The Astrophysical Journal, Vol. 158, October 1969

EVIDENCE OF TIDAL EFFECTS IN SOME PULSATING STARS

II. 16 LACERTAE AND β CEPHEI

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Received 1968 December 26; revised 1969 March 10

ABSTRACT

Published observations of two β CMa stars are analyzed. The star 16 Lac is a single-line spectroscopic binary with a circular orbit of $12^{d}096$ period; the primary star pulsates in a primary period of $0^{d}169166$. In addition, two nonradial modes are excited, apparently by coupling to the primary pulsation through rotation and through a tidal resonance at 0.2 orbital period, respectively. The star β Cep is also found to be a single-line spectroscopic binary with an orbit of large eccentricity of $10^{d}893$ period. The orbital velocity of the primary is small because of a large primary-to-secondary mass ratio; the orbital elements are therefore rather uncertain, but they show that the primary pulsation ($P_1 = 0^{d}19049$) undergoes a strong, nonresonant amplitude modulation around the time of periastron passage, with maximum amplitude probably occurring at maximum tidal compression.

A simplified theory of resonant and nonresonant tidal modulation is presented. A summary is given of presently available information relating to tidal modulations of five β CMa stars and three δ Sct stars.

I. INTRODUCTION

It has been shown previously (Fitch 1967; hereinafter referred to as Paper I) that the long-period modulations observed in the light variation of the δ Sct star CC And and in the radial-velocity variation of the β CMa (β Cep) star σ Sco appear to originate in disturbances of the normal pulsation mode of each star by tidal deformations produced in the outer layers of each primary by a faint companion. In a continuation of this work, we here present the results of analyses of published observations of the light and velocity variations of the β CMa star 16 Lac (§ II) and of the velocity variation of β Cep itself (§ III). In § IV a summary is given of the currently available information on the tidal modulations of the pulsation modes of the β CMa and δ Sct stars.

II. 16 LAC

a) Analysis of the Observational Data

The star 16 Lac was announced by Struve and Bobrovnikoff (1925) as a single-line spectroscopic binary of $12^{d}3106$ period, but its intrinsic variability as a β CMa star was discovered by Walker (1951*a*). The observations used here are the radial-velocity measures obtained on twenty-six nights in 1951 by Struve *et al.* (1952) and on seven nights in 1952 and 1954 by McNamara (1957), together with the data contained in the differential blue-light curves (16 Lac minus 14 Lac) published by Walker (1951*a*, 1952*a*, 1954). Because the original lists of the photoelectric measures were lost (Walker, personal communication), it was necessary to read the observations from photographic enlargements of the published light curves. For the benefit of interested workers, these measures, which are necessarily less accurate than the originals, have been submitted for inclusion in the IAU(27). RAS file of the Royal Astronomical Society library. All of Walker's blue-magnitude observations were employed, except that the two nights of 1951 August 10 and 11 were omitted because of the very low signal-to-noise ratio evident in the light curves for these nights. The light curves published by Miczaika (1952) were not used because they were made without a filter, and on many nights they appear to have a very low signal-to-noise ratio.

A preliminary periodogram analysis (Wehlau and Leung 1964; Paper I) confirmed the primary pulsation period ($P_1 = 0^d169168$) reported by previous workers, and a frequency $f_1 = 5.9113$ cycles/day (c/d) was initially adopted as best fitting these observations. With this period, the light and/or velocity measures on each night were fitted individually by least squares as in Paper I to obtain the semiamplitude A_1, phase-zero point α_1, and mean value ($\langle V \rangle$ in velocity or $\langle l_{16}/l_{14} \rangle$ in luminosity) which best represented a sinusoidal variation (i.e., $A_1 \sin 2\pi\,[f_1(t - \text{J.D. } 2433000.0) + \alpha_1]$) on each night, and the resulting values are displayed in Tables 1 and 2, together with the nightly mean time of observation $\langle t \rangle$. The fittings to the light variation were performed in luminosity units, and the semiamplitudes listed in columns (3) and (7) of Table 1 are given in percentage of mean light for the night. Since the nightly luminosity means vary strongly in an

TABLE 1

AMPLITUDES AND PHASE-ZERO POINTS OF THE LIGHT VARIATION OF 16 LACERTAE

$\langle t \rangle$ (J.D. 2430000+)	$\langle l_{16}/l_{14} \rangle$	A_1 (%)	α_1 (periods)	$\langle t \rangle$ (J.D. 2430000+)	$\langle l_{16}/l_{14} \rangle$	A_1 (%)	α_1 (periods)
3504.89	1.747	4.92	0.643	3904.80	1.654	2.82	0.535
3505.88	1.755	4.55	.564	3910.76	1.670	1.41	.699
3519.92	1.727	2.99	.667	3917.77	1.716	3.65	.641
3521.90	1.684	4.34	.697	3925.84	1.645	1.35	.536
3523.87	1.726	2.47	.695	4231.93	1.707	2.65	.550
3527.93	1.718	2.76	.528	4234.89	1.633	1.71	.521
3535.90	1.743	1.74	.049	4235.88	1.673	1.28	.833
3870.89	1.646	2.58	.537	4236.89	1.669	2.73	.653
3871.91	1.677	2.42	.540	4237.90	1.660	1.10	.760
3872.88	1.660	2.20	.526	4238.88	1.668	3.79	.715
3873.87	1.651	1.25	.601	4239.88	1.673	1.82	.612
3883.85	1.663	3.33	.622	4240.91	1.704	3.51	.741
3889.87	1.688	2.25	.509	4241.90	1.708	4.89	.605
3897.84	1.696	2.69	0.678	4242.91	1.691	2.25	0.658

TABLE 2

AMPLITUDES AND PHASE-ZERO POINTS OF THE RADIAL-VELOCITY VARIATION OF 16 LACERTAE

$\langle t \rangle$ J.D. 2430000+)	$\langle V \rangle$ (km sec^{-1})	A_1 (km sec^{-1})	α_1 (periods)	$\langle t \rangle$ (J.D. 2430000+)	$\langle V \rangle$ (km sec^{-1})	A_1 (km sec^{-1})	α_1 (periods)
3869.85	−20.0	14.4	0.843	3939.80	−35.1	17.3	0.833
3870.88	−14.9	13.9	.821	3940.80	−36.5	14.6	.819
3871.85	− 5.5	14.1	.823	3941.71	−31.8	10.9	.807
3872.83	+ 5.6	13.4	.807	3961.69	−11.3	10.7	.874
3873.88	+11.5	12.0	.825	3962.80	−23.3	12.2	.790
3874.89	+ 7.8	11.6	.834	3963.69	−31.5	10.2	.872
3875.80	+ 2.8	9.7	.842	3964.70	−35.5	14.6	.903
3876.91	−10.6	13.1	.886	3965.70	−31.3	11.8	.867
3883.84	− 7.6	17.3	.867	3966.70	−26.5	15.5	.904
3889.86	−22.7	12.8	.810	4253.75	−32.3	14.7	.944
3897.82	+ 6.5	15.6	.889	4258.75	+ 0.5	20.0	.906
3904.82	−36.5	15.5	.821	4909.90	−28.8	18.9	.851
3910.74	+ 8.0	10.1	.882	4914.89	+11.5	19.1	.838
3917.75	−32.3	18.0	.880	4966.84	−24.8	13.8	.812
3925.84	−18.5	12.0	.810	4977.81	−12.1	21.3	.949
3937.85	−18.0	21.5	.839	4984.79	− 6.2	10.3	0.910
3938.82	−27.9	17.1	0.814				

apparently erratic fashion and since the comparison star 14 Lac has been reported to be variable (Walker 1952*a*, 1953), these variations in mean light level were attributed to 14 Lac and were compensated for by applying night corrections deduced from the mean values of the luminosity ratio. Note that this arbitrary correction procedure precludes the possibility of finding any light variation which depends solely on the orbital motion.

The nightly velocity means $\langle V \rangle$ in Table 2 were fitted by least squares in our spectroscopic binary program to yield the orbital elements and their mean errors listed in Table 3. These values agree very closely with the 1951 preliminary values given by Struve *et al.* (1952), except for T and ω, which are quite uncertain in this nearly circular orbit. The comparison of observed and computed mean velocity variation is shown in Figure 1.

After the data were prewhitened for the primary pulsation f_1 and for either the nightly

TABLE 3

ORBITAL ELEMENTS OF 16 LACERTAE

$P = 12^{d}097 \pm 0^{d}001$
$T = \text{J.D.}\ 2433841^{d}3 \pm 1^{d}3$
$\gamma = -13.0 \pm 0.4$ km sec^{-1}
$K_1 = 23.0 \pm 0.6$ km sec^{-1}
$e \doteq 0.035 \pm 0.03$
$\omega = 104° \pm 38°$
$a_1 \sin i = 3.82 \times 10^6$ km
$f(M) = 0.0152\ M_\odot$

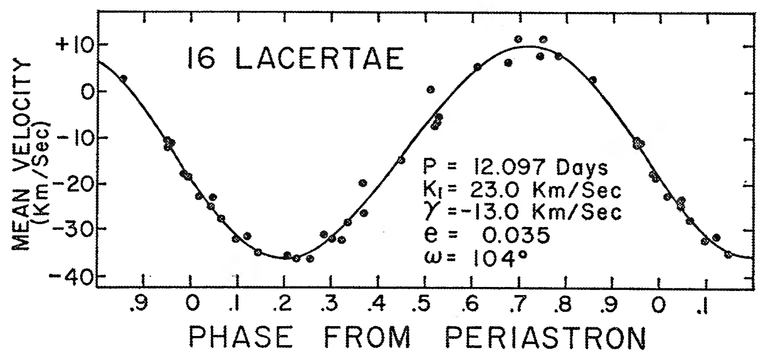

FIG. 1.—Observed and computed orbital-velocity variation of 16 Lac

luminosity corrections or the orbital velocity variation (frequency $f_0 = 0.08267$ c/d), respectively, further periodogram analysis disclosed the existence of two more periodic variations (frequencies $f_2 = 5.8529$ and $f_3 = 5.4998$ c/d) in both light and velocity. The results of fitting all of these frequencies by least squares to the observations are shown in Tables 4 and 5 for various subdivisions of the data, where the two groupings of the 1951 velocity measures correspond to observations concurrent and not concurrent with the 1951 light measures, respectively (cf. Tables 1 and 2).

Walker (1952*a*) pointed out that the primary pulsation has a much larger phase shift in light than in velocity and that the light amplitude approaches zero before the velocity amplitude; his findings are confirmed by the data of Tables 1 and 2, which show that on eleven nights with simultaneous light and velocity measures, the amplitude and phase variations of f_1 are about twice as strong in light as in velocity. As Tables 4 and 5 demonstrate, this is caused by the difference in the velocity-to-light amplitude ratios for the three frequencies, which are about 6.2, 3.0, and 2.0 km sec^{-1} percent mean light, respectively. The other important point evident in Tables 4 and 5 is that, while the terms f_1

and f_2 are of approximately constant amplitude and phase, the term f_3 is very weak during the first part of the 1951 observing season but fairly strong at other times. Since this behavior is shown by both light and velocity measures, it is certainly real and suggests the probability of long-period interference effects arising from a close-frequency doublet.

In a first version of this paper, the velocity data as published by McNamara (1957) were overlooked, and only the one 1952 velocity curve obtained by McNamara and published by Walker (1954) was included with the 1951 velocity measures of Struve *et al.* (1952). Consequently, none of the periods were as well determined as they are here, and it appeared that both f_2 and f_3 were doublets. Further, the extent of the ambiguities attending the precise frequency determination for terms of a low signal-to-noise ratio

TABLE 4

BLUE-LIGHT VARIATION OF 16 LACERTAE

Year	Blue-Light Range (%)			Phase-Zero Point (Periods)			M.E. of 1 Obs.	No. of Obs.
	$2A_1$	$2A_2$	$2A_3$	α_1	α_2	α_3		
1950	4.8	2.3	3.1	0.590	0.281	0.565	1.6	242
1951	4.6	2.4	0.3	.617	.242	.567	0.7	473
1952	4.5	2.5	2.7	.618	.225	.550	0.8	452
1950, 1952	4.5	2.5	2.7	.611	.231	.560	1.1	694
1950, 1951, 1952	4.7	2.4	1.9	0.612	0.235	0.560	1.0	1167

TABLE 5

RADIAL-VELOCITY VARIATION OF 16 LACERTAE

Year	Velocity Range (km sec^{-1})			Phase-Zero Point (Periods)			M.E. of 1 Obs.	No. of Obs.
	$2A_1$	$2A_2$	$2A_3$	α_1	α_2	α_3		
1951 before J.D. 2433930	27.5	6.8	0.8	0.857	0.503	0.895	3.5	234
1951 after J.D. 2433930	29.3	7.2	4.1	.852	.548	.864	3.5	208
1952, 1954	26.3	6.7	7.6	.872	.462	.850	4.8	76
1951, 1952, 1954	28.6	7.3	2.9	0.858	0.507	0.891	3.7	518

(S/N, where we adopt S/N = amplitude A of a term/mean error of one observation as determined by residuals from the adopted solution) was not fully appreciated. When S/N is large, the periodogram side-lobe pattern (cf. Paper I, Fig. 1) produced by the interrupted time sequence of observations usually causes little trouble; but when S/N is small, the combination of accidental errors with the lobe pattern often results in spurious peaks showing the largest amplitudes. Therefore, any low S/N frequencies derived from a single set of observations should be accepted with reservation. To discriminate against spurious frequencies, we here adopt the following criteria, all of which must be (nearly) satisfied before the term is accepted as real: (1) An amplitude maximum as determined by least-squares fitting to frequencies in that spectral region must occur at the adopted frequency. (2) An error minimum as determined in the same way must also occur. (3) Since the pulsation properties of all terms presumably obey the same physical laws at the surface of the star, we require that the relations of velocity versus light phase be the same for all terms. The primary pulsation f_1 shows maximum light at maximum compression, so we require that $\alpha(\text{RV}) - \alpha(\text{light}) = +0.25$ for all terms. (4) Members of a very close-frequency doublet are assumed to be free and forced frequencies in the same kind of pulsation mode, so that they are required to show the same velocity-to-light amplitude ratio.

Application of these criteria to the complete set of light and velocity measures proved that the term f_2 is a singlet having no apparent connection with tidal perturbations, and it allowed the frequency improvement of f_1 and f_2 to $f_1 = 5.91134$ and $f_2 = 5.85286$ c/d ($P_1 = 0^d.169166$ and $P_2 = 0^d.170857$). These values agree precisely with those found by McNamara (1957). For f_3, only the doublet members $f_3 = 5.49990$ and $f_1 - 5f_0 = 5.49799$ c/d ($P_3 = 0^d.181821$ and $P_{res} = 0^d.181885$) were found to satisfy all four criteria, as shown by Table 6. However, the adopted solution is not unique in the sense that a distinctly different doublet $f_4 = 5.50250$ c/d $\approx f_3 + 1$ cycle/year and $f_5 = 5.50692$ c/d satisfies criteria (1), (2), and (3) equally well, though it does not satisfy criterion (4), and, if real, it seems very difficult to explain theoretically. Our subsequent discussion assumes that the solution in Table 6 is correct and that the alternative doublet (f_4, f_5) is spurious.

TABLE 6

Adopted Solution for 16 Lacertae

Frequency (c/d)	Description	Range 2A (km sec⁻¹)	Range 2A (% Light)	α (Periods) R.V.	α (Periods) Light	Signal/Noise R.V.	Signal/Noise Light
0.08267.......	f_0	46.0	....	0.992	.....	6.3	...
0.16534.......	$2f_0$	2.3	....	.473	.....	0.3	...
5.91134.......	f_1	28.6	4.64	.817	0.575	3.9	2.4
5.85286.......	f_2	7.2	2.40	.547	.276	1.0	1.2
5.49990.......	f_3	4.3	1.96	.725	.471	0.6	1.0
5.49799.......	f_1-5f_0	2.2	1.16	0.914	0.664	0.3	0.6

b) *Tidal-Modulation Theory*

To explain the doublet, we consider the set of pulsation modes of a star as a system of N coupled linear oscillators, in which the state of the kth oscillator (of natural frequency f_k) is specified by the generalized coordinate q_k and on which an external periodic force F of frequency f_0 acts. We assume we can expand each component of F in a Fourier series harmonic in f_0, so that, neglecting nonlinear terms in the coordinates q_k, we have

$$\frac{dq_j^2}{dt^2} + f_j^2 q_j = F_j + \sum_{k=1}^{N} F_{jk} q_k \qquad (j = 1,2,\ldots,N)\,, \quad (1)$$

where

$$F_{jk} = \sum_{m=0}^{\infty} (a_{jkm} \sin mf_0 t + b_{jkm} \cos mf_0 t) \quad (2)$$

and F_j is given by a similar expression. Proceeding by successive approximations, we can easily show that the periodic solutions of these equations complete to the second order may be written in the form

$$q_j = A_j \cos (f_j t + \alpha_j) + \sum_{m=0}^{\infty} A_{jm} \cos (mf_0 t + \alpha_{jm})$$
$$+ \sum_{k=1}^{N} \sum_{m=0}^{\infty} A_{jkm} \left\{ \frac{\cos [(f_k - mf_0)t + \alpha_{jkm}]}{f_j^2 - (f_k - mf_0)^2} + \frac{\cos [(f_k + mf_0)t + \beta_{jkm}]}{f_j^2 - (f_k + mf_0)^2} \right\} . \quad (3)$$

Thus, the total variation as a linear function of the q_k will contain the natural frequencies f_k and the impressed frequencies mf_0, as well as the combination terms $(f_k \pm mf_0)$, but of the latter, the most important are apt to be those (if any) for which a near resonance exists (i.e., $|f_k \pm mf_0| \approx f_j$, so that the denominator of one of the last two expressions in eq. [3] becomes very small). This explanation of resonance in such a system is rigorous if m is an integer, but it is well known that resonance may also occur when m is any rational number, provided it is the quotient of two small integers. In the present case, with P_0 the orbital period in a (nearly) circular orbit, the tidal perturbations are (nearly) the same at intervals of $P_0/2$, P_0, $3P_0/2$, etc., so that the strongest perturbing frequencies should be $2f_0, f_0, 2f_0/3$, etc.

Since $f_1 - f_2 - 2f_0/3 = +0.00337$ c/d $= 1$ cycle/year $+ 0.00064$ c/d, and since in the first version of this paper selection criteria (3) and (4) were not applied, it seemed that the doublet $(f_2, f_1 - 2f_0/3)$ was present with $f_2 = 5.8496$ and $f_1 - 2f_0/3 = 5.8561$ c/d. While we now find that f_2 is actually a singlet ($=5.85286$ c/d), the data of Tables 4, 5, and 6 demonstrate the reality of the doublet near 5.500 c/d.

c) Discussion

Schmalberger (1960) has shown that the β CMa variability probably arises because of a relatively brief instability associated with internal structural changes following hydrogen exhaustion in the core. This conclusion has been contested by Stothers (1965), whose arguments were based partly on the statistics of O–B stars but primarily on the ratios of radial-pulsation periods which van Hoof (1961*a,b;* 1962*a,b,c,d,e;* 1964) claims to have identified in the β CMa stars. However, a critical examination of van Hoof's procedures and results shows that only in the case of ν Eri (1961*a,b*) is there much evidence for some of the many radial-pulsation modes claimed, and even here all the overtones are in some doubt, because van Hoof did not prewhiten the data for the strong frequencies first identified before he calculated mean light curves in the weaker modes claimed. According to Christy (1966*a*), for $T_e \gtrsim 7500°$ there is no evidence of radial-pulsational instability such as that which produces the Cepheid instability strip, so it may be that part of the β CMa driving mechanism is centrally located; in any case, the He II ionization zone is likely to provide a significant contribution to the total negative dissipation required for maintenance of pulsation, and efficiency considerations suggest that the primary pulsation is (nearly) radial in character, at least in the deep interior. Theoretical investigations of the radial pulsations of Cepheid-strip variables by Christy (1966*a,b*), by Baker and Kippenhahn (1962, 1965), by King, Cox, and Eilers (1966), and by others have shown that linear adiabatic calculations give the possible pulsation periods with errors not exceeding a few percent, but that determination of pulsational instability in any mode requires at least a linear, nonadiabatic computation, while theoretical assessment of the limiting radial-pulsation amplitudes and phase relations can only be done by means of the full nonlinear, nonadiabatic computations such as those carried out by Christy (1966*a*) for RR Lyrae variables.

Evidence summarized in Paper I favored the interpretation of β CMa stars as having relatively low rotational velocities for their spectral types, but recent work by Hill (1967) has shown that some members of the group are rapid rotators. Certainly $v \sin i$ is small in 16 Lac (5 km sec^{-1} according to McNamara and Hansen [1961]), even though the customary procedure for determining rotational velocities by measurement of line profiles at the sharpest line phase (Huang and Struve 1955) may lead to underestimates of the projected rotational velocities (Ledoux 1951; Clement 1965*b;* Christy 1967). The fairly large orbital velocity suggests that the orbital (and therefore probably also the rotational) inclination is fairly large, so the absolute rotational velocity is probably small for a B2 IV star.

Linear adiabatic calculations carried out by Chandrasekhar and Lebovitz (1962*a, b*) for rigidly rotating gaseous masses led them to suggest (1962*c*) that the beat phenomena observed in some β CMa stars arise from the rotational coupling of a purely radial R-mode with an axisymmetric nonradial S-mode; in order to ensure that any pulsational instability will excite the S-mode as well as the normally expected R-mode, they postulated a zero-rotation R-S degeneracy occurring at a critical value of γ (≈ 1.6) such that the resulting normal modes are both highly nonradial, even in a nonrotating star. Their theory was carried to a second approximation by Clement (1965*a*) and applied to the observations of Böhm-Vitense (1963) and Clement (1965*b*), who found that the theoretical rotational velocities necessary to explain the observed frequency splitting are in some stars a factor of 3 or 4 larger than observed. The calculations have recently been extended by Tassoul and Ostriker (1968) to the case of nonuniformly rotating bodies; they find that a low enough value of Γ in a large enough region of the star will lead to a dynamical instability such as is encountered in the case of radial, linear, adiabatic calculations for $\Gamma \leq \frac{4}{3}$. But, as is the case for radial pulsations, accurate theoretical prediction of absolute excitation strength for nonradial modes would appear to require a full nonlinear, nonadiabatic calculation, and such a problem is at present probably prohibitively difficult for nonradial pulsations.

The differing velocity-to-light amplitude ratios observed in the various terms in 16 Lac (cf. Table 6) imply that (at most) only f_1 can be a radial-pulsation mode. The small velocity-to-light amplitude ratio of the (f_3,$f_1 - 5f_0$)-doublet may be understood if 16 Lac is viewed at intermediate inclination while the doublet members are essentially equatorial waves, for then the components of their velocities on the line of sight can be relatively small even though the apparent light variations are appreciable. If we ignore the uncertain effects caused by averaging the motions in the various modes over the apparent stellar disk, we can take the squares of the observed velocity amplitudes as proportional to the pulsational energy in each mode, and we find that the terms f_1, f_2, f_3, and $f_1 - 5f_0$ contain 91.6, 5.8, 2.1, and 0.5 percent of the total pulsational energy, respectively. The existence of the resonance doublet of very low energy suggests that f_3 is a nonradial mode of inherently positive dissipation excited by coupling to the primary pulsation through the accident of orbital resonance, and that the rest of the possible nonradial modes (Cowling 1941; Ledoux 1951; Tassoul and Ostriker 1968) in this rotating star (other than f_2) are positively damped to nonexcitation.

The necessity for the R-S degeneracy (as distinct from the rotational coupling of the R- and S-modes) postulated by Chandrasekhar and Lebovitz (1962*c*) does not seem to have been clearly demonstrated, and we would like to suggest instead that, when nonadiabatic and nonlinear effects can be properly allowed for, one may find that the S-mode is adequately excited by energy transfer from the R-mode through rotational coupling in the absence of the R-S degeneracy, so that in the zero rotation limit the R-mode is spherically symmetric while the nonexcited S-mode is axially symmetric. If this suggestion is correct, it requires that in 16 Lac, f_1 and f_2 are the nearly radial R-mode and the nearly volume-conserving axisymmetric S-mode, respectively. For example, quadratic interpolations in Tables 1 and 2 of Clement (1965*a*) show that the R-S degeneracy occurs at $\gamma = 1.5718$ for an $n = 3$ polytrope and that $f^2{}_R = f^2{}_S = 0.04618$ (in units of $4\pi G\rho_c$) at zero rotation, whereas the observed frequency splitting ($1 - f_2{}^2/f_1{}^2 = 0.01969$) may be obtained in the zero rotation limit at $\gamma = 1.5795$, with $f_R{}^2 = 0.04737$ and $f_S{}^2 = 0.04644$. Clearly, some intermediate value of γ will allow the observed frequency splitting at the observed rotational velocity, and will thus eliminate the major objection to the theory while it also precludes a definitive test based on rotational velocities.

III. β CEPHEI

In his review paper on the β CMa stars, Struve (1955) remarked that β Cep is singly periodic, even though in 1953 Struve *et al.* announced it as showing velocity amplitude that varied strongly. In this latter paper, the authors presented 713 velocity measures obtained on twenty-eight nights in 1950–1952, and we have employed here all but two of these velocities, discarding only the measures at J.D. 2433549.682 and at J.D. 2433552.775 as being very discordant from the run of the rest of the velocities.

Preliminary periodogram analysis gave $f_1 = 5.24965$ c/d, in excellent agreement with the value of 5.24961 c/d predicted by equation (3) of Struve *et al.* (1953) (the period is slowly increasing), and the data for twenty-seven individual nights were fit to this frequency to yield the values of $\langle V \rangle$, A_1, and α_1 listed in Table 7 (there were too few observations on J.D. 2434177 to permit a meaningful fit). The symbols have the same significance as for 16 Lac in § II, including the time-zero point at J.D. 2433000.0. Table 7 shows the phase variation to be generally small (0.09 P_1 maximum range except for the nights of 2434164 and 2434166, where inspection of the velocity curves suggests the phase is poorly determined), but shows the amplitudes to vary strongly, with maximum amplitude falling near maximum $\langle V \rangle$ (i.e., J.D. 2434166, 2434199, and 2434210). A modulation period near 11 days is thus indicated, and plots of amplitude versus phase

TABLE 7

AMPLITUDES AND PHASE-ZERO POINTS OF THE RADIAL-VELOCITY VARIATION OF β CEPHEI

$\langle t \rangle$ (J.D. 2430000+)	$\langle V \rangle$ (km sec^{-1})	A_1 (km sec^{-1})	α_1 (Periods)	$\langle t \rangle$ (J.D. 2430000+)	$\langle V \rangle$ (km sec^{-1})	A_1 (km sec^{-1})	α_1 (Periods)
3549.79.......	−4.6	10.8	0.36	4166.90.....	−2.2	7.9	0.32
3550.76.......	−2.3	10.9	.36	4167.89.....	−3.3	9.3	.43
3551.77.......	−4.3	13.0	.37	4169.85.....	−6.3	13.2	.38
3552.76.......	−2.2	13.5	.35	4184.86.....	−5.9	15.1	.40
3553.72.......	−4.9	12.9	.37	4196.85.....	−1.5	9.8	.40
3562.81.......	−4.1	13.5	.40	4197.85.....	+1.4	10.5	.42
3569.82.......	−2.8	11.8	.40	4198.86.....	+2.1	13.1	.41
3590.70.......	−5.2	12.9	.39	4199.86.....	+4.5	8.1	.39
3597.71.......	−3.5	14.2	.41	4210.84.....	+0.1	7.6	.36
3924.73.......	−4.5	12.9	.40	4217.83.....	−5.0	14.9	.42
4161.90.......	−5.7	14.5	.40	4225.80.....	−5.8	13.7	.44
4163.89.......	−3.3	12.6	.42	4238.75.....	−3.9	14.4	.41
4164.90.......	−1.3	11.2	.48	4247.86.....	−4.8	13.3	0.43
4165.90.......	−1.1	14.3	0.41				

for the 1952 data show that the period must lie between 10.7 and 11.1 days, while periodogram calculations on all the data give a pronounced maximum near frequency 0.091 c/d (reflecting the variation in $\langle V \rangle$). The precise value of the orbital period is difficult to establish, because the variations in $\langle V \rangle$ are small and because the time spacing of the observations is very poor for period determination. Most spectra were obtained around full moon in September 1950 and June and July 1952, and observations were made on only one night in 1951, so there is a very large uncertainty in the biennial cycle count and the corresponding period.

Least-squares fitting of the velocities to a sine curve in frequencies near 5.24965 c/d yielded $f_1 = 5.2497$ c/d as the best frequency for these measures; then a series of doubly harmonic expansions in frequencies ($jf_1 \pm kf_0$; $j = 0,1$; $k = 0,1,2,3,4$) were fitted by least squares for a variety of frequencies f_0 in the range 0.0899–0.0922 c/d. The residuals from these fits showed at $f_0 = 0.0918$ c/d a sharp minimum in the mean error of a single observation ($=3.3$ km sec^{-1}), so this value was adopted as the correct orbital frequency, with the corresponding solution displayed in Table 8; the nightly mean velocities from Table 7 were fitted with our spectroscopic binary program to the orbital period $P_0 = 10^{d}.893$. This last fitting was done with and without the one anomalously high velocity at 2434199.86, with the results illustrated in Figures 2 and 3, respectively, where the upper curve represents the computed solution for orbital motion and the lower curve is the estimated real variation in pulsation amplitude with orbital phase. In Figure 3 the obvious correlation of amplitude extremes with phase angle $v + \omega$ (and therefore with tide-raising potential; cf. Paper I) led to the adoption of the second set (one observation omitted), with the corresponding elements given in Table 9. The relatively large uncertainties in the elements reflect the very small amplitude K_1. Since the observed amplitude modulation suggests an intermediate-to-large orbital inclination, the very small mass function probably results from a very large mass ratio M_1/M_2. If our adopted

TABLE 8

RADIAL-VELOCITY VARIATION OF β CEPHEI

Frequency (c/d)	Description	$2A$ (km sec⁻¹)	α (Periods)	Frequency (c/d)	Description	$2A$ (km sec⁻¹)	α (Periods)
0.0918.....	f_0	5.5	0.11	5.2497......	f_1	24.1	0.20
0.1836.....	$2f_0$	3.6	.89	5.3415......	f_1+f_0	2.1	.42
4.8825.....	f_1-4f_0	1.7	.80	5.4338......	f_1+2f_0	1.3	.37
4.9743.....	f_1-3f_0	2.8	.55	5.5251......	f_1+3f_0	0.4	.03
5.0661.....	f_1-2f_0	0.6	.08	5.6169......	f_1+4f_0	1.8	0.67
5.1579.....	f_1-f_0	0.9	0.98				

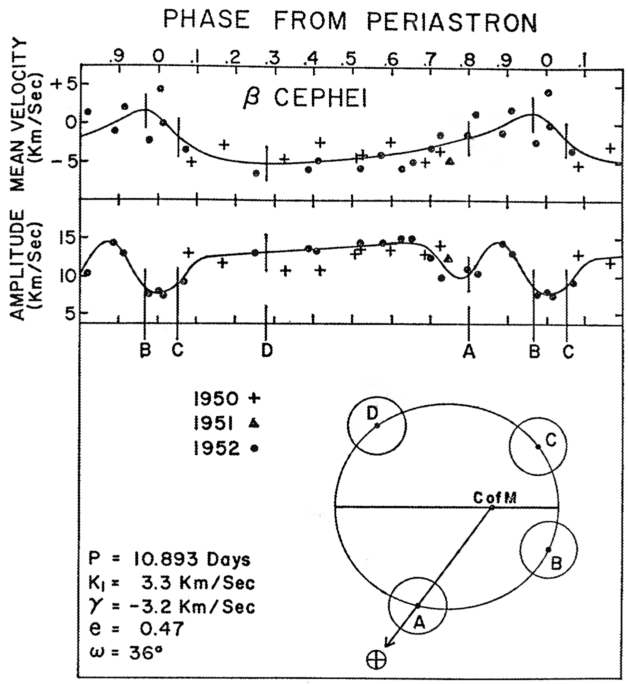

FIG. 2.—Observed and computed orbital-velocity variation of β Cep (*top*), based on mean velocities from all twenty-seven nights. The observed and smoothed pulsation-amplitude variation (*center*) shown in relation to the aspect angle of the primary in its orbit about the center of mass (*bottom*).

solution is correct, then in this star, minimum pulsation amplitude occurs at (or near) maximum tidal extension and maximum amplitude at maximum compression.

IV. STARS WITH MODULATED PULSATIONS

Most of the presently available information concerning modulation phenomena in the β CMa and δ Sct stars is summarized in Table 10, where the data are generally self-explanatory. In column (10), b and y signify blue and yellow light measures, respectively, and blank entries in the table mean only that definitive information does not exist or was not located by the author. The data for β Cep, 16 Lac, σ Sco, and CC And have been presented in this paper and in Paper I, except the orbital eccentricity of CC And, which was recently estimated by comparison of the observed phase and amplitude modulations with tidal potential calculations.

a) β *Canis Majoris Stars*

In ν Eri, the light and velocity variations are quite complex, and van Hoof (1961*a*,*b*) attributed them to the simultaneous excitation of five radial-pulsation modes and

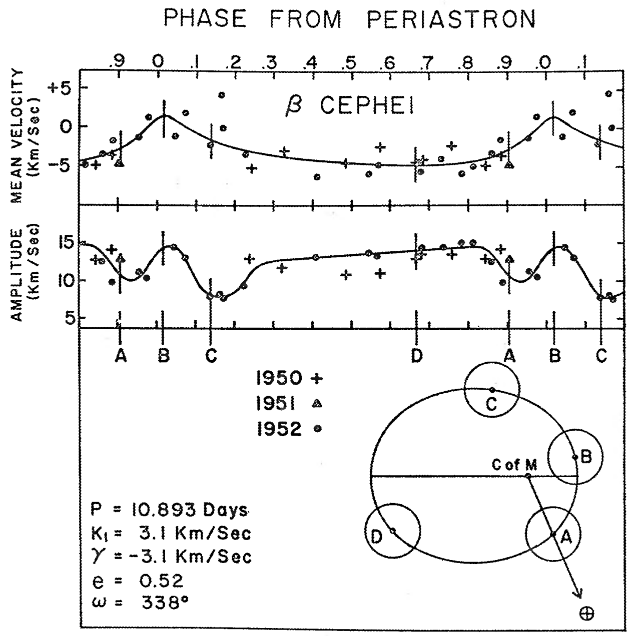

FIG. 3.—Observed and computed orbital-velocity variation of β Cep (*top*), with orbital elements determined after omission of the one anomalously high velocity near phase 0.16. Otherwise, description is same as for Fig. 2.

TABLE 9

ORBITAL ELEMENTS OF β CEPHEI

$P = 10^{d}.893$ (assumed)
$T =$ J.D. $2433555^{d}.4 \pm 0^{d}.6$
$\gamma = -3.1 \pm 0.3$ km sec^{-1}
$K_1 = 3.1 \pm 0.8$ km sec^{-1}
$e = 0.52 \pm 0.15$
$\omega = 338° \pm 18°$
$a_1 \sin i = 4.0 \times 10^5$ km
$f(M) = 2.1 \times 10^{-5}\, M_\odot$

resonance interactions between these modes. However, Opolski and Ciurla (1962*b*) and the present author (Paper I) suggested that van Hoof's published results appeared instead to indicate modulation of the primary pulsation by a regular disturbance of $15^{d}.79$ period. Van Hoof's very extensive magnitude measurements are still not available for analysis, so the suggestion is not yet verified or disproved, but work in progress on the published velocity measures (Struve *et al.* 1952; Struve and Abhyankar 1955) has already shown that, if van Hoof's periodogram analysis is basically correct, then the star changed markedly in modulation characteristics in the 5 years between the velocity and light measures. Thus, it may be that tidal-resonance phenomena such as 16 Lac shows are also present in ν Eri.

The star 12 Lac was included here because Opolski and Ciurla (1961, 1962*a*) found a 9-day modulation period accompanied by a mean velocity variation which we would interpret as representing orbital motion with a very small velocity amplitude. If their period is correct, it does not agree with the 26-day period found by Barning (1962) in his analysis of the light variation.

b) *δ Scuti Stars*

The star 14 Aur is a single-line binary with a well-defined orbit of small eccentricity, and work in progress at the Paris Observatory (Chevalier 1968) has shown a complex interaction phenomenon similar to that of 16 Lac, in which the light variation is much more strongly modulated than is the velocity variation. This suggests that tidal resonances are also operative in 14 Aur.

TABLE 10

MODULATION CHARACTERISTICS OF SOME PULSATING STARS

			Orbital Characteristics				Pulsation Characteristics		
Star*	Spectral Type	V (mag)	P (Days)	K_1 (km sec^{-1})	e	ω	P (Days)	$2A_1$ (km sec^{-1})	$2A_1$ (mag)
β Cep...	B2 III	3.32	10.893	≈ 3	0.5	338°	0.1905	24.1	
ν Eri....	B2 III	4.12	15.79?				.1735	33	0.114*b*
12 Lac..	B2 III	5.18	8.876?	≈ 3			.1931	39	.073*y*
16 Lac..	B2 IV	5.54	12.097	23.0	0.035	104	.1692	28.6	.049*b*
σ Sco...	B1 III	3.08	33.13	34.7	0.40	301	.2468	93.2	.040*y*
CC And.	F0–F2 II–III	9.34	10.466		0.12		.1249	18	.140*y*
14 Aur..	A9 V	5.05	3.789	21.6	0.033	20°	.122		.056*y*
δ Del...	F2 IV	4.50	40.5		Large		.1345	2.6	.05*y*
							0.1568		0.059*b*

* Beta Cep: Nonresonant tidal modulation in radial velocity is well defined in amplitude but poorly defined in phase (McNamara and Hansen 1961; Hill 1967; this paper). ν Eri: Two or more additional modes are excited, at least one of which is of variable strength (van Hoof 1961*a*,*b*; McNamara and Hansen 1961; Opolski and Ciurla 1962*b*; Paper I; Struve and Abhyankar 1955; Struve *et al.* 1952; Walker 1951*b*, 1952*b*; work in progress). 12 Lac: Shows strong phase and amplitude modulation in light and velocity (Jerzykiewicz 1963; McNamara and Hansen 1961; Opolski and Ciurla 1961, 1962*a*; Struve 1950; Struve and Zebergs 1955). 16 Lac: Two nonradial modes are coupled to primary pulsation through rotation and tidal resonance (McNamara and Hansen 1961; this paper). σ Sco: Nonresonant tidal modulation in radial velocity is well defined in phase but poorly defined in amplitude; strong amplitude modulation of the light variation (van Hoof 1966; McNamara and Hansen 1961; Paper I). CC And: Shows strong nonresonant tidal modulation in phase and amplitude of light and velocity variations (Fitch 1960; Paper I; Wilson and Walker 1956). 14 Aur: Shows modulation characteristics similar to 16 Lac; eccentricity is small and tidal resonances are possible (Chevalier 1968; Danziger and Dickens 1967; Harper 1938). δ Del: Very complex light and velocity variations; both components are probably variables (Eggen 1956; Kuhi and Danziger 1967; Leung and Wehlau 1967; Preston 1966; Struve, Sahade, and Zebergs 1957; Wehlau and Leung 1964).

Preston (1966) has announced δ Del as a double-line binary of long period and large eccentricity. He has also suggested that both components may be variable. If his surmise is correct, then this star provides in principle a unique opportunity for studying the influence of external perturbations on the pulsation properties of stars.

Sufficiently extensive and accurate observations, preferably simultaneous in light and velocity, of δ Del or other selected members of these two groups should provide a wealth of information on the dynamical properties of these stars. However, as the discussions in §§ II and III indicate, the measures must be extremely numerous in order to delimit clearly the nature of the variations.

An anonymous referee pointed out the earlier omission of some velocity data on 16 Lac, and their inclusion here has led to a significant improvement in both the analysis and the interpretation of the observations. It is a pleasure to thank Messrs. Clifford Jewsbury and Ashit Sanyal for card punching the observations of 16 Lac and β Cep, respectively, and Mr. Robert Dukes for developing our spectroscopic-binary reduction program. I am indebted to Drs. R. F. Christy, W. J. Cocke, R. J. Weymann, and R. E. Williams for helpful conversations. The calculations were performed either on the

IBM 1401-7072 system at the University of Arizona Numerical Analysis Laboratory or, more recently, on the CDC 6400 at our new Computer Center, and the assistance of the personnel of these laboratories is gratefully acknowledged.

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