

Icarus and the Determination of Astronomical Constants

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(Received 4 December 1968)

Radar and optical observations of the minor planet (1566) Icarus have been analyzed in an attempt to verify the predictions of general relativity using both the Schwarzschild nonisotropic and isotropic metrics, and to estimate the dynamical oblateness of the sun, the mass of Mercury, and other parameters. Identical values of the relativity parameters λ_S and λ_I , which are introduced as multipliers of the relativistic effects due to Schwarzschild nonisotropic and isotropic line elements, respectively, were obtained if the elements of both the Earth and Icarus were adjusted. A value of $\lambda_S = 1$ corresponds to motion in general relativity with the nonisotropic Schwarzschild line element, while $\lambda_I = 1$ corresponds to motion with the isotropic line element. The values $\lambda_S = \lambda_I = 0$ correspond to Newtonian motion. Values of λ_S were obtained over the interval $0.87 \pm 0.08 \leq \lambda_S \leq 1.26 \pm 0.11$. The nonuniform distribution of observations over the orbits of Icarus and the Earth seriously affects the suitability of the Icarus data in verifying general relativity. The estimate of the dynamical oblateness of the sun is $J_2 = (1.8 \pm 2.0) \times 10^{-5}$ and the estimated mass of Mercury is $m_{\text{Hg}}^{-1} = 5\,934\,000 \pm 65\,000$. All error estimates are expressed in terms of the standard deviation (mean error) estimated from the data.

I. INTRODUCTION

THE minor planet (1566) Icarus, discovered by W. Baade in 1949, has been an object of great interest to the astronomical community because of its close approaches to the Earth and Mercury, its large relativistic advance of perihelion, its possible role in estimating the solar oblateness, and the mathematical challenges associated with treating a highly perturbed and eccentric orbit. The pioneering efforts of S. Herrick (1953) in developing mathematical tools with which to treat Icarus, together with the investigations conducted by Gilvarry (1953) and by Shapiro *et al.* (1968) concerning the general relativistic advance of perihelion, those by Brans and Dicke (1961) and by Dicke and Peebles (1965) concerning general relativity, and a scalar-tensor theory, and those by Dicke (1965), by Shapiro (1965), and by Dicke and Goldenberg (1967) concerning solar oblateness, have in large part contributed to the great interest shown in Icarus during the 1968 opposition when it approached within 6 million kilometers of the Earth.

The Jet Propulsion Laboratory (JPL) became especially interested in Icarus when it appeared that radar Doppler-shift observations of Icarus could be made at the NASA/JPL Goldstone, California tracking stations. With the cooperation of Herrick of the University of California at Los Angeles, G. Null and D. Holdridge of JPL performed the preliminary analysis, made accuracy studies, and generated radar predictions with which Icarus was successfully observed by Goldstein (1968). Shapiro *et al.* (1968) conducted a similar effort to enable the tracking of Icarus by the Haystack Microwave Research Facility at Tyngsboro, Massachusetts.

This paper presents the results of an analysis of the photographic observations of Icarus from 1949 to 1968 and the seven Doppler-shift observations made at the Goldstone station in June 1968. The observations have been analyzed in an attempt to confirm the predictions of general relativity using both the Schwarzschild

nonisotropic and isotropic line elements (Eddington 1960), and to estimate the dynamical oblateness of the sun and the mass of Mercury. The paper is divided into four major sections as follows: Observational Data, Ephemerides and Differential Corrections, Solutions, and Conclusions.

II. OBSERVATIONAL DATA

A polar-coordinate plot depicting the orbits of Icarus and the Earth projected onto the ecliptic is shown in Fig. 1. Most of the observations of Icarus were made during the months of June and July when Icarus approaches the Earth on its trajectory away from the sun. The observed photographic positions of Icarus are listed in Table I which contains an assigned observation number, the observatory, the time of observation in Universal Time (year, month, day, and fraction of a day past 0^h U.T.), the topocentric astrometric right ascension and declination of Icarus referred to the mean equator and equinox of 1950.0, the residuals to the "best-fit" trajectory (to be discussed later) in $\cos\delta\Delta\alpha$ and $\Delta\delta$, and the source of the observation. Table II contains the Doppler data: the observation number, the tracking station, the epoch of reception in U.T., the observed Doppler shift in Hz, and the residual in Hz. The station locations are listed in Table III, which contains the assigned observatory number, the geocentric distance in km, the longitude (measured east from Greenwich) in degrees, the geocentric latitude in degrees, and the observatory name. Station locations are taken from the 1968 American Ephemeris or communicated by the observer when not in the A.E. The geocentric distance and latitude were computed with an Earth equatorial radius of 6378.160 km and a flattening of 1/298.25, as recommended by the International Astronomical Union (Fricke 1966). The station locations employed for the Doppler observations (Station 6) contain the positions of the transmitting antenna Deep-Space Station 13 (DSS-13) and

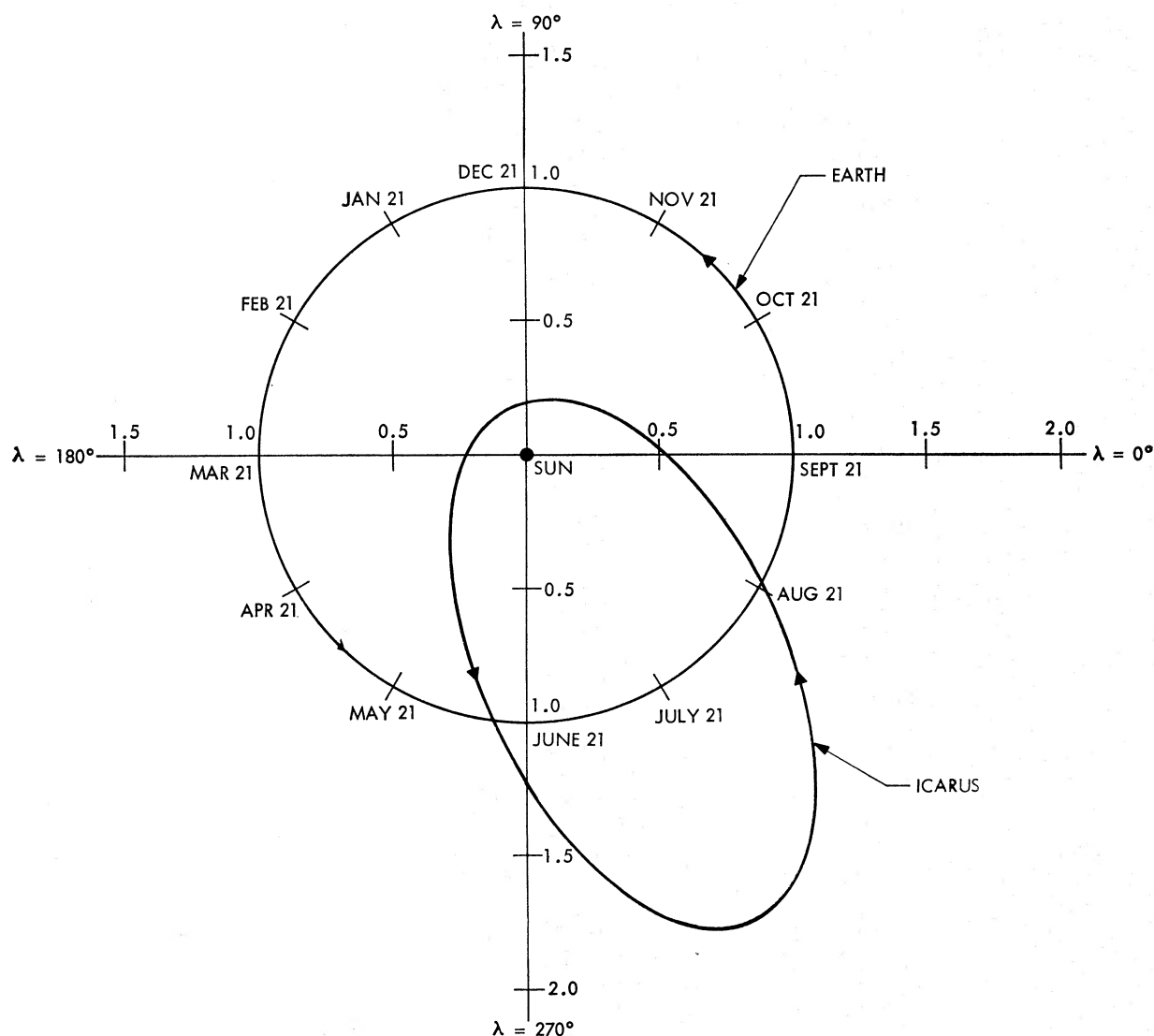


FIG. 1. Two-dimensional plot of the orbits of Icarus and the Earth projected onto the ecliptic: The ecliptic longitude λ and distance from the sun in astronomical units are indicated, together with the approximate dates at which the Earth appears at the positions marked on its orbit.

the receiving antenna, DSS-14, which are located approximately 14 miles apart. The DSS locations listed in the table are preliminary but adequate for the Icarus experiment. The transmitting station, DSS-13, is an 85-ft parabolic antenna, while the receiving station, DSS-14, is a 210-ft parabolic antenna. The transmitter frequency used in the Icarus experiment was 2388 MHz and the random noise was on the order of 0.06 m/sec. Further details may be found in a paper by Goldstein (1968).

There are 154 photographic positions employed in the analysis (several others were deleted because of large residuals which were evidently due to inaccurate reference star positions, difficulty in tracking or timing etc.). Of the 154 optical pairs employed, 134 were made during the months of June and July. Ninety-one

observations were made in 1968, while the remaining 63 cover the period 1949–1967. During the June 1968 close approach to the Earth, the geocentric motion of Icarus was approximately $1''$ per second. The very concentrated distribution of observations along the orbits of Icarus and the Earth was a major factor in seriously limiting the information which could be obtained concerning general relativity, as will be demonstrated later.

III. EPHEMERIDES AND DIFFERENTIAL CORRECTIONS

The trajectory of Icarus was integrated numerically using a second-sum twelfth-order backward-differences Gauss-Jackson n -body numerical-integration computer program developed by C. L. Lawson of JPL. The motions of Icarus and the eight major planets (ex-

TABLE I. Photographic observations of Icarus.

No.	Obs.	Date (U.T.)		α (1950.0)	δ (1950.0)	$\cos\delta\Delta\alpha$	$\Delta\delta$	Reference*	
1	11	1949	6 27	0.238890	16 ^h 41 ^m 8 ^s 13	-27°51'53".7	0".52	0".42	MPC 270
2	11	1949	6 29	0.230560	16 31 54.65	-28 16 20.5	1.56	0.81	MPC 270
3	11	1949	7 1	0.315620	16 24 15.33	-28 35 5.3	-0.20	-0.45	MPC 270
4	11	1949	7 13	0.206600	16 3 10.48	-29 26 9.1	1.33	0.16	MPC 270
5	11	1949	7 14	0.217010	16 2 26.90	-29 28 39.0	1.44	-0.25	MPC 270
6	11	1949	7 23	0.195830	17 0 1.70	-29 47 23.4	0.82	0.05	HAC 1017
7	11	1949	7 24	0.187500	16 0 6.74	-29 49 16.5	0.47	0.26	HAC 1017
8	9	1949	7 28	0.206330	16 0 59.85	-29 56 51.0	0.93	0.11	HAC 1017
9	9	1949	7 28	0.223690	16 1 0.01	-29 56 52.4	-0.43	0.46	HAC 1017
10	11	1950	8 3	0.191670	13 54 21.03	-12 1 25.9	0.29	-0.73	HAC 1096
11	11	1950	8 3	0.209720	13 54 25.84	-12 2 6.5	1.14	0.72	HAC 1096
12	11	1950	8 4	0.191670	13 58 42.56	-12 39 36.0	1.09	-1.19	HAC 1096
13	11	1950	8 5	0.191320	14 2 56.28	-13 16 5.0	-0.19	0.50	HAC 1096
14	11	1950	8 5	0.209030	14 3 0.65	-13 16 42.1	-0.56	1.17	HAC 1096
15	12	1952	6 23	0.145140	1 27 17.60	-16 20 42.0	-1.18	0.86	MPC 937
16	12	1952	7 25	0.137500	4 29 6.10	-9 5 3.0	0.70	-0.94	MPC 937
17	12	1952	7 25	0.152430	4 29 13.20	-9 4 35.0	-1.71	1.08	MPC 937
18	12	1953	8 8	0.107990	1 47 4.69	-53 34 31.7	1.30	1.62	MPC 1041
19	12	1953	8 10	0.089560	1 57 54.08	-55 57 48.3	1.59	1.18	MPC 1041
20	12	1953	8 13	0.108640	2 18 18.18	-59 48 1.5	-0.78	0.28	MPC 1041
21	12	1953	8 13	0.139870	2 18 32.41	-59 50 27.8	0.13	0.45	MPC 1041
22	2	1953	8 14	0.099460	2 26 23.17	-61 5 48.7	-0.63	0.70	MPC 1041
23	2	1953	8 15	0.096730	2 35 23.54	-62 24 45.9	-1.09	0.62	MPC 1041
24	2	1953	8 18	0.115550	3 9 39.17	-66 22 6.7	-1.33	-0.57	MPC 1041
25	12	1954	7 30	0.943720	22 8 14.22	-50 14 46.0	-0.64	-0.11	MPC 1220
26	12	1954	7 30	0.981240	22 8 6.35	-50 16 7.6	1.26	1.65	MPC 1220
27	12	1954	8 23	0.899630	19 57 51.54	-59 59 38.7	-1.37	0.90	MPC 1220
28	12	1954	8 23	0.938150	19 57 36.40	-59 59 48.8	-0.35	1.44	MPC 1220
29	1	1957	7 1	0.313290	19 55 39.22	-37 34 35.1	-1.35	0.16	MPC 1660
30	1	1957	7 1	0.341060	19 55 32.37	-37 35 7.6	-1.28	0.08	MPC 1660
31	5	1958	5 15	0.440350	23 0 41.02	1 42 29.5	0.97	0.36	MPC 2038
32	5	1958	5 15	0.455580	23 0 37.64	1 41 50.1	0.33	1.37	MPC 2038
33	5	1958	5 23	0.418960	22 29 32.15	-4 43 55.6	1.20	-0.48	MPC 2038
34	5	1958	5 23	0.443200	22 29 25.71	-4 45 12.3	0.42	0.18	MPC 2038
35	12	1958	6 18	0.122980	19 42 54.87	-29 52 36.5	-0.09	-0.04	MNASSA 17, No. 7
36	12	1958	6 18	0.142080	19 42 45.50	-29 53 24.5	-0.04	0.22	MNASSA 17, No. 7
37	12	1958	6 22	0.126680	19 11 12.62	-32 30 0.2	-1.34	0.05	MNASSA 17, No. 7
38	5	1958	6 23	0.373840	19 1 44.37	-32 56 22.0	-1.13	-0.12	MPC 2038
39	5	1958	6 23	0.385610	19 1 39.02	-32 56 40.5	-0.43	0.33	MPC 2038
40	8	1958	6 25	0.407660	18 46 50.46	-33 45 31.3	0.21	-0.01	AJ 65, 164
41	8	1958	6 25	0.425720	18 46 42.45	-33 45 54.3	-1.59	-0.20	AJ 65, 164
42	8	1958	6 25	0.445160	18 46 34.06	-33 46 18.5	-0.80	-0.03	AJ 65, 164
43	5	1958	7 8	0.258070	17 34 35.09	-35 38 45.0	-1.55	0.62	MPC 2038
44	5	1958	7 8	0.279880	17 34 29.64	-35 38 45.4	-0.91	0.53	MPC 2038
45	12	1965	6 29	0.965540	21 23 5.22	-36 40 29.9	1.08	-0.42	MPC 2581
46	12	1965	6 30	0.003040	21 23 0.26	-36 41 23.7	-0.98	-0.62	MPC 2581
47	12	1965	7 2	0.044590	21 18 35.70	-37 29 47.5	-0.67	-0.05	MPC 2581
48	12	1965	7 2	0.082090	21 18 30.39	-37 30 41.2	-1.88	-0.56	MPC 2581
49	12	1965	7 28	0.890350	19 52 4.92	-45 52 14.4	1.14	-0.51	MPC 2581
50	12	1965	7 28	0.925070	19 51 56.79	-45 52 34.1	-0.71	-0.82	MPC 2581
51	12	1965	7 28	0.966740	19 51 47.31	-45 52 56.9	-0.14	-0.77	MPC 2581
52	12	1965	7 29	0.892080	19 48 20.57	-46 1 9.7	0.51	0.11	MPC 2581
53	12	1965	7 29	0.926810	19 48 12.54	-46 1 27.5	-0.63	-0.19	MPC 2581
54	12	1966	6 22	0.934030	20 49 58.62	-34 20 58.2	1.01	-0.68	MPC 2698
55	12	1966	6 22	0.981250	20 49 50.62	-34 22 6.2	-1.05	0.20	MPC 2698
56	12	1966	6 23	0.034720	20 40 41.70	-34 23 23.5	-1.18	0.38	MPC 2698
57	12	1966	6 23	0.084030	20 49 33.51	-34 24 34.6	-0.65	0.23	MPC 2698
58	12	1966	7 16	0.858510	19 26 34.16	-41 45 49.1	-0.25	-0.30	MPC 2698
59	12	1966	7 16	0.932120	19 26 17.00	-41 46 31.6	-0.12	0.41	MPC 2698
60	12	1966	7 21	0.875070	19 7 53.11	-42 23 48.1	-1.34	-0.15	MPC 2698
61	12	1966	7 21	0.934960	19 7 40.14	-42 24 7.7	1.76	-0.45	MPC 2698
62	14	1967	6 8	0.353510	20 58 22.10	-25 33 8.9	-0.80	0.13	PER
63	14	1967	6 8	0.378510	20 58 16.45	-25 34 5.1	0.37	-0.57	PER
64	14	1968	6 11	0.430590	4 3 14.70	64 50 30.9	0.61	1.00	IAUC 2077
65	14	1968	6 11	0.438230	4 3 25.03	64 54 5.1	0.51	1.09	IAUC 2077
66	8	1968	6 12	0.215266	4 29 48.64	71 42 50.6	0.67	1.73	AJ 73, 747
67	8	1968	6 12	0.218700	4 30 0.16	71 44 51.6	0.11	0.62	IAUC 2078
68	11	1968	6 12	0.465590	4 47 5.43	74 18 34.0	-0.03	1.00	IAUC 2088
69	11	1968	6 14	0.186750	13 21 3.74	73 39 19.2	-0.63	-1.01	IAUC 2088
70	8	1968	6 14	0.223146	13 25 40.26	72 56 32.2	-1.16	2.96	AJ 73, 747
71	8	1968	6 14	0.257216	13 29 38.92	72 15 46.4	0.15	-1.62	AJ 73, 747
72	5	1968	6 14	0.269570	13 30 56.12	72 0 41.4	-0.74	0.17	IAUC 2088
73	5	1968	6 14	0.269800	13 30 57.70	72 0 24.4	-0.49	-0.10	IAUC 2088

TABLE I (continued)

No.	Obs.	Date (U.T.)		α (1950.0)	δ (1950.0)	$\cos\delta\Delta\alpha$	$\Delta\delta$	Reference ^a	
74	8	1968	6 14	0.278698	13 ^h 32 ^m 1 ^s 42	71°49'46".8	0".14	-2".14	AJ 73, 747
75	5	1968	6 14	0.298030	13 34 0.09	71 25 58.2	-1.18	0.22	IAUC 2088
76	5	1968	6 14	0.298180	13 34 1.18	71 25 47.5	-0.49	0.56	IAUC 2088
77	5	1968	6 14	0.298290	13 34 1.89	71 25 40.1	-0.41	1.26	IAUC 2088
78	15	1968	6 15	0.017390	14 18 55.62	55 44 37.3	-2.09	-0.95	IAUC 2080
79	11	1968	6 15	0.196640	14 24 55.87	51 46 12.8	-0.25	0.14	IAUC 2088
80	11	1968	6 15	0.199420	14 25 0.43	51 42 30.8	-0.04	0.45	IAUC 2088
81	11	1968	6 15	0.202200	14 25 4.91	51 38 48.6	-0.37	0.59	IAUC 2088
82	11	1968	6 15	0.204980	14 25 9.50	51 35 6.1	0.39	0.35	IAUC 2088
83	11	1968	6 15	9.207760	14 25 13.94	51 31 23.6	-0.12	0.18	IAUC 2088
84	11	1968	6 15	0.210530	14 25 18.45	51 27 41.6	0.28	-0.34	IAUC 2088
85	11	1968	6 15	0.213310	14 25 22.85	51 23 59.6	-0.33	-0.02	IAUC 2088
86	11	1968	6 15	0.216090	14 25 27.37	51 20 17.5	0.30	0.11	IAUC 2088
87	11	1968	6 15	0.218870	14 25 31.79	51 16 35.6	0.01	0.42	IAUC 2088
88	8	1968	6 15	0.306308	14 27 47.49	49 20 15.0	-0.13	1.12	AJ 73, 747
89	8	1968	6 15	0.320197	14 28 7.98	49 1 50.5	0.77	-0.05	AJ 73, 747
90	13	1968	6 15	0.909720	14 39 47.14	36 46 41.3	0.37	-1.95	IAUC 2078
91	13	1968	6 15	0.911110	14 39 48.33	36 45 5.8	0.48	-0.26	IAUC 2078
92	11	1968	6 16	0.175090	14 43 31.76	31 51 20.4	0.47	0.02	IAUC 2088
93	8	1968	6 16	0.342708	14 45 18.96	28 56 16.6	0.38	2.13	AJ 73, 747
94	8	1968	6 16	0.355208	14 45 26.58	28 43 35.4	-0.10	1.87	AJ 73, 747
95	13	1968	6 16	0.923610	14 50 42.13	20 6 47.3	-0.48	1.61	IAUC 2080
96	13	1968	6 16	0.925000	14 50 42.71	20 5 39.8	-0.61	2.10	IAUC 2080
97	11	1968	6 18	0.179270	14 57 56.07	6 43 32.3	-0.39	-0.22	IAUC 2088
98	8	1968	6 18	0.262362	14 58 13.88	6 2 57.4	-0.10	-0.06	AJ 73, 747
99	8	1968	6 18	0.266539	14 58 14.71	6 0 56.5	-0.54	-0.90	AJ 73, 747
100	3	1968	6 19	0.057640	15 1 4.94	0 35 8.6	-1.36	-0.94	IAUC 2088
101	3	1968	6 19	0.058320	15 1 5.03	0 34 54.0	-1.51	-0.93	IAUC 2088
102	3	1968	6 19	0.114410	15 1 13.23	0 15 0.6	-0.70	-0.98	IAUC 2088
103	3	1968	6 19	0.115100	15 1 13.25	0 14 46.7	-1.89	-0.34	IAUC 2088
104	8	1968	6 19	0.212876	15 1 32.23	- 0 20 55.6	-0.10	-0.28	AJ 73, 747
105	8	1968	6 19	0.216348	15 1 32.71	- 0 22 5.9	-0.21	-0.00	AJ 73, 747
106	11	1968	6 19	0.236340	15 1 35.04	- 0 28 44.3	0.36	0.61	IAUC 2088
107	3	1968	6 20	0.089920	15 3 46.05	- 4 36 59.6	-1.47	-0.71	IAUC 2088
108	3	1968	6 20	0.091970	15 3 46.29	- 4 37 29.9	-1.06	0.29	IAUC 2088
109	3	1968	6 20	0.094030	15 3 46.49	- 4 38 2.0	-1.27	-0.38	IAUC 2088
110	8	1968	6 20	0.238889	15 4 5.56	- 5 15 41.5	-0.95	-0.04	AJ 73, 747
111	11	1968	6 20	0.241080	15 4 5.40	- 5 16 9.5	-0.25	-0.83	IAUC 2088
112	8	1968	6 20	0.247222	15 4 6.42	- 5 17 42.8	-0.67	-0.10	AJ 73, 747
113	15	1968	6 20	0.973300	15 5 31.32	- 7 56 9.9	-0.06	0.60	IAUC 2083
114	15	1968	6 21	0.013470	15 5 35.05	- 8 3 59.5	-0.61	-0.46	IAUC 2083
115	3	1968	6 21	0.112490	15 5 47.35	- 8 21 21.6	-0.07	-1.04	IAUC 2088
116	3	1968	6 21	0.116630	15 5 47.64	- 8 22 7.3	-0.62	0.09	IAUC 2088
117	3	1968	6 21	0.119390	15 5 47.89	- 8 22 38.8	-0.15	-0.23	IAUC 2088
118	14	1968	6 21	0.211200	15 5 58.00	- 8 41 3.4	0.77	0.88	PER
119	14	1968	6 21	0.213290	15 5 58.22	- 8 41 26.0	1.66	1.28	PER
120	8	1968	6 21	0.213540	15 5 58.94	- 8 41 35.2	0.08	-0.35	AJ 73, 747
121	8	1968	6 21	0.220490	15 5 59.48	- 8 42 51.7	0.08	-0.40	AJ 73, 747
122	10	1968	6 21	0.862418	15 7 1.41	-10 31 1.3	-0.97	2.62	IAUC 2080
123	7	1968	6 21	0.922220	15 7 5.60	-10 40 18.2	0.16	0.87	IAUC 2088
124	7	1968	6 21	0.935420	15 7 6.67	-10 42 21.2	1.60	-1.85	IAUC 2088
125	7	1968	6 21	0.942360	15 7 6.98	-10 43 21.8	-1.41	0.60	IAUC 2088
126	4	1968	6 22	0.065350	15 7 19.39	-11 0 24.2	-0.45	0.08	IAUC 2083
127	4	1968	6 22	0.068820	15 7 19.60	-11 0 55.0	-0.57	-0.06	IAUC 2083
128	4	1968	6 22	0.072290	15 7 19.80	-11 1 26.2	-0.84	-0.63	IAUC 2083
129	8	1968	6 22	0.209740	15 7 32.20	-11 22 31.0	-0.36	0.02	AJ 73, 747
130	8	1968	6 22	0.218770	15 7 32.76	-11 23 48.1	-0.43	-0.14	AJ 73, 747
131	8	1968	6 23	0.211110	15 8 50.87	-13 29 24.1	-0.46	0.10	AJ 73, 747
132	8	1968	6 23	0.218750	15 8 51.28	-13 30 15.6	-0.34	0.32	AJ 73, 747
133	8	1968	6 24	0.211810	15 9 59.13	-15 11 16.9	-0.92	0.04	AJ 73, 747
134	8	1968	6 24	0.221530	15 9 59.58	-15 12 10.4	-0.86	-0.01	AJ 73, 747
135	8	1968	6 25	0.212150	15 11 0.14	-16 34 50.0	-0.70	-0.26	AJ 73, 747
136	8	1968	6 25	0.221880	15 11 0.54	-16 35 34.4	-0.73	-0.35	AJ 73, 747
137	8	1968	6 26	0.213190	15 11 56.03	-17 44 38.2	0.12	-0.70	AJ 73, 747
138	8	1968	6 26	0.225000	15 11 56.48	-17 45 23.9	0.08	-1.17	AJ 73, 747
139	8	1968	6 27	0.205230	15 12 47.78	-18 43 17.4	-0.43	-0.80	AJ 73, 747
140	8	1968	6 27	0.219120	15 12 48.29	-18 44 2.8	-0.35	-0.73	AJ 73, 747
141	3	1968	6 28	0.036210	15 13 29.53	-19 25 19.0	1.07	-0.40	IAUC 2094
142	3	1968	6 28	0.038750	15 13 29.63	-19 25 26.5	1.23	-0.50	IAUC 2094
143	3	1968	6 28	0.044240	15 13 29.78	-19 25 42.4	0.67	-0.42	IAUC 2094
144	8	1968	6 28	0.207650	15 13 37.31	-19 34 10.4	-0.34	0.24	AJ 73, 747
145	8	1968	6 28	0.221530	15 13 37.86	-19 34 49.8	0.55	0.04	AJ 73, 747
146	8	1968	6 29	0.215970	15 14 25.12	-20 18 36.4	-0.97	-0.70	AJ 73, 747

TABLE I (continued)

No.	Obs.	Date (U.T.)			α (1950.0)	δ (1950.0)	$\cos\delta\Delta\alpha$	$\Delta\delta$	Reference ^a
147	8	1968	6 29	0.237510	15 ^h 14 ^m 25 ^s .83	−20°19′29″.6	−1″.45	−1″.01	AJ 73, 747
148	8	1968	6 30	0.250000	15 15 12.60	−20 58 26.8	−0.65	−1.38	AJ 73, 747
149	8	1968	6 30	0.273610	15 15 13.40	−20 59 17.0	−1.02	−1.06	AJ 73, 747
150	8	1968	7 1	0.293750	15 15 59.61	−21 33 49.6	−0.61	−0.66	AJ 73, 747
151	14	1968	7 2	0.170180	15 16 40.09	−22 0 27.1	1.05	−0.15	PER
152	14	1968	7 2	0.194490	15 16 40.91	−22 1 8.8	1.03	0.17	PER
153	14	1968	8 15	0.156240	15 59 10.31	−29 0 3.3	0.18	−0.89	PER
154	14	1968	8 15	0.202770	15 59 13.58	−29 0 14.4	−0.27	−0.83	PER

* Abbreviations employed:

AJ: Astronomical Journal
HAC: Harvard Announcement Card
IAUC: International Astronomical Union Circular
MNASSA: Monthly Notices Astronomical Society South Africa
MPC: Minor Planet Circular
PER: Private communication from E. Roemer

cluding Pluto) were integrated simultaneously with the nonisotropic Schwarzschild line element in a heliocentric 1950.0 equatorial reference frame at quarter-day steps using the standard predictor-corrector formulas and 16 decimal-digit arithmetic. The Earth and the moon were treated as separate bodies and the lunar ephemeris was read from magnetic tape. The epoch initially employed was 1949 June 27 (Julian Ephemeris Date J. E. D. 2433094.5) but results are presented in this paper for the standard epochs 1949 June 13 (J.E.D. 2433080.5) and 1969 January 19 (J.E.D. 2440240.5). The planetary positions were taken from JPL Development Ephemeris Number 40 (DE-40) which resulted from an n -body fit to optical observations of the planets made at the U. S. Naval Observatory from 1949 to 1967 and radar observations of Mercury, Venus, and Mars from 1964 to 1967 (Melbourne and O'Handley 1968). The planetary masses employed in DE-40 and in the preliminary stages of this research are (Mulholland 1967):

Body	Inverse mass
Sun	1
Mercury	6 017 000
Venus	408 504
Earth+moon	328 900
Mars	3 098 500
Jupiter	1 047.3908
Saturn	3 499.2
Uranus	22 930
Neptune	19 071
Earth/moon	81.30

The set of computer programs developed by Lawson (Melbourne and O'Handley 1968) and known as the Solar System Data Processing System (SSDPS) was

TABLE II. Radar observations of Icarus.

No.	Obs.	Date (U.T.)			Doppler shift (Hz)	Residual (Hz)	
1	6	1968	6	14	0.22917	115417.119	0.563
2	6	1968	6	14	0.93056	— 10324.100	1.131
3	6	1968	6	15	0.18750	— 61207.180	1.976
4	6	1968	6	15	0.40278	—104441.890	0.974
5	6	1968	6	16	0.06944	—202453.520	—0.576
6	6	1968	6	16	0.27083	—234709.969	0.174
7	6	1968	6	16	0.41667	—255404.750	0.775

employed, in modified form, to reduce the Icarus data and to numerically integrate the trajectories.

The partial derivatives relating the observables to the modeled parameters were obtained from the numerical integration of the variational equations by means of computer programs described by Lieske (1968). The partial derivatives obtained from the variational equations were compared with the difference-quotients computed by differencing the coordinates of two trajectories obtained by slightly altering the parameter of interest. The comparison was satisfactory and the variational equations were employed in further analyses.

The effects of general relativity were approximated by considering only the distortions of Euclidean space due to the presence of the sun. The corresponding line element is referred to as a one-body line element. In the trajectory calculations, the Schwarzschild non-

TABLE III. Observatory locations.

No.	R (km)	λ_E (degrees)	ϕ' (degrees)	Name
1	6373.575	295°45'33.3	-31°42'66.0	Bosque Alegre, Argentina
2	6371.544	18.47658	-33.75600	Cape of Good Hope, S. Africa, Royal Obs.
3	6374.059	288.84167	-32.80789	Cerro el Roble, U. of Chile Obs.
4	6372.817	295.80350	-31.25028	Cordoba, Argentina
5	6373.407	248.24125	+35.01012	Flagstaff, Arizona, U. S. Naval Obs. Station
6	6372.177	243.20599	+35.06660	Goldstone, Calif., Transmitter DSS-13 (Venus site)
	6372.053	243.11135	+35.24441	Goldstone, Calif., Receiver DSS-14 (Mars site)
7	6365.625	11.71250	+50.79246	Jena, Germany K. Schwarzschild Obs.
8	6371.617	238.35446	+37.15497	Mt. Hamilton, Calif. Lick Obs.
9	6373.179	241.94029	+34.03784	Mt. Wilson, Calif. Obs. Carnegie Inst.
10	6368.365	7.30042	+43.52919	Nice, France
11	6373.438	243.13600	+33.17975	Palomar Mountain, Calif.
12	6375.682	28.22875	-25.63793	Pretoria, S. Africa, Radcliffe Obs.
13	6367.736	20.24500	+48.99848	Skalná Pleso, Czechoslovakia
14	6372.057	249.28333	+32.22614	Tucson, Arizona, U. of Arizona Catalina Station
15	6365.466	4.35821	+50.60999	Uccle, Belgium

isotropic line element (Eddington 1960, p. 85) was employed. In polar coordinates the one-body Schwarzschild nonisotropic line element may be written

$$ds^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2, \quad (1)$$

where

$$m = GM_{\odot}/c^2 \cong 1.48 \text{ km}, \quad (2)$$

and where G is the square of the Gaussian gravitational constant $k = 0.01720 20989 5$ rad/day, $M_{\odot}$ is the mass of the sun, and c is the speed of light in astronomical units (a.u.) per day. The values employed for the speed of light and the astronomical unit are 299792.5 km/sec and 149 597 896 km, respectively (Melbourne and O'Handley 1968). The perturbative acceleration $\ddot{\mathbf{r}}_S$ for the Schwarzschild nonisotropic line element, which may be added to the Newtonian equations of motion $\ddot{\mathbf{r}}_N$, was taken from Brouwer and Clemence (1961a), and is given by

$$\ddot{\mathbf{r}}_S = A_S \mu r^{-3} \mathbf{r} + B_S \dot{\mathbf{r}}, \quad (3)$$

where, if m_p is the mass of the body relative to the sun,

$$\mu = k^2(1 + m_p),$$

$$A_S = \frac{2m}{r} - \frac{2v^2}{c^2} + \frac{2(\mathbf{r} \cdot \dot{\mathbf{r}})^2}{c^2 r^2} + \frac{(\mathbf{r} \cdot \dot{\mathbf{r}})^2}{c^2 r^2 (1 - 2m/r)}, \quad (4)$$

and

$$B_S = \frac{2m(\mathbf{r} \cdot \dot{\mathbf{r}})}{r^3(1 - 2m/r)}. \quad (5)$$

In the preceding equations, $\mathbf{r}$ and $\dot{\mathbf{r}}$ denote the heliocentric Schwarzschild coordinates in a.u. and velocities in a.u. per day, respectively, while $v^2 = |\dot{\mathbf{r}}|^2$ and the small parameter m is defined in Eq. (2).

The coordinates $(\bar{r}, \bar{\theta}, \bar{\varphi}, \bar{t})$ in the so-called isotropic line element (Eddington 1960, p. 93)

$$ds^2 = (1 - \frac{1}{2}m/\bar{r})^2(1 + \frac{1}{2}m/\bar{r})^{-2}c^2 d\bar{t}^2 - (1 + \frac{1}{2}m/\bar{r})^4(d\bar{r}^2 + \bar{r}^2 d\bar{\theta}^2 + \bar{r}^2 \sin^2 \bar{\theta} d\bar{\varphi}^2) \quad (6)$$

are related to the Schwarzschild nonisotropic coordinates (r, θ, φ, t) in the following manner, originally due to Weyl:

$$\begin{aligned} r &= (1 + \frac{1}{2}m/\bar{r})^2 \bar{r} \cong \bar{r} + m, \\ \theta &= \bar{\theta}, \\ \varphi &= \bar{\varphi}, \\ t &= \bar{t}. \end{aligned} \quad (7)$$

If one should numerically integrate the equations of motion using the one-body isotropic line element given above, the perturbative acceleration $\ddot{\mathbf{r}}_I$, analogous to the nonisotropic acceleration $\ddot{\mathbf{r}}_S$ in Eq. (3), is given by

$$\ddot{\mathbf{r}}_I = A_I \mu r^{-3} \mathbf{r} + B_I \dot{\mathbf{r}}, \quad (8)$$

where

$$A_I = 4mr^{-1} - v^2 c^{-2} \quad (9)$$

and

$$B_I = 4m(\mathbf{r} \cdot \dot{\mathbf{r}})r^{-3}. \quad (10)$$

In the preceding equations, $\mathbf{r}$ and $\dot{\mathbf{r}}$ refer to the heliocentric isotropic coordinates and velocities, respectively. The isotropic equations were originally obtained for n bodies by de Sitter (1916), while the perturbative accelerations corresponding to the one-body line element of Eq. (6) have more recently been given by Moyer (1968) and Anderson (1968).

The effects of the solar oblateness J_2 were obtained from a potential of the form

$$U = -k^2 r^{-1} [1 - J_2 (R_{\odot}/r)^2 (-\frac{1}{2} + \frac{3}{2} \sin^2 \beta)], \quad (11)$$

where $R_{\odot}$ is the equatorial radius of the sun in km and β is the latitude relative to the solar equator. The adopted value is $R_{\odot} = 6.9598 \times 10^5$ km (Allen 1963). The dynamical oblateness of the sun J_2 is related to the dynamical flattening f by the formula

$$\frac{3}{2}J_2 = f - \frac{1}{2}R_{\odot}\omega_{\odot}^2 g_{\odot}^{-1} = f - 1.0 \times 10^{-5}, \quad (12)$$

where $\omega_{\odot}$ is the sun's angular velocity of rotation and $g_{\odot}$ is its surface gravity. Dicke and Goldenberg (1967) have estimated the visual flattening as

$$f = (5.0 \pm 0.7) \times 10^{-5},$$

which would correspond to a value $J_2 = (2.7 \pm 0.5) \times 10^{-5}$ if the visual flattening were identical with the dynamical flattening.

The perturbative acceleration due to the J_2 term of the potential given in Eq. (11) may be written in an (Earth) equatorial coordinate system as

$$J_2 \ddot{\mathbf{r}}_J = -\frac{3}{2}J_2 k^2 R_{\odot}^2 r^{-4} [(1 - 5 \sin^2 \beta) r^{-1} \mathbf{r} + 2\mathbf{C} \sin \beta], \quad (13)$$

where

$$r \sin \beta = xC_x + yC_y + zC_z,$$

and where

$$\begin{aligned} C_x &= \sin I_{\odot} \sin \Omega_{\odot} &= +0.12194, \\ C_y &= -\sin I_{\odot} \cos \Omega_{\odot} \cos \epsilon - \cos I_{\odot} \sin \epsilon &= -0.42454, \\ C_z &= -\sin I_{\odot} \cos \Omega_{\odot} \sin \epsilon + \cos I_{\odot} \cos \epsilon &= +0.89716. \end{aligned}$$

In the preceding equations, $I_{\odot}$ and $\Omega_{\odot}$ denote the inclination and longitude of the ascending node of the solar equator on the ecliptic, while ϵ denotes the obliquity of the ecliptic. Numerical values of $I_{\odot}$ and $\Omega_{\odot}$ were taken from Allen (1963, p. 179) and the mean obliquity of the ecliptic at 1950.0 used throughout this investigation is $\epsilon = 23^{\circ} 44' 57.8''$.

The equations of motion involving Newtonian motion, solar oblateness, and general relativity using either the Schwarzschild nonisotropic or isotropic line element may schematically be written as

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_N + \lambda_S \ddot{\mathbf{r}}_S + \lambda_I \ddot{\mathbf{r}}_I + J_2 \ddot{\mathbf{r}}_J, \quad (14)$$

where $\ddot{\mathbf{r}}_N$ denotes the Newtonian n -body equations of motion, $\ddot{\mathbf{r}}_S$ denotes the contribution due to the Schwarzs-

child nonisotropic line element given by Eq. (3), $\ddot{\mathbf{r}}_I$ denotes the contribution due to the isotropic line element given by Eq. (8), and where $\ddot{\mathbf{r}}_J$ denotes the contribution (except for the multiplicative factor J_2) due to the solar oblateness given by Eq. (13). The factors λ_S and λ_I have been introduced in the manner of Shapiro *et al.* (1968) as multipliers of the effects due to the Schwarzschild nonisotropic and isotropic line elements, respectively. Clearly, one of the parameters is set to zero in actual computations. If $\lambda_I=0$, then $\lambda_S=0$ corresponds to Newtonian motion, while $\lambda_S=1$ corresponds to motion in general relativity using the Schwarzschild nonisotropic line element of Eq. (1). If $\lambda_S=0$, then $\lambda_I=0$ corresponds to Newtonian motion, while $\lambda_I=1$ corresponds to motion in general relativity using the isotropic line element of Eq. (6). Examination of the terms A and B appearing in Eqs. (3) and (8) indicates that for a given set of initial conditions, the integrated effects of $\ddot{\mathbf{r}}_I$ and $\ddot{\mathbf{r}}_S$ will be quite different. However, either metric should be capable of predicting the same observable. It will be shown that they indeed do predict identical observables (to 0".001), provided that the initial conditions are allowed to vary in changing from one metric to the other.

In the trajectory calculations the values $\lambda_S=1$, and $\lambda_I=J_2=0$ were employed, so that the motion of Icarus and the planets was computed using the Schwarzschild nonisotropic line element. Variational equations were integrated numerically to obtain $\partial\mathbf{r}/\partial\lambda_S$, $\partial\mathbf{r}/\partial\lambda_I$, and $\partial\mathbf{r}/\partial J_2$ for Icarus and the Earth using $\lambda_S=\lambda_I=J_2=0$. This was done so that the vector $\mathbf{V}$ defining $\partial\mathbf{r}/\partial\lambda_S$, $\partial\mathbf{r}/\partial\lambda_I$, or $\partial\mathbf{r}/\partial J_2$, and in general given by

$$\ddot{\mathbf{V}} = \mathbf{A}\mathbf{V} + \mathbf{B}\dot{\mathbf{V}} + \mathbf{C}, \quad (15)$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{\partial \ddot{\mathbf{r}}}{\partial \mathbf{r}} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial \ddot{\mathbf{r}}}{\partial \dot{\mathbf{r}}} \end{bmatrix},$$

and

$$\mathbf{C} = \begin{bmatrix} \frac{\partial \ddot{\mathbf{r}}}{\partial q} \end{bmatrix}$$

with $q = \lambda_S$, λ_I , or J_2 , could be simplified to the form

$$\ddot{\mathbf{V}} = \mathbf{A}_1\mathbf{V} + \mathbf{C}, \quad (16)$$

where $\mathbf{A}_1$ is the relatively simple 3×3 matrix given by Lieske (1968). The negligible error introduced into the variational equations is on the order of $m \partial\mathbf{r}/\partial q$ for $\partial\ddot{\mathbf{r}}/\partial\lambda_S$ and $\partial\ddot{\mathbf{r}}/\partial\lambda_I$, and is on the order of $J_2' \partial\mathbf{r}/\partial J_2$ for $\partial\ddot{\mathbf{r}}/\partial J_2$, where $m \cong 10^{-8}$ and $J_2' = J_2 R_\odot^2 (\text{a.u.}) \cong 10^{-9}$.

The data then may be analyzed to estimate λ_I by "turning off" the Schwarzschild parameter λ_S by setting $\Delta\lambda_S = -1$ in the normal equations and estimating λ_I by modifying the normal equations in the manner outlined by Lieske (1968). Similarly, it is simple

to obtain the effect one would have obtained by actually integrating the trajectory with the isotropic metric merely by setting $\Delta\lambda_S = -1$ (to go from motion with the Schwarzschild nonisotropic line element to Newtonian motion) and $\Delta\lambda_I = +1$ (to go from Newtonian motion to motion with the isotropic line element).

In addition to the three parameters λ_S , λ_I , and J_2 already mentioned, the variations with respect to small changes in the initial position and velocity components of Icarus (six parameters) and the Earth (six parameters), together with the variations of the coordinates of Icarus and the Earth with respect to changes in the masses of Mercury, Venus, and Earth+moon (three parameters) also were integrated numerically. The conditional equations for a bias in right ascension $\Delta\alpha_0$ and declination $\Delta\delta_0$, together with the partial derivative of the observables with respect to a change in the speed of general precession in longitude p , were computed analytically. Thus, a total of 21 parameters was available for analysis. The conditional equations for $\Delta\alpha_0$ and $\Delta\delta_0$ are defined in the sense that $-\Delta\alpha_0$ and $-\Delta\delta_0$ are to be added to the observed right ascension and declination. The partial derivatives of the coordinates of Icarus and the Earth with respect to changes in their initial position and velocity components were converted to partial derivatives of the coordinates with respect to the Eckert-Brouwer Set III parameters (Brouwer and Clemence 1961b, p. 241) by means of a transformation matrix similar to that given by Lieske (1968).

The normal equations were accumulated for the photographic observations in $\cos\delta\Delta\alpha$ and $\Delta\delta$ and the Doppler observations by employing weights of 1", 1", and 1 Hz, respectively. The least-squares solution was obtained by the Gauss-Jordan elimination technique for various sets of parameters and *a priori* nominal changes (e.g., "turning off" the Schwarzschild parameter and "turning on" the isotropic parameter by setting $\Delta\lambda_S = -1$, and $\Delta\lambda_I = +1$). All error estimates are expressed in terms of standard deviations (mean errors) calculated from the data.

IV. SOLUTIONS

The trajectory obtained by Null and Warner (1968) in their analysis of the pre-1968 data was employed in the preliminary analysis of the 1949-1968 data. Initially, only the six elements describing the orbit of Icarus and the mass of Mercury were estimated, and the integrations were performed using the nonisotropic Schwarzschild line element. When the seven Doppler observations were included with the 308 optical data, the estimate for the mass of Mercury changed markedly from its value when only the optical data were considered. In addition, the mean-square residual increased slightly (relative to its value obtained from analyzing only the optical data) when the seven Doppler data were included. No attempt was made to

TABLE IV. Initial conditions for Icarus.^a

Coordinate	Epoch 1949 June 13 (J.E.D. 2433080.5)	Epoch 1969 Jan. 19 (J.E.D. 2440240.5)
Equator and equinox of 1950.0 ^b		
x (a.u.)	-9.23655 65525 22831 D-02	+1.04085 47887 71752 D-00
y (a.u.)	-1.00650 70770 91298 D-00	-1.16489 99145 47210 D-00
z (a.u.)	-4.13378 38100 36122 D-01	-1.01317 10505 65478 D-00
$\dot{x}$ (a.u./day)	+7.29293 33437 23731 D-03	+1.84665 18751 76362 D-03
$\dot{y}$ (a.u./day)	-1.17521 48692 43108 D-02	+6.03015 00823 87009 D-03
$\dot{z}$ (a.u./day)	-8.72369 84512 76600 D-03	+1.88362 42410 92679 D-03
Ecliptic and equinox of 1950.0		
a (a.u.)	1.07777 00284 3217	1.07788 75084 5703
e	0.82667 04094 57439	0.82657 83894 70799
I (deg)	22.99378 70493 743	22.94541 19231 332
Ω (deg)	87.77236 90013 944	87.63076 50397 777
ω (deg)	30.87852 13036 842	31.04025 22337 458
M_0 (deg)	43.55662 05622 183	230.84591 20787 99

^a Obtained with Schwarzschild nonisotropic line element and $m_{\text{g}}^{-1} = 5\,934\,000$.

^b D denotes power of 10, so that 1.23D-04 means 1.23×10^{-4} .

reduce the photographic observations individually to a common system since almost all of the pre-1968 observations contained no mention of what reference star catalogues were employed. However, if a declination bias $\Delta\delta_0$ was estimated together with the mass of Mercury and the six elements of Icarus, then the optical-only and the optical-plus-Doppler solutions were consistent in their estimates of the mass of Mercury, and the mean-square residual decreased slightly when the Doppler data were included. The declination bias obtained was $\Delta\delta_0 = -0''.29 \pm 0''.13$, so that $-\Delta\delta_0$ should be added to the observed declinations. Except for suggesting the existence of a declination bias, the seven Doppler data had little influence on the solutions. The trajectory of Icarus was then reintegrated using the same conditions of DE-40 for the eight major planets and the moon, in conjunction with the mass of Mercury obtained in the preceding analysis, $m_{\text{g}}^{-1} = 5\,934\,000$. The mass of Mercury is in very close agreement with the value $m_{\text{g}}^{-1} = 5\,935\,000 \pm 45\,000$ obtained by Shapiro (1968) from radar observations of the planets.

The initial conditions of Icarus at two epochs for this "best" trajectory are listed in Table IV. The initial

conditions are consistent with the preceding mass of Mercury, the Schwarzschild nonisotropic line element, and a declination bias of $-\Delta\delta_0 = +0''.29$. The residuals listed in Tables I and II correspond to the observed minus computed ($O-C$) positions obtained by numerically integrating the initial conditions listed in Table IV, together with a declination bias of $-\Delta\delta_0 = +0''.29$ (the observed declinations in Table I have not been corrected by $-\Delta\delta_0$, however). Several attempts were made at estimating a right-ascension bias, but since the estimates varied quite drastically depending upon what parameters were estimated, no right-ascension bias was included in further analyses. It also was not possible to obtain consistent estimates of the correction to the general precession in longitude, so no corrections were included for precession errors. Similarly, no significant improvement of the masses of Venus and the Earth-moon system could be obtained from the Icarus data.

We were primarily interested in obtaining estimates of the relativity parameters λ_S and λ_I , and observing the consequences of employing either the isotropic or nonisotropic line elements. In their analysis of the pre-1968 data, Shapiro *et al.* (1968) observed a 30 percent reduction in the sum of squares of the residuals in fitting the data with the Schwarzschild isotropic line element compared with pure Newtonian motion. In addition, they estimated the parameter λ_I and obtained $\lambda_I = 0.97 \pm 0.20$. Their error analysis (including only errors in the orbit of Icarus) indicated that the standard error of λ_I should decrease to ± 0.08 if the 1968 photographic and radar data were included. We were concerned with the possible effects of small changes in the ephemeris of the Earth on estimates of the relativity parameters, since small changes in the orbit of the Earth are greatly magnified in the geocentric position of Icarus due to its close approach.

The results of fitting the optical and Doppler data to estimate the relativity parameters are presented in Table V. All solutions in the table were obtained using

TABLE V. Solutions for the relativity parameters λ_S and λ_I .

Case	Parameters ^a	SOS ^b	λ_S	λ_I
1	E, I (12)	197.42	1	-
2	E, I (12)	197.41	-	1
3	E, I (12)	284.98	0	0
4	E except $\Delta a/a$, I (11)	198.64	1	-
5	E except $\Delta a/a$, I (11)	255.11	-	1
6	E, I, λ_S (13)	193.43	1.26 ± 0.11	-
7	E, I, λ_I (13)	193.43	-	1.26 ± 0.11
8	E except $\Delta a/a$, I, λ_S (12)	198.41	1.04 ± 0.07	-
9	E except $\Delta a/a$, I, λ_I (12)	220.22	-	0.65 ± 0.05
10	E only $\Delta a/a$, I, λ_S (8)	224.51	0.98 ± 0.08	-
11	E only $\Delta a/a$, I, λ_I (8)	224.47	-	0.98 ± 0.08
12	I, λ_S (7)	224.72	0.96 ± 0.07	-
13	I, λ_I (7)	266.34	-	0.48 ± 0.04

^a E and I indicate that the solutions involve estimating the six Eckert-Brouwer Set III parameters for the Earth and Icarus, respectively.

^b Denotes the weighted sum of squares of linearized residuals.

$m_{\odot}^{-1} = 5\,934\,000$ and $J_2 = 0$. The table contains a case number, the parameters estimated (E indicates that the six Eckert-Brouwer Set III parameters of the Earth were adjusted, while I refers to the six parameters describing the orbit of Icarus), the value of the Schwarzschild parameter λ_S or the isotropic parameter λ_I which was adopted or estimated, and the sum of squares (SOS) of the linearized residuals. The number of parameters estimated is listed in parentheses. For example, Case 1 contains a summary of a 12-parameter solution in which the six elements of the Earth (E) and Icarus (I) were adjusted using the nonisotropic Schwarzschild line element ($\lambda_S = 1$). Case 2 is similar, but the isotropic line element ($\lambda_I = 1$) was used. It may be noted that Cases 1 and 2 resulted in almost identical sums of squares. The individual linearized residuals for Cases 1 and 2 were also identical to 0".001, so that the Schwarzschild nonisotropic and isotropic line elements do predict identical observable quantities. Case 3 represents a 12-parameter Newtonian fit ($\lambda_S = \lambda_I = 0$) to the data, which indicates that the final sum of squares of either relativistic line element is 30 percent less than the Newtonian solution. The 11-parameter solutions listed in Cases 4 and 5 (in which corrections to the six elements of Icarus and the Earth, excluding $\Delta a/a$ of the Earth, were estimated) indicate that the parameter a plays a major role in describing the difference between the Schwarzschild and isotropic line elements. Cases 6–11 indicate that identical values of the parameters λ_S and λ_I and sum of squares could be obtained only if the semimajor axis of the Earth was adjusted. Cases 12 and 13 indicate the results if only the orbit of Icarus and λ_S or λ_I were adjusted. The estimate for λ_S given in Case 12 is somewhat dependent upon the mass of Mercury employed, primarily because of the close approach (0.09 a.u.) of Icarus to Mercury in April 1968. If the mass of Mercury is held fixed at $m_{\odot}^{-1} = 6\,017\,000$, then the resultant value of λ_S analogous to Case 12 is $\lambda_S = 0.87 \pm 0.07$.

It is apparent that identical estimates of λ can be obtained with a standard error (0.08) similar to that estimated by Shapiro *et al.* (1968) for a given set of solved-for parameters, but that the estimated values of λ systematically vary over the range

$$0.87 \pm 0.08 \leq \lambda \leq 1.26 \pm 0.11$$

depending upon which, if any, elements of the Earth ephemeris are estimated from the Icarus data. Because of the nonuniform distribution of observations mentioned earlier, there are good arguments for not adjusting the elements of the orbit of the Earth. However, it seems to us that if one desires to "verify" general relativity by employing observations of Icarus, then he must be willing to allow the Earth to travel on whatever trajectory (Newtonian, relativistic, or some other form of motion) the data favors. Unless one performs a "global" solution involving observations of

Icarus together with radar and optical observations of other bodies (in order to obtain more physically realistic corrections to the orbit of the Earth), we feel that observations of *just* Icarus tend to confirm a general relativistic type of motion with an uncertainty of approximately 30 percent rather than the 8 percent indicated by any specific solution in Table V.

From Table V it is apparent that adjusting the parameters a of the Earth and Icarus has the greatest influence in enabling one to predict identical observables using either the Schwarzschild nonisotropic or isotropic line elements. We have intentionally refrained from referring to the parameter a as the "semimajor axis" because we are unwilling (and unable) to attach a geometric significance to it. Mathematically, however, the parameter a is well defined if one chooses to employ the usual formula for the osculating value of the semimajor axis given by

$$a = \mu r (2\mu - rv^2)^{-1}. \quad (17)$$

One can predict the difference between the osculating values of the Schwarzschild "semimajor axis" a_S and the isotropic "semimajor axis" a_I , for a fixed value of $\mu = k^2(1+m_p)$, by means of the transformation given in Eq. (7). It is well known that the radial coordinate r of the Schwarzschild nonisotropic line element is approximately 1.48 km larger than the isotropic radial coordinate $\bar{r}$. The transformation from the isotropic position vector $\bar{\mathbf{r}}$ and velocity vector $\dot{\bar{\mathbf{r}}}$ (velocities are expressed in coordinate time) to the Schwarzschild nonisotropic position and velocity vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ may be written

$$\mathbf{r} = (1 + \frac{1}{2}m/\bar{r})^2 \bar{\mathbf{r}} \quad (18)$$

$$\dot{\mathbf{r}} = (1 + \frac{1}{2}m/\bar{r})^2 \dot{\bar{\mathbf{r}}} - m\bar{r}^{-2} \dot{\bar{\mathbf{r}}}(1 + \frac{1}{2}m/\bar{r}) \bar{\mathbf{r}},$$

where $m = GM_{\odot}c^{-2} \cong 1.48$ km. To first order in m the difference between the osculating values of a_S and a_I is obtained from Eq. (17):

$$a_S - a_I = 2m \left(\frac{a}{r} \right)^2 \left[1 + \frac{a(1-e^2)}{r} \right], \quad (19)$$

where a , e , and r appearing on the right-hand side may be taken as either the isotropic or Schwarzschild osculating semimajor axis, eccentricity, and radial distance, respectively. The relation may also be written in closed form using the true anomaly f :

$$a_S - a_I = 2m(2 + 5e \cos f + 4e^2 \cos^2 f + e^3 \cos^3 f)(1 - e^2)^{-2}. \quad (20)$$

Thus, although the difference between the Schwarzschild and isotropic radial coordinates is $m = 1.48$ km, the osculating values of a determined from Eq. (19) should differ by $4m = 5.92$ km for circular orbits. The preceding relations were independently derived by J. D. Anderson. It is of interest to note that the actual

TABLE VI. Solutions involving the mass of Mercury.

Case ^a	Parameters ^b	SOS ^c	$m_{\text{g}}^{-1} - 5\,934\,000$	$\Delta\delta_0$	λ_{S}	$10^5 J_2$
14	7	224.7			0.96 ± 0.07	
15	8	220.8		$-0''.29 \pm 0''.13$	1.00 ± 0.07	
16	8	220.3		$-0''.31 \pm 0''.12$		1.8 ± 2.0
17	9	218.9		$-0''.28 \pm 0''.13$	0.83 ± 0.12	6.0 ± 3.7
18	7	228.5	+54 000		0.89 ± 0.07	
19	8	222.6	+54 000	$-0''.36 \pm 0''.13$	0.94 ± 0.07	
20	8	222.8	+54 000	$-0''.41 \pm 0''.12$		1.7 ± 2.0
21	9	216.2	+54 000	$-0''.34 \pm 0''.12$	0.64 ± 0.12	10.9 ± 3.6
22	7	224.4	$-22\,000 \pm 31\,000$		1.00 ± 0.08	
23	9	220.8	$+700 \pm 31\,000$	$-0''.29 \pm 0''.13$		
24	9	220.3	$-250 \pm 27\,000$	$-0''.31 \pm 0''.13$		1.8 ± 2.0
25	10	214.2	$+143\,700 \pm 56\,000$	$-0''.46 \pm 0''.14$	0.26 ± 0.25	20.2 ± 6.6
26 ^d	7	220.8		$-0''.29 \pm 0''.13$		

Notes

^a Cases 14–17 obtained with $m_{\text{g}}^{-1} = 5\,934\,000$; Cases 18–21 obtained with $m_{\text{g}}^{-1} = 5\,988\,000$; Cases 22–25 obtained with m_{g}^{-1} allowed to vary.

^b All solutions also involve estimating the six elements of Icarus.

^c Denotes weighted sum of squares of linearized residuals.

^d Nominal trajectory listed in Table IV.

differences between the Schwarzschild and isotropic osculating values of a which were obtained in fitting the actual Icarus data in Cases 1 and 2 of Table V (the epoch employed in fitting the data was J.E.D. 2433094.5) were

$$a_{\text{S}} - a_{\text{I}} = 5.66 \pm 0.40 \text{ km for the Earth}$$

and

$$a_{\text{S}} - a_{\text{I}} = 2.66 \pm 0.90 \text{ km for Icarus,}$$

while the values predicted from Eq. (19) are 5.67 km and 2.72 km for the Earth and Icarus, respectively.

In the subsequent solutions for other parameters such as the solar oblateness and the mass of Mercury listed in Table VI, only the orbit of Icarus and the relevant parameters were estimated because of the difficulty in obtaining realistic corrections to the orbit of the Earth. In Table VI solutions for the mass of Mercury, λ_{S} , and the solar oblateness J_2 are presented. In Cases 14–17 the mass of Mercury is held fixed at the value obtained in the preceding analysis, $m_{\text{g}}^{-1} = 5\,934\,000$. In Cases 18–21 the mass of Mercury is fixed at $m_{\text{g}}^{-1} = 5\,988\,000$, which is close to the value of $m_{\text{g}}^{-1} = 5\,983\,000 \pm 25\,000$ estimated by Melbourne and O'Handley (Melbourne *et al.* 1968, p. 36) from an analysis of the sum-of-squares curve obtained by fitting radar observations of Mercury, Venus, and Mars. In Cases 22–25 the mass of Mercury was allowed to vary together with the other parameters listed. From the table it is seen that the effects of the solar oblateness coefficient J_2 and the mass of Mercury were reasonably separable (viz., Cases 16, 20, and 24), but that if the relativity parameter λ_{S} was estimated together with the mass of Mercury and the solar oblateness, the normal equations became ill conditioned and unreasonable results were obtained. It appears that the estimates of the solar oblateness J_2 and the relativity parameter λ_{S} are both obtained from the perihelion advance and hence cannot be separated.

The solar oblateness also contributes an as-yet undetectable regression of the node of Icarus on the ecliptic of approximately $\Delta\Omega = -0''.6$ per century for a flattening of 5×10^{-5} (Dicke 1965).

If the solutions involving the estimation of both λ_{S} and J_2 are rejected because of the separability difficulty, then from Cases 16, 20, and 24 the best estimate of the solar oblateness (assuming that general relativity is valid) is

$$J_2 = (1.8 \pm 2.0) \times 10^{-5},$$

which neither conclusively demonstrates nor disproves a dynamical solar oblateness comparable to the visual flattening.

V. CONCLUSIONS

The fundamental difficulties, due to the nonuniform distribution of observations of Icarus, in obtaining corrections to the orbits of Icarus and the Earth severely limit the precision with which observations of Icarus may be used in attempts to verify general relativity. The data suggest that identical values of the Schwarzschild and isotropic parameters may be obtained in the interval

$$0.87 \pm 0.08 \leq \lambda \leq 1.26 \pm 0.11,$$

and that only by considering observations of Icarus and other bodies could one reliably improve the estimate of the relativity parameter. It would be very difficult, if at all possible, to discriminate between general relativistic motion and, for example, the Brans-Dicke scalar-tensor theory, especially since the latter theory contains an arbitrary parameter which conceivably could be adjusted to fit whatever value of λ one estimates.

Most of the Icarus observations were made during the months of June and July and all future close approaches to the Earth will occur in June, as indicated in Table

TABLE VII. Close approaches ($\rho \leq 0.15$ a.u.) of Icarus to the Earth, Mercury, and Venus.

Earth				Mercury				Venus			
Year	Month	Day	ρ (a.u.)	Year	Month	Day	ρ (a.u.)	Year	Month	Day	ρ (a.u.)
1949	6	11	0.10	1954	12	11	0.12	1952	9	29	0.10
1968	6	14	0.05	1958	4	18	0.15	1965	1	19	0.10
1987	6	21	0.15	1964	12	21	0.12	1977	5	11	0.10
1996	6	11	0.10	1968	4	25	0.10	1989	9	1	0.12
2015	6	16	0.05	1973	12	18	0.12	2001	12	22	0.15
				1977	4	24	0.15				
				1983	12	31	0.12				
				1987	5	6	0.10				
				1992	12	28	0.12				
				1996	5	3	0.15				
				2003	1	8	0.12				
				2006	5	14	0.10				

VII, which was obtained by extending the trajectory listed in Table IV to A.D. 2000 and employing two-body formulas to estimate the distances and dates of closest approach. The seven Doppler observations were useful in suggesting a declination bias in the optical data, but otherwise had little effect on the solutions. The use of the nonisotropic Schwarzschild line element and the assumption that no adjustment of the ephemeris of the Earth is necessary leads to an estimate of the dynamical solar oblateness given by

$$J_2 = (1.8 \pm 2.0) \times 10^{-5},$$

which is at best marginal in demonstrating the existence of an oblate sun via the dynamical effects on Icarus.

The mass of Mercury is perhaps the best-determined astronomical constant obtainable from the Icarus data. The solutions involving the mass of Mercury and the elements of Icarus listed in Table VI indicate a standard error of approximately $\pm 30\,000$ in the inverse mass. However, analysis of the covariance matrix obtained in estimating corrections to the orbits of Icarus and the Earth indicates a standard error of $\pm 65\,000$ and the value adopted in this investigation is

$$m_{\text{g}}^{-1} = 5\,934\,000 \pm 65\,000.$$

The strength of the Mercury solution primarily comes from the close approach of Icarus to Mercury which occurred just prior to the well-observed 1968 opposition. The data presented in Table VII concerning future close approaches of Icarus to Mercury suggest that future observations of Icarus may eventually provide one of the best means of determining the mass of Mercury.

ACKNOWLEDGMENTS

We are indebted to Professor S. Herrick and Professor P. Herget and to B. G. Marsden who forwarded the Icarus observations to us. We thank E. Roemer and the numerous observers who kindly relayed their 1968 observations almost immediately after they were made.

D. B. Holdridge (JPL), M. J. Sykes (JPL), R. Reed (IBM), and D. Dannenfeldt (IBM) were indispensable in providing computer programming support. J. D. Anderson of JPL contributed helpful advice. Collaboration with I. I. Shapiro of MIT in the earlier stages of providing radar predictions was valuable. All calculations were performed on the IBM 7094/7044 computer at the Jet Propulsion Laboratory. This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.

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