda^2\Omega^2 + 80\Omega^4)V_{12} + \Omega(5\lambda^4 - 40\lambda^2\Omega^2 + 16\Omega^4)(V_{11} - V_{22})]\} \\
&= \frac{GI_{11}(\pi G\rho)^{5/2}}{c^5}\{2[2\lambda(\lambda^2 - 12\Omega^2)(\Omega^2 - 2B_{11}) - \tfrac{3}{5}\lambda^5 + 8\lambda^3\Omega^2 + 16\lambda\Omega^4]V_{12} \\
&\quad - \tfrac{1}{2}[-8\Omega(3\lambda^2 - 4\Omega^2)(\Omega^2 - 2B_{11}) + 8\lambda^4\Omega - \tfrac{128}{5}\Omega^5](V_{11} - V_{22})\} \\
&\qquad\qquad \{[(1,2) + (2,1)]\text{-terms}\}
\end{aligned} \tag{41}$$

and

$$\frac{GI_{11}(\pi G\rho)^{5/2}}{c^5}\{2(\Omega^2-2B_{11})[\lambda(\lambda^2-12\Omega^2)(V_{11}-V_{22})-4\Omega(3\lambda^2-4\Omega^2)V_{12}]$$
$$-4\Omega[(\lambda^4-24\lambda^2\Omega^2+16\Omega^4)V_{12}+4\lambda\Omega(\lambda^2-4\Omega^2)(V_{11}-V_{22})]$$
$$-\tfrac{3}{5}[\lambda(\lambda^4-40\lambda^2\Omega^2+80\Omega^4)(V_{11}-V_{22})-4\Omega(5\lambda^4-40\lambda^2\Omega^2+16\Omega^4)V_{12}]\}$$
$$=\frac{GI_{11}(\pi G\rho)^{5/2}}{c^5}\{[2\lambda(\lambda^2-12\Omega^2)(\Omega^2-2B_{11})-\tfrac{3}{5}\lambda^5+8\lambda^3\Omega^2+16\lambda\Omega^4](V_{11}-V_{22}) \tag{42}$$
$$+[-8\Omega(3\lambda^2-4\Omega^2)(\Omega^2-2B_{11})+8\lambda^4\Omega-\tfrac{128}{5}\Omega^5]V_{12}\}$$
$$\{[(1,1)-(2,2)]\text{-terms}\} .$$

Now letting

$$Q_1=2\lambda(\lambda^2-12\Omega^2)(\Omega^2-2B_{11})-\tfrac{3}{5}\lambda^5+8\lambda^3\Omega^2+16\lambda\Omega^4$$

and

$$Q_2=-8\Omega(3\lambda^2-4\Omega^2)(\Omega^2-2B_{11})+8\lambda^4\Omega-\tfrac{128}{5}\Omega^5 , \tag{43}$$

we can rewrite the expressions (41) and (42) in the forms

$$\frac{GI_{11}(\pi G\rho)^{5/2}}{c^5}[2Q_1V_{12}-\tfrac{1}{2}Q_2(V_{11}-V_{22})] \qquad \{[(1,2)+(2,1)]\text{-terms}\} \tag{44}$$

and

$$\frac{GI_{11}(\pi G\rho)^{5/2}}{c^5}[Q_1(V_{11}-V_{22})+Q_2V_{12}] \qquad \{[(1,1)-(2,2)]\text{-terms}\} . \tag{45}$$

Including the foregoing terms in the equations given in E.F.E. (eqs. [44] and [45] on p. 83), we obtain the pair of equations

$$[\lambda^2+2(2B_{11}-\Omega^2)]V_{12}+\lambda\Omega(V_{11}-V_{22})+D[2Q_1V_{12}-\tfrac{1}{2}Q_2(V_{11}-V_{22})]=0 \tag{46}$$

and

$$[\tfrac{1}{2}\lambda^2+(2B_{11}-\Omega^2)](V_{11}-V_{22})-2\lambda\Omega V_{12}+D[Q_1(V_{11}-V_{22})+Q_2V_{12}]=0 , \tag{47}$$

where

$$D=\frac{GI_{11}(\pi G\rho)^{3/2}}{c^5} \tag{48}$$

is a non-dimensional constant. Equations (46) and (47) can be rewritten in the forms

$$[\lambda^2+2(2B_{11}-\Omega^2)+2DQ_1]V_{12}+(\lambda\Omega-\tfrac{1}{2}DQ_2)(V_{11}-V_{22})=0 \tag{49}$$

and

$$[\lambda^2+2(2B_{11}-\Omega^2)+2DQ_1](V_{11}-V_{22})-4(\lambda\Omega-\tfrac{1}{2}DQ_2)V_{12}=0 . \tag{50}$$

Equations (49) and (50) lead to the characteristic equation

$$[\lambda^2+2(2B_{11}-\Omega^2)+2DQ_1]^2+4(\lambda\Omega-\tfrac{1}{2}DQ_2)^2=0 . \tag{51}$$

Writing

$$\lambda=i\sigma , \tag{52}$$

we can factorize equation (51) to give

$$\sigma^2-2\sigma\Omega-2(2B_{11}-\Omega^2)-D(2Q_1+iQ_2)=0 \tag{53}$$

and a similar equation with $-\Omega$ in place of Ω. With Q_1 and Q_2 given by equations (43), we verify that

$$2Q_1 + iQ_2 = -4i(2\Omega - \sigma)^3[(2B_{11} - \Omega^2) + \tfrac{1}{10}(2\Omega - \sigma)(4\Omega + 3\sigma)] . \tag{54}$$

Accordingly, equation (53) becomes

$$\sigma^2 - 2\sigma\Omega - 2(2B_{11} - \Omega^2) + 4iD(2\Omega - \sigma)^3[(2B_{11} - \Omega^2) + \tfrac{1}{10}(2\Omega - \sigma)(4\Omega + 3\sigma)] . \tag{55}$$

Letting

$$\sigma = \sigma_0 + \Delta\sigma , \tag{56}$$

where σ_0 is a characteristic frequency in the absence of radiation reaction, we find from equation (55) that

$$\Delta\sigma = -2iD\frac{(2\Omega - \sigma_0)^3}{\sigma_0 - \Omega}[(2B_{11} - \Omega^2) + \tfrac{1}{10}(2\Omega - \sigma_0)(4\Omega + 3\sigma_0)] + O(D^2) . \tag{57}$$

By making use of the equation satisfied by σ_0, we find that equation (57) can be reduced to the remarkably simple form

$$\Delta\sigma = -\tfrac{2}{5}iD\frac{(2\Omega - \sigma_0)^5}{\sigma_0 - \Omega} . \tag{58}$$

In the absence of radiation reaction, equation (55) allows two roots:

$$\sigma_0^{(1)} = \Omega - (4B_{11} - \Omega^2)^{1/2} \quad \text{and} \quad \sigma_0^{(2)} = \Omega + (4B_{11} - \Omega^2)^{1/2} . \tag{59}$$

It is known (E.F.E., p. 99) that viscosity induces a secular instability of the mode $\sigma_0^{(1)}$ which becomes neutral at the point of bifurcation where $\Omega^2 = 2B_{11}$. We shall now show that radiation reaction induces a similar instability of the mode $\sigma_0^{(2)}$ which acquires a frequency 2Ω at the same point.

For the mode $\sigma_0^{(2)}$, equation (58) gives

$$i\Delta\sigma = -\tfrac{2}{5}D\frac{[(4B_{11} - \Omega^2)^{1/2} - \Omega]^5}{(4B_{11} - \Omega^2)^{1/2}} . \tag{60}$$

From equation (60) it follows that *while the mode $\sigma_0^{(2)}$ is damped prior to the point of bifurcation at $\Omega^2 = 2B_{11}$, it is amplified in the interval* $4B_{11} > \Omega^2 > 2B_{11}$. And the e-folding time of the instability is given by

$$\tau = \frac{1}{D(\pi G\rho)^{1/2}} \frac{5(4B_{11} - \Omega^2)^{1/2}}{2[\Omega - (4B_{11} - \Omega^2)^{1/2}]^5} . \tag{61}$$

A brief listing of this time in the unit $(\pi G\rho)^{-1/2}/D$ is given in Table 1.

Equation (60) should be contrasted with the equation

$$i\Delta\sigma = -\frac{5\nu}{a_1^2}\frac{(4B_{11} - \Omega^2)^{1/2} - \Omega}{(4B_{11} - \Omega^2)^{1/2}} \tag{62}$$

which gives the effect of a kinematic viscosity (ν) of the fluid on the mode $\sigma_0^{(1)}$.

We have thus shown that *radiation reaction, like viscosity, makes the Maclaurin spheroid secularly unstable past the point of bifurcation. But the mode that is made unstable by radiation reaction is not the same one that is made unstable by viscosity.*

Finally, we may note that the singularity at $\Omega^2 = 4B_{11}$ which the solution (60) exhibits is spurious. It arises solely from the inadmissibility of seeking a solution for

$\Delta\sigma$ of $O(D)$ in the neighborhood of this point: when $\Omega^2 = 4B_{11}$ and $\sigma_0 = \Omega$, equation (55) directly gives

$$\sigma = \Omega \pm (0.4D\Omega^5)^{1/2}(1 - i) \quad \text{at} \quad \Omega^2 = 4B_{11} . \tag{63}$$

VI. CONCLUDING REMARK

The principal fact that has emerged from the present analysis is that radiation reaction *can* induce secular instabilities. This fact must clearly have an important bearing on what may happen during the last stages of a gravitational collapse. Some additional comments bearing on this question are made in the paper following this one (Chandrasekhar 1970*b*).

TABLE 1

THE *e*-FOLDING TIME OF THE SECULAR INSTABILITY INDUCED GRAVITATIONAL RADIATION*

e	τ	e	τ
0.81267....	∞	0.89.......	1.700×10^3
0.82.......	6.785×10^8	0.90.......	7.328×10^2
0.83.......	8.100×10^6	0.91.......	3.249×10^2
0.84.......	7.262×10^5	0.92.......	1.436×10^2
0.85.......	1.323×10^5	0.93.......	6.033×10^1
0.86.......	3.461×10^4	0.94.......	2.171×10^1
0.87.......	1.122×10^4	0.95.......	3.885×10^0
0.88.......	4.173×10^3	0.95289....	0

* τ is listed in the unit $(\pi G\rho)^{-1/2}/D$.

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