

Generalized Cassini's Laws

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(Received 31 October 1968; revised 7 January 1969)

All of the possible stable, coplanar configurations of a planet's or satellite's spin vector, orbit normal, and the vector designating the precession of the orbit are shown to represent extremes in the energy associated with the body's orientation. It is shown that such a stable configuration yields a relation between the moment differences $(C-A)/C$ and $(B-A)/C$, and bounds on their magnitudes if the spin angular velocity is commensurate with the orbital mean motion. Only a value of $(C-A)/C$ is determined by a stable configuration for noncommensurate rotation. The effect of $(B-A)/C$ on the value of $(C-A)/C$ for the observed lunar configuration is less than the uncertainty introduced by observational error. Mercury is likely to be in one of two possible stable configurations. The value of $(C-A)/C$ as a function of the obliquity of the equator plane to the orbit plane is derived for both configurations, and the effects of axial asymmetry, although small for the likely values of $(C-A)/C$ for Mercury, are explicitly displayed for the entire range of $(C-A)/C$.

I. INTRODUCTION

IN 1643 Cassini described the rotation of the moon in his three famous laws. The first noted that the rotation rate was synchronous with the orbital mean motion such that one side always faced the Earth. The other two laws state the facts that the inclination of the lunar equator plane to the ecliptic is constant, and that the spin axis and the normals to the ecliptic and orbit plane remain coplanar.

Colombo (1966) showed that Cassini's second and third laws were generalizable for an axially symmetric planet with an arbitrary spin angular velocity. The present paper approaches the problem from a more easily understood point of view and completes the generalization of Cassini's second and third laws to include the effects of axial asymmetry and rotation rates which are commensurate with the orbital mean motion. Some of the conclusions reached here do not agree with those of Colombo.

The Hamiltonian of a slightly aspherical planet or satellite whose spin is commensurate with its orbital motion is developed in a reference frame which precesses with the orbit. If angular momentum and energy are approximately conserved, the precession of the spin axis relative to the coordinate system fixed in the orbit plane is determined by the intersection of a sphere and either a parabolic cylinder (Colombo 1966) or a triaxial ellipsoid depending on the order retained in the expansions in eccentricity. The spin axis is fixed relative to the precessing orbit when the energy has an extreme value. These fixed positions represent the generalization of Cassini's second and third laws. The theory is applied to the moon and to Mercury, and relations between $(C-A)/C$ and $(B-A)/C$ are developed and their absolute magnitude determined for stable, coplanar configurations of the body's spin vector, orbital angular-momentum vector and orbit precessional angular velocity.

* Now with Department of Physics, University of California at Santa Barbara. Supported in part by The National Aeronautics and Space Administration under Grant NSG 216.

II. DEVELOPMENT OF THE HAMILTONIAN

The coordinate systems and angles to be used throughout are displayed in Fig. 1, the common origin of the systems being at the center of mass of the planet or satellite. The XYZ system is oriented with the Z axis being normal to the orbit plane, and X axis oriented along the ascending node of the orbit on the plane which is perpendicular to the orbit precessional angular velocity $\mathbf{y}$. Unit vectors along the XYZ axes are denoted respectively by $\mathbf{IJK}$. The principal axes fixed in the planet or satellite are xyz with unit vectors $\mathbf{ijk}$. The inclination of the orbit plane relative to $-\mathbf{y}$ is ι ; θ is the obliquity of the equator plane to the orbit plane; φ is the longitude of the node of the equator plane on the orbit plane measured from the X axis; and ψ is the longitude of the x axis measured from the node.

The moments of inertia of the spinning body about the $x y z$ axes are, respectively, $A-B-C$ in the order of

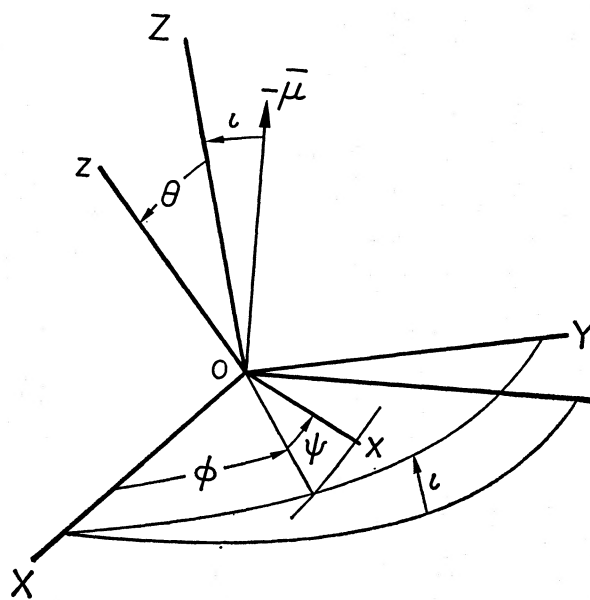


FIG. 1. Coordinate systems.

increasing magnitude. The orbit plane is assumed to precess at the uniform angular velocity $\mathbf{u}$ such that the kinetic energy associated with the spin is given by

$$T = \frac{1}{2}A(\omega_x + \mu_x)^2 + \frac{1}{2}B(\omega_y + \mu_y)^2 + \frac{1}{2}C(\omega_z + \mu_z)^2, \quad (1)$$

where ω is the angular velocity relative to the precessing coordinate system XYZ . Both ω and $\mathbf{u}$ are resolved into their components along the principal axes in Eq. (1). As the potential energy does not involve velocities, the generalized momenta associated with ω_i are

$$\alpha_i = \frac{\partial T}{\partial \omega_i} = A_i(\omega_i + \mu_i). \quad (2)$$

If we neglect the small deviation from principal axis rotation induced by the gravitational torques $\{\omega_x, \omega_y\} = O[(C-A)\sin\theta n^2/(\psi A)]$, $\alpha \approx \alpha_z$, and the Hamiltonian is given by

$$H = \sum_i \alpha_i \omega_i - T + V \approx \frac{1}{2} \frac{\alpha^2}{C} - \alpha \cdot \mathbf{u} + V, \quad (3)$$

where V is that part of the potential energy of the spinning body in the gravitational field of its primary *which depends on the body's orientation*:

$$V = -\frac{3GM}{2r^5} \{ (C-A)(\mathbf{r} \cdot \mathbf{k})^2 + (B-A)(\mathbf{r} \cdot \mathbf{j})^2 \} + O\left(\frac{1}{r^4}\right), \quad (4)$$

where M is the mass of the primary and $\mathbf{r}$ is the vector distance from the center of mass of the planet or satellite to that of the primary.

In the XYZ system,

$$\mathbf{r} = r[\cos(\omega + f)\mathbf{I} + \sin(\omega + f)\mathbf{J}], \quad (5)$$

where ω is the longitude of the pericenter of the primary orbit measured from the I axis, and f is the true anomaly. If the scalar products in Eq. (4) are expressed in terms of the Euler angles defined in Fig. 1, we have

$$\frac{(\mathbf{r} \cdot \mathbf{k})^2}{r^2} = \frac{1}{2} \sin^2\theta [1 - \cos 2(\omega - \varphi + f)], \quad (6)$$

$$\begin{aligned} \frac{(\mathbf{r} \cdot \mathbf{j})^2}{r^2} = & -\frac{1}{4} \sin^2\theta [1 + \cos 2\psi - \cos 2(\omega - \varphi + f)] \\ & -\frac{1}{4} (1 + \cos^2\theta) \cos(2\omega - 2\varphi + 2f) \cos 2\psi \\ & -\frac{1}{2} \cos\theta \sin(2\omega - 2\varphi + 2f) \sin 2\psi + \frac{1}{2}. \end{aligned} \quad (7)$$

We are not concerned here with short-period effects as their amplitudes are negligible. These terms can be eliminated by averaging V over an orbit period, which involves averages of Eqs. (6) and (7) multiplied by a^3/r^3 , where a is the semimajor axis of the orbit. The slowly varying parameters are held constant during the averaging. The averaging procedure is equivalent to selecting only the long-period and secular terms in the

expansion of the Hamiltonian. If ψ is not commensurate with the orbital mean motion, the terms in V involving ψ average to zero over a time interval sufficiently long. We shall assume that the planet or satellite is in such a spin resonance, since the nonresonance results can be obtained by simply dropping the appropriate terms.

If we account for the relative motion of the node of the equator plane and the pericenter, a nearly resonant angular velocity can be represented by

$$\psi = pn + q(\omega - \varphi) + \gamma, \quad (8)$$

where p and q are half-integers, and $\gamma \ll n$ is a small increment in the angular velocity. The factor p is determined by the resonance in question ($p=1$ for moon, $\frac{3}{2}$ for Mercury), and q by the requirement that a pendulum equation be obtained for the variation in γ (Goldreich and Peale 1966). Integration of Eq. (8) yields

$$\psi = pM + q(\omega - \varphi) + \gamma, \quad (9)$$

where M is the mean anomaly. The constant of integration is chosen such that the angle between the x axis and the vector directed from the origin to the pericenter is nearly minimal when $\gamma = M = 0$.

Substituting Eq. (9) into Eq. (7), dropping the constant term in Eq. (7), and subsequently averaging over an orbit period with γ and other slowly varying quantities held constant, we have

$$\begin{aligned} \left\langle \frac{a^3}{r^5} (\mathbf{r} \cdot \mathbf{j})^2 \right\rangle = & -\frac{1}{4} \sin^2\theta \{ (1 - e^2)^{-\frac{3}{2}} + H_3(p, e) \cos 2[q(\omega - \varphi) + \gamma] \} \\ & -\frac{1}{8} (1 + \cos\theta)^2 [H_1(p, e) + H_2(p, e)] \\ & \times \cos 2[(q-1)(\omega - \varphi) + \gamma] \\ & -\frac{1}{8} (1 - \cos\theta)^2 [H_1(p, e) - H_2(p, e)] \\ & \times \cos 2[(q+1)(\omega - \varphi) + \gamma], \end{aligned} \quad (10)$$

where the angular brackets denote average value, and

$$\begin{aligned} H_1(p, e) = & \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \cos 2f \cos 2pM \, dM, \\ H_2(p, e) = & \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \sin 2f \sin 2pM \, dM, \\ H_3(p, e) = & \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \cos 2pM \, dM. \end{aligned}$$

The $H(p, e)$ are easily evaluated with the aid of Cayley's tables (1859). If we retain terms only up to order e^2 , $H_1(p, e) = H_2(p, e)$ and only their sum is used. The sum $H_1 + H_2$ and H_3 are shown in Table I for two values p . The term $\langle a^3/r^3 \rangle$ in Eq. (10) was evaluated directly

without using the Cayley expansions. Comparison of Eqs. (6) and (7) yields directly the remaining term in V :

$$\left\langle \frac{a^3}{r^5} (\mathbf{r} \cdot \mathbf{k})^2 \right\rangle = \frac{\sin^2 \theta}{2(1-e^2)^{3/2}}. \quad (11)$$

If we use the expressions from Table I, but neglect $H_3(1, e)$ since it is multiplied by $\sin^2 \theta$, then substitution of Eqs. (11) and (10) into Eq. (4), and Eq. (4) into Eq. (3) gives the averaged Hamiltonian

$$\langle H \rangle = \frac{1}{2}(\alpha^2/C) - \alpha \cdot \mathbf{u} + (C-A)\xi \sin^2 \theta - (B-A) \times [\zeta(1+\cos \theta)^2 + \frac{1}{2}\xi \sin^2 \theta], \quad (12)$$

where

$$\xi = \frac{3}{4}(n^2/(1-e^2)^{3/2});$$

$$\zeta = \frac{3}{16}n^2[H_1(p, e) + H_2(p, e)].$$

In Eq. (12), $q=1$ for the proper form, and γ has been set to zero, which corresponds to completely damped librations.

III. CASSINI'S SECOND AND THIRD LAWS

Within a spin resonance or in the absence of energy dissipation, α is a constant of the motion. Since H is also a constant of the motion we can write

$$R(C-A)Z_k^2 + (B-A)[S(1+Z_k)^2 - \frac{1}{2}RZ_k^2] + \sin \iota Y_k + \cos \iota Z_k = D, \quad (13)$$

where

$$D = (\frac{1}{2}\alpha^2/C + \xi(C - \frac{1}{2}A - \frac{1}{2}B) - \langle H \rangle)/\alpha\mu$$

is a constant of the motion, and

$$R = (\xi/\alpha\mu), \quad S = (\zeta/\alpha\mu).$$

In Eq. (13), we have used the components of $\mathbf{u}$ and $\mathbf{k}$ in the XYZ system:

$$\mathbf{u} = \mu(\sin \iota \mathbf{J} + \cos \iota \mathbf{K}),$$

$$\mathbf{k} = X_k \mathbf{I} + Y_k \mathbf{J} + Z_k \mathbf{K}. \quad (14)$$

Equation (13) is identical with the expression obtained by Colombo (1966) if $B=A$ or $\zeta=0$. The angular momentum about the spin axis is conserved such that

$$\alpha^2[X_k^2 + Y_k^2 + Z_k^2] = \alpha^2 = \text{constant}. \quad (15)$$

As with the axisymmetric case (Colombo 1966), the intersection of the surfaces defined by Eqs. (13) and (15) describes the path the spin vector must trace on the unit sphere as it precesses under the action of the gravitational torques. Since Eq. (13) can be written in the form

$$[Z_k - F_1]^2 = \frac{-\sin \iota}{R(C - \frac{1}{2}A - \frac{1}{2}B) + S(B-A)} \times \left[Y_k - \frac{D}{\sin \iota} - F_2 \right], \quad (16)$$

TABLE I.

	$p=1$	$p=\frac{3}{2}$
$H_1(p, e) + H_2(p, e)$	$1 - \frac{5}{2}e^2 + 0(e^4)$	$\frac{7}{2}e + 0(e^3)$
$H_3(p, e)$	$9/4e^2 + 0(e^4)$	$0(e^3)$

where F_i are functions of the parameters, a change in D changes the position of the vertex of the parabola but does not change its axis or shape. Since R and S are negative for cases of interest, the parabolas always open toward increasing Y_k . All of the possible trajectories of the precessing spin axis on the unit sphere are thus determined by the intersection of a parabolic cylinder with a fixed shape as it assumes a range of positions along its axis. A change in D corresponds to a change in the energy associated with the spin and orientation of the body. The limited range of D which allows intersection of the parabolic cylinder with the sphere corresponds to the limited range of energy available from changes in orientation.

It should be pointed out that keeping the neglected terms in Eq. (10) changes the parabolic cylinders [Eqs. (13) and (16)] into triaxial ellipsoids. Observational accuracy does not justify the considerable complication of the analysis.

It is clear that the limits of D correspond to points where the parabolic cylinders are just tangent to the sphere. In this case the spin vector is fixed in the YZ plane and shares the angular velocity $\mathbf{u}$ of the XYZ system. Cassini's second and third laws for the moon note the observational fact that $\mathbf{k}$, $\mathbf{K}$, and $\mathbf{u}$ remain coplanar, which indicates that the moon has assumed one of the extreme values of D . As D involves the energy of the spin and orientation and angular momentum, any process which interchanges energy and angular momentum between the spin and orbital motions will drive D to an extreme value. Tidal interaction provides a mechanism of energy transfer which can either add or subtract energy associated with the spin and orientation (e.g., see Peale and Gold 1965; Goldreich and Peale 1968).

A necessary condition at an extreme value of D is that the partial derivatives of D with respect to the components of $\mathbf{k}$ must vanish, subject to the condition of Eq. (15). D is independent of X_k , so substitution of

$$Y_k = \pm (1 - X_k^2 - Z_k^2)^{1/2}$$

into Eq. (13) yields D in terms of quasi-independent variables Z_k and X_k , and incorporates the condition. The partial derivatives then yield the relations

$$2R(C-A)Z_k Y_k + (B-A)[2S(1+Z_k) - RZ_k]Y_k + Y_k \cos \iota - Z_k \sin \iota = 0; \quad (17)$$

$$X_k = 0,$$

which must be satisfied if $\mathbf{u}$, $\mathbf{K}$, $\mathbf{k}$ are to remain co-

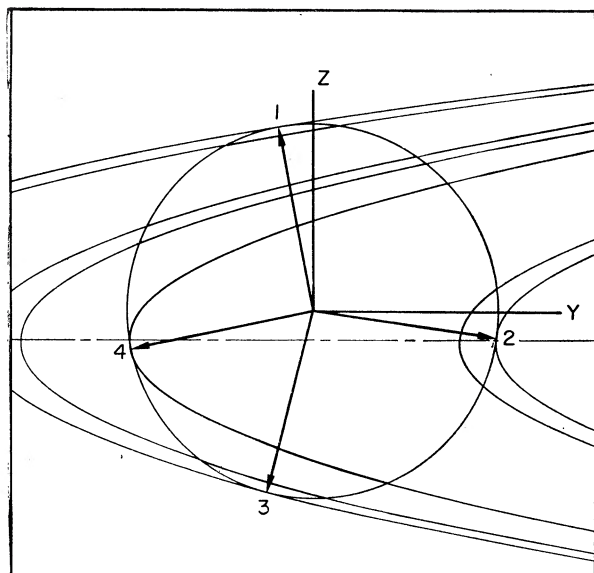


FIG. 2. Schematic of possible precession trajectories of Mercury's spin axis.

planar. Equation (17) can also be derived from the fact that coplanar precession of $\mathbf{K}$ and $\mathbf{k}$ about $\mathbf{u}$ requires

$$\frac{d\alpha}{dt} = -\frac{\partial \langle V \rangle}{\partial \theta} \mathbf{I} = \mathbf{u} \times \alpha.$$

Since Z_k and Y_k are invariant in the first of Eqs. (17), these relations represent the second and third of Cassini's laws generalized to an arbitrary spin resonance or to nonresonant rotation if $S=0$. Hence, if the orbit inclination ι and the obliquity θ of the planet or satellite can be determined, the first of Eqs. (17) yields a relation between $(B-A)/C$ and $(C-A)/C$ and a limit on their range of values.

Additional insight into the generalization of Cassini's laws can be gained if we substitute $Z_1 = \cos\theta_1$, $X_1 = \sin\theta_1$ into Eq. (17) and rearrange to the form

$$[R(C - \frac{1}{2}A - \frac{1}{2}B) + S(B-A)]\sin 2\theta_1 + S(B-A)\sin\theta_1 = \sin(\iota - \theta_1). \quad (18)$$

Since $\mu < 0$, R and S are both < 0 and hence $\theta_1 > \iota$ or $\theta_1 < 0$. This means that the spin axis can never remain in the coplanar configuration if it lies between the orbit normal $\mathbf{K}$ and the negative of the orbital precessional angular velocity $-\mathbf{u}$. This restriction is necessary physically, since the precession of $\mathbf{k}$ about $\mathbf{K}$ due to gravitational torques when $\mathbf{k}$ is between $\mathbf{K}$ and $-\mathbf{u}$ is in a direction opposite that necessary to keep $\mathbf{k}$ coplanar with $\mathbf{K}$ and $\mathbf{u}$.

Colombo (1966) pointed out that a spin rate which is commensurate with the orbital mean motion is not necessary for coplanar precession of $\mathbf{k}$ and $\mathbf{K}$ about $\mathbf{u}$. However, Eq. (18) shows that estimates of a planet's or satellite's deformation from spherical symmetry, from

observations of the pertinent angles are affected by deviations from axial asymmetry, if a commensurate spin is involved. However, the magnitude of this effect is shown to be small in the application of the theory to the moon, and for likely values of $(C-A)/C$, in the application to Mercury. More importantly, the fixed positions of the spin axis in the precessing system being determined by extremes in the energy associated with the body's orientation is in disagreement with Colombo's assertion that these positions are determined by minimizing the internal dissipation.

IV. MOON

The moon is the only body in the solar system which is known to have assumed an extreme value of the energy associated with a well-determined orientation of its spin axis. From Table I, $H_1 + H_2 \approx 1 - 5/2e^2$ for the moon such that Eq. (17) can be written

$$\begin{aligned} & \frac{3n}{2\mu} (1 + \frac{3}{2}e^2) Z_k Y_k \frac{(C-A)}{C} \\ & + \frac{3n}{8\mu} [1 - \frac{5}{2}e^2 - (1 + \frac{1}{2}e^2) Z_k] Y_k \left(\frac{B-A}{C} \right) \\ & + Y_k \cos \iota - Z_k \sin \iota = 0, \quad (19) \end{aligned}$$

where the expansion of $(1-e^2)^{-\frac{1}{2}}$ has been used. Equation (19) represents the analytic generalization of the numerical relation between $\theta_1 - \iota$, $(C-A)/C$, and $(B-A)/C$ obtained by Eckhardt (1965). However, substitution of the values of the angles and eccentricity (I. A. U. Commission 17, 1964)

$$\begin{aligned} \iota &= 5^\circ 9' 16'' \pm 23'', \\ \theta_1 - \iota &= 1^\circ 30' 54'' \pm 30'', \\ \theta_1 &= 6^\circ 40' 10'' \pm 53'', \\ e &= 0.0549, \end{aligned}$$

where θ_1 is derived from the other two given angles, yields a ratio of the coefficient of $(B-A)/C$ to that of $(C-A)/C$ of about 4×10^{-3} . Hence, the value of $(C-A)/C$ determined by Eq. (19) is virtually unaffected by the deviation from axial symmetry as B is varied over its maximum range $A \leq B \leq C$. In fact,

$$6.19 \times 10^{-4} \leq C-A/C \leq 6.21 \times 10^{-4}, \quad (20)$$

with the given values of the angles and $A \leq B \leq C$, whereas

$$6.1 \times 10^{-4} \leq C-A/C \leq 6.3 \times 10^{-4} \quad (21)$$

is the range allowed by the uncertainty in the observations. Hence, neither Eq. (19) nor Eckhardt's numerical relation can be used to gain information about $(B-A)/C$ from the position of the spin axis. As observational accuracy is increased, higher-order terms

must be retained in V in order to obtain a sufficiently accurate representation of $(C-A)/C$ vs $(B-A)/C$ for a given set of data.

V. MERCURY

Mercury exhibits a generalization of Cassini's first law, since it is almost certainly stable with a spin angular velocity of precisely $1.5n$ (Pettengill and Dyce 1965; Carmichel and Dollfus 1968; Smith and Reese 1968; Goldreich and Peale 1968). It is probable (since the orbit precesses) that the generalization of the second and third laws discussed above for $p = \frac{3}{2}$ is also applicable. All that is necessary is that sufficient time elapse for the energy to assume one of the extreme values. Unfortunately, the position of Mercury's spin axis is not well determined such that a good estimate of its permanent deformation is not yet possible. The approach to the problem will thus differ somewhat from that to the moon. In the process of determining the position of Mercury's spin axis as a function of its deformation, more details of the generalized Cassini's laws will emerge.

From Eq. (16) we see that the intersection of the common axis of a family of parabolic cylinders with the Z axis is given by

$$Z_k^0 = - \left[\frac{B-A}{\frac{7}{8}e} + \frac{\mu}{n} \cos \tau \right] \times \left[\left(1 + \frac{3}{2}e^2 \right) \frac{C-A}{C} + \left(-\frac{1}{2} + \frac{7}{8}e - \frac{3}{4}e^2 \right) \frac{B-A}{C} \right]^{-1},$$

where the values of H for $p = \frac{3}{2}$ (Table I) have been used. For large differences in the moments of inertia, Z_k^0 approaches zero for $B=A$ and $-.253$ for $B=C$, where $e=0.206$ for Mercury's orbit. The parabola axis rises with decreasing moment differences; however, since $\mu/n \approx -10^{-6}$ for Mercury (Brouwer and Clemence 1961), $(C-A)/C$ must be less than 10^{-6} before Z_k^0 can approach unity. Hence, for a wide range of probable values of $(C-A)/C$ [recall $(C-A)/C \approx 6.2 \times 10^{-4}$ for the moon], the family of parabolas will appear approximately as in Fig. 2. (Figure 2 corresponds to Colombo's Fig. 3, but the latter is not correct. The error arises from an incorrect sign on the orbital precession which reverses the orientation of the parabolas.) Four possible tangent points of the parabolic cylinders with the unit sphere, and hence, four fixed spin states are possible. However, state 4 in Fig. 2 is not stable (as can be seen by the trajectory of the spin axis which passes through it).

If we start with the relatively large differences of moments of inertia for which Fig. 2 applies, and decrease these moment differences, the rising parabola axis (and spreading of the branches) forces position 1 away from $\mathbf{K}$ and position 4 toward $\mathbf{K}$. The former becomes less stable and finally ceases to exist as a stable

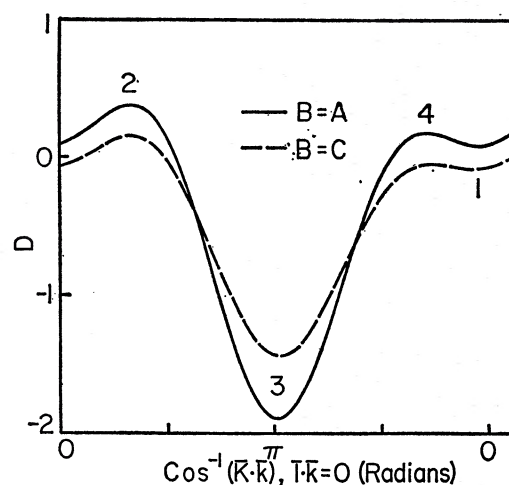


FIG. 3. Extremes in energy function corresponding to stable coplanar configurations for Mercury $(C-A)/C = 1.7 \times 10^{-6}$.

solution of Eq. (17) when positions 1 and 4 coincide. For smaller values of $(C-A)/C$, only positions 2 and 3 allow coplanar precession of $\mathbf{K}$, $\mathbf{k}$, $\mathbf{u}$ with position 2 approaching the $-\mathbf{u}$ direction. In Fig. 3, D is shown as a function of the angle between $\mathbf{K}$ and $\mathbf{k}$ when the latter is in the YZ plane for a relatively small value of $(C-A)/C = 1.7 \times 10^{-6}$. The extremes correspond to the states 1 through 4 in Fig. 2. The decreased stability of state 1 as it approaches state 4 is evident. The two extremes degenerate into an inflection when the two states coincide.

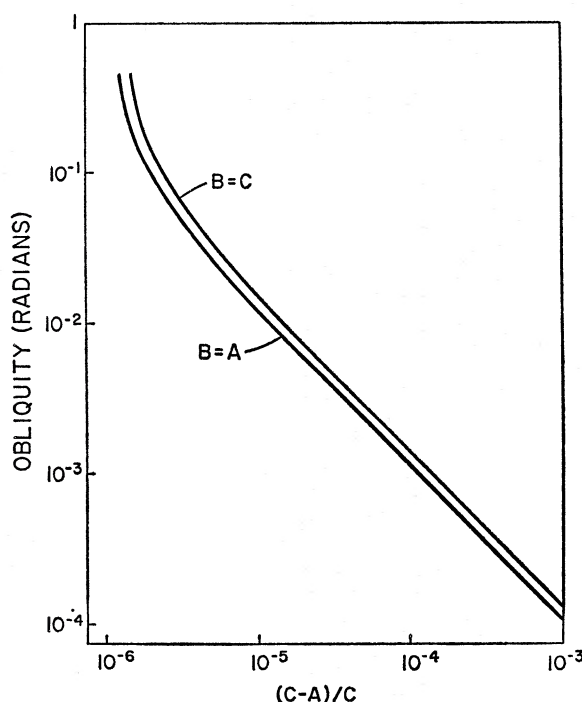
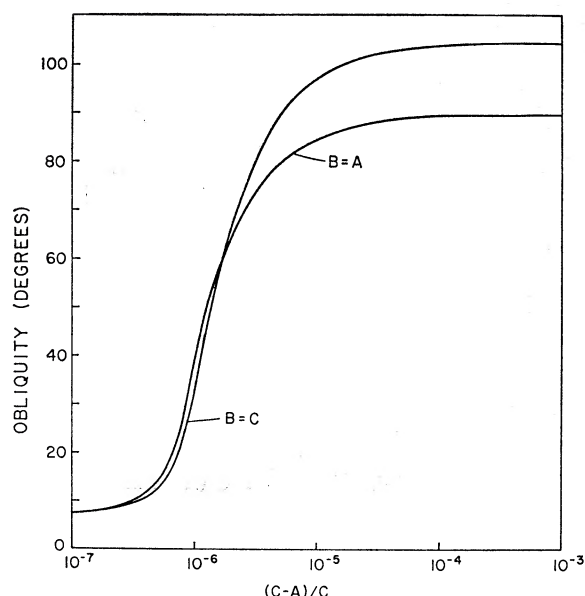


FIG. 4. Obliquity of Mercury's equator plane vs $(C-A)/C$; state 1.

FIG. 5. Obliquity of Mercury's equator plane vs $(C-A)/C$; state 2.

Although the behavior of the solutions of Eq. (17) is here applied to Mercury, the discussion is completely general. Regardless of the spin, a similar variation of the extremes of D with the magnitude of the asphericity of the body occurs with the ratio of μ/n to $(C-A)/C$ determining the number of possible extremes. For the moon $|\mu/n| \gg (C-A)/C$, and only two extremes exist corresponding to states 2 and 3 in Fig. 2, with the moon occupying state 2 at a position close to $-\mathbf{u}$.

For applications of the theory to Mercury, we shall use $i=7^\circ$, $e=.206$. The roots of Eq. (17) yield the positions of the spin vector as a function of the parameters. In Fig. 4 we have shown the obliquity of Mercury's equator plane to its orbit plane for state 1 as a function of $(C-A)/C$ for the two extreme values of B . The effect of the deviation from axial symmetry is much more important here than for the moon, but the position of the spin axis is not markedly affected by $B-A$ over much of the range of $(C-A)/C$. The points where the slopes of the curves become infinite correspond to the loss of stability for state 1. Mercury cannot occupy state 1 if $(C-A)/C \leq 10^{-6}$.

The dependence of the obliquity on $(C-A)/C$ for state 2 is shown in Fig. 5. The spin vector approaches $-\mathbf{u}$ asymptotically as the planet's deformation is decreased, which is a necessary consequence of the forbidden occupancy by $\mathbf{k}$ of the region between $\mathbf{K}$ and $-\mathbf{u}$ for coplanar precession.

A marked contrast between states 1 and 2 when the spin vector is respectively close to the orbit normal, or the negative of the precessional angular velocity is evident from Figs. 4 and 5 and from the discussion of the behavior of the parabola axis as a function of $(C-A)/C$. Increasing the planet's or satellite's permanent deformation from spherical symmetry decreases

the obliquity in state 1 but increases the obliquity in state 2. This behavior has a simple physical explanation.

In Fig. 6 the points K, k, μ represent, respectively, the endpoints of the vectors $\mathbf{K}, \mathbf{k}, -\mathbf{u}$ on the unit sphere. The diagrams are then considered to be drawn on the unit sphere with the scale sufficiently small, such that great circles can be approximated by straight lines. Fig. 6(a) represents state 1 with the coplanar precession about $-\mathbf{u}$ indicated by the velocity vectors $\dot{\mathbf{K}}$ and $\dot{\mathbf{k}}$. The precessional angular velocity of $\mathbf{k}$ about $\mathbf{K}$ is given by

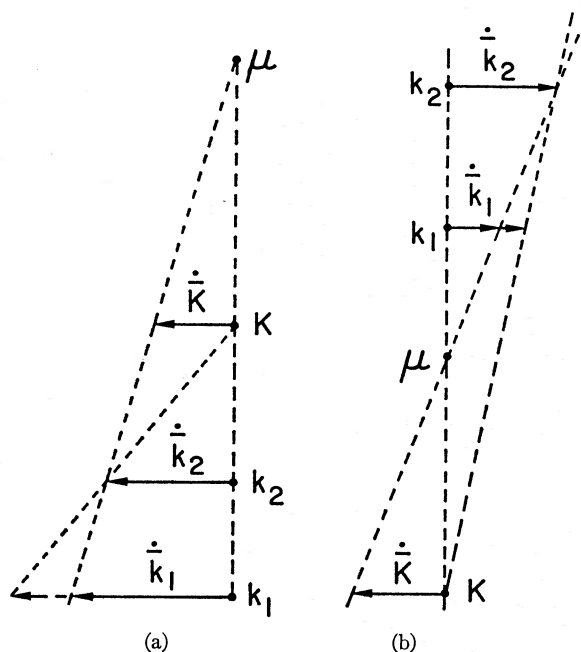
$$\dot{\mathbf{k}} = -\frac{3n^2(C-A)}{2\psi C}(\mathbf{K} \cdot \mathbf{k})(\mathbf{K} \times \mathbf{k}).$$

With $\mathbf{K} \cdot \mathbf{k} \approx 1$,

$$\dot{\mathbf{k}} \approx \omega \times \mathbf{k},$$

where $\omega = -1.5n^2(C-A)/(\psi C)\mathbf{K}$. Hence, $\dot{\mathbf{k}}$ in Fig. 6 is approximately proportional to the separation of K and k . If we increase $\dot{\mathbf{k}}$ to $\dot{\mathbf{k}}'$ by increasing $(C-A)/C$, the only way the coplanar precession can be restored is to move k closer to K , i.e., decrease the obliquity. This change is indicated in Fig. 6(a) by the position of k_2 which corresponds to the increased value of $(C-A)/C$.

State 2 is shown in Fig. 6(b) with the coplanar precession indicated by $\dot{\mathbf{K}}$ and $\dot{\mathbf{k}}_1$. However, in this case coplanar precession is restored after increasing $(C-A)/C$, only by increasing the distance kK , which increases the obliquity. The contrasting behavior of states 1 and 2 are thus easily understood.

FIG. 6. Schematic for physical explanation of the dependence of the obliquity on $(C-A)/C$ for states 1 and 2.

VI. DISCUSSION

We have shown that invariant coplanar configurations in the generalization of Cassini's second and third laws simply represent extremes in the energy associated with the spin axis orientation. The stable coplanar configurations determine a relation between $(B-A)/C$ and $(C-A)/C$, and place relatively narrow bounds on their magnitudes. However, the deviation from axial symmetry has little effect on the determination of $(C-A)/C$ for the moon and, for likely values of this ratio, for Mercury.

Mercury's spin axis is very approximately perpendicular to its orbit plane, so as seen in Figs. 4 and 5, it must be in either state 1 with $(C-A)/C \gg |\mu/n|$ or in state 2 with $(C-A)/C \lesssim |\mu/n|$. Since these states are separated by at least 7° , it should be possible in the near future to determine which is occupied and place some bounds on Mercury's deformation. Unfortunately, it is

likely that $10^{-5} < (C-A)/C < 10^{-3}$ for Mercury such that the spin axis will be in state 1 and differ by no more than a few minutes of arc from the orbit normal. The small absolute range in the obliquity means that only the determination of a lower bound on $(C-A)/C$ is likely to be feasible in the reasonably near future.

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