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ELLIPTIC POLARIZATION OF SYNCHROTRON RADIATION

M. P. C. LEGG AND K. C. WESTFOLD
Monash University, Clayton, Victoria, Australia
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ABSTRACT

Earlier calculations on the polarization characteristics of a distribution of ultrarelativistically gyrating electrons emitting synchrotron radiation have been extended to the next order of approximation, and appropriate corrections made for the fundamental frequency at a field point. Calculations by Roberts and Komesaroff of the degree of circular polarization have been corrected, and comparisons made with observations, particularly of the Crab Nebula Here it appears possible that recent 38 Mc/s observations are consistent with the results obtained and a magnetic field of magnitude 10^{-4} gauss; further calculations are necessary before valid comparisons can be made with recently reported optical measurements.

I. INTRODUCTION

In the original investigation (Westfold 1959*a;* hereinafter referred to as "Paper I") of the polarization characteristics of the synchrotron radiation from electrons moving with ultrarelativistic velocities βc in a magnetic field, it was shown that for radiation of frequency f from a single electron the principal axes of the polarization ellipse are parallel and perpendicular to the projection of the direction $\boldsymbol{k}$ of the external magnetic field $\boldsymbol{B}_0$ on to the plane transverse to the direction of observation $\boldsymbol{n}$. Its sense of description is right-handed (RH) when the direction of motion $\boldsymbol{\tau}$ of the electron passes close to $\boldsymbol{n}$ on the opposite side to $\boldsymbol{B}_0$, within an angular distance $\psi = O(\xi)$, where $\xi = \sqrt{(1 - \beta^2)} \ll 1$ (Fig. 1), and left-handed (LH) if $\boldsymbol{\tau}$ passes close to $\boldsymbol{n}$ on the same side as $\boldsymbol{B}_0$. The major axis is perpendicular to the projection of $\boldsymbol{B}_0$ for ψ close to zero, but as ψ increases, the form of the ellipse varies through circular to elliptic with the major axis parallel to the projection of $\boldsymbol{B}_0$. For ψ greater than $O(\xi)$, the emission in the direction $\boldsymbol{n}$ remains negligible throughout the motion. In terms of the Stokes parameters I, M, C, S (or I, Q, U, V) referred to base vectors $\boldsymbol{i}_1, \boldsymbol{i}_2$ parallel and perpendicular to the projection of $\boldsymbol{B}_0$, $C = 0$, $S \gtrless 0$ according as $\psi \gtrless 0$, and M is negative for ψ close to zero and passes through zero to positive values as ψ increases.

Next, the contributions to the emissivity from an isotropic[1] velocity distribution of gyrating electrons were summed, these being negligible for electrons passing $\boldsymbol{n}$ at angular distances ψ greater than $O(\xi)$. To the first approximation, being odd in ψ, contributions S to the fourth Stokes parameter for the equal number of electrons having positive and negative ψ cancel, and the contributions M from electrons with ψ close to zero predominate, so that the second Stokes parameter is negative. Thus the resultant emission is linearly polarized in the direction $\boldsymbol{i}_2$ perpendicular to the projection of $\boldsymbol{B}_0$ on to a plane transverse to $\boldsymbol{n}$.

From these considerations it appears that in a higher approximation, for ξ large enough, it is possible for the resultant emission from a distribution of electrons to remain elliptically polarized provided that (*a*) the fourth Stokes parameter S for a single electron is no longer an odd function of ψ or (*b*) the number of contributing electrons having ψ positive is no longer equal to the number of electrons having ψ negative.

It can be seen at once that proviso *b* is satisfied, for, since it is proportional to the distance $2\pi \sin \alpha$ of $\boldsymbol{\tau}$ from the axis through $\boldsymbol{k}$ (Fig. 1), the number of electrons whose directions of motion pass within a small positive angular distance ψ of $\boldsymbol{n}$ is greater than the

[1] In Paper I this was incorrectly specified as uniform with respect to α, the angle between $\boldsymbol{\tau}$ and $\boldsymbol{B}_0$.

number within an equal negative angular distance of $\boldsymbol{n}$ when the angle θ between $\boldsymbol{k}$ and $\boldsymbol{n}$ is in the first quadrant and less when θ is in the second quadrant. We should therefore expect, on this account alone, that the fourth Stokes parameter corresponding to the emissivity should be positive or negative according as θ is in the first or second quadrant. This is indeed confirmed by calculations made for monoenergetic electrons by Roberts and Komesaroff (1965) in connection with the decimetric radiation from the planet Jupiter. They find, further, that the ratio of the fourth to the first Stokes parameter, the degree of circular polarization, is proportional to ξ, which is itself proportional to $[(B_0 \sin \theta)/f]^{1/2}$. In principle, then, a measurement of the fourth Stokes parameter enables an estimate of the magnitude of $\boldsymbol{B}_0$ to be obtained. From their own nugatory measurements at 960 Mc/s they inferred that the field in the Jovian Van Allen belt was not

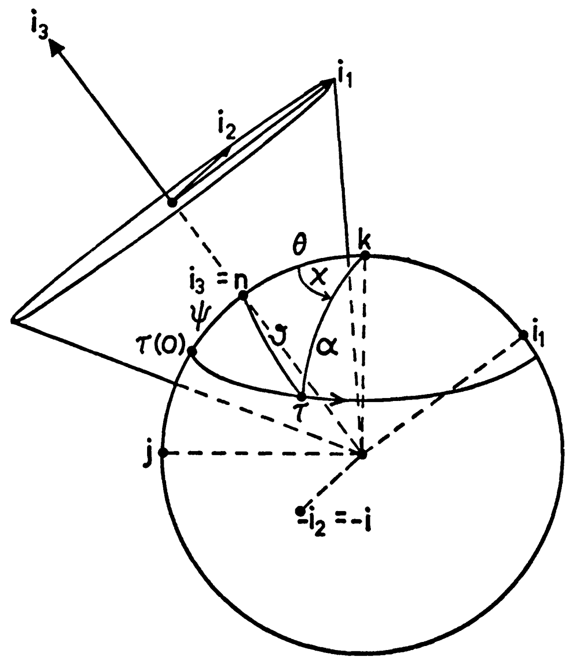

FIG. 1.—Directions of motion and of observation, relative to the magnetic field, and associated base vectors appropriate to the radiation field at the point of observation.

greater than 0.2 gauss. Berge (1965) claims to have measured degrees of circular polarization of up to 10 per cent in the 21-cm radiation from Jupiter, which, by inclusion of a revised pitch-angle distribution in the calculations of Roberts and Komesaroff, corresponds to a magnetic field of 1.7 gauss.

In order that reliable estimates of the fourth emissivity Stokes parameter may be made, it is necessary to calculate this parameter for the emission from a single electron to the next order in ξ. Then, if proviso *a* is satisfied, this will lead to a new formula for the emissivity parameter. It is the object of the present paper to examine this matter in detail. Following the formulation of a previous paper (Westfold 1959*b*), we define an emission-polarization tensor for a single electron. The components of this tensor consist of linear combinations of the four Stokes parameters. We calculate these components to an approximation of one higher order in ξ than was used in Paper I. An emissivity-polarization tensor is then obtained, to the same order of approximation, by summing contributions from the individual members of a distribution of gyrating electrons.

Finally, as has been pointed out by Epstein and Feldman (1967), in Paper I the spectrum of the radiation field from a single electron was incorrectly taken as a superposition of harmonics of the fundamental frequency $\omega_B/2\pi$, the gyrofrequency of the motion of the source, instead of the Doppler shifted frequency at a field point, $\omega_B/(2\pi \sin^2 \alpha)$. However, considerations recently advanced by Scheuer (1968), and independently by Ginzburg, Sazanov, and Syrovatskii (1968), demonstrate that this correction is annulled by a further compensating factor $\sin^2 \alpha$ when the emissivity at a fixed point is calculated. All these considerations have been incorporated into the present paper, but to a higher approximation appropriate to our purpose. The final results have already been presented in a preliminary report of this work (Legg and Westfold 1967).

II. THE SPECTRUM OF THE FIELD FROM A SINGLE ELECTRON

The period of the motion with respect to a field point $\boldsymbol{r}$ is the interval of local time t during which the electron, whose position at the retarded time t' is $\boldsymbol{r}_1(t')$, completes a period with respect to the angle of gyration $\chi = \omega_B t'$ about the direction $\boldsymbol{k}$ of the magnetic field (Fig. 1). According to equations (8) and (15) of Paper I and those following equation (16), we have, for $r_1' \ll r$,

$$\omega_B(t - r/c) = \omega_B(t' - \boldsymbol{n}\cdot\boldsymbol{r}_1'/c) = a\chi - b \sin \chi \,, \tag{1}$$

where, as in Le Roux (1961),

$$a = 1 - \beta' \cos \alpha \cos (\alpha - \psi) \,, \quad b = \beta' \sin \alpha \sin (\alpha - \psi) \,, \tag{2}$$

and β' is the magnitude of $\boldsymbol{\beta}' = d\boldsymbol{r}_1(t')/cdt$.

It follows that the period in t is $2\pi a/\omega_B$ instead of $2\pi/\omega_B$, which was used in Paper I. The corresponding time intervals at the field point and source are related by the equation

$$dt = (a - b \cos \chi)dt' \,,$$

which, by equations (2), is equivalent to equation (10) of Paper I. The part adt' represents the interval during which radiation, emitted during the interval dt', would be received at a field point as a consequence of the uniform parallel motion of the electron; during each circuit of the periodic perpendicular motion, dt' is alternately diminished and augmented by the part $-(b \cos \chi dt')$. Since the source of this radiation is, in fact, attributable only to the circular perpendicular component of the motion, confined to regions small compared with the distance from the field point, we should regard the source as the electron in its projected circular orbit described with speed $\beta'_\perp c$, and the electron-velocity component $\boldsymbol{\beta}'_\parallel c$ simply as the velocity of this source. The factor a therefore determines the difference between the periods at the source and at the field point as a Doppler effect. We proceed to consider the spectrum of the radiation field in terms of the fundamental period $2\pi a/\omega_B$.

In the notation of Paper I the electric vector of the nth harmonic is

$$\begin{aligned} \boldsymbol{E}_n = \frac{\mu e c}{8\pi^2 r}\frac{\omega_B}{a} \exp\left(in\frac{\omega_B}{a}\frac{r}{c}\right) \int_{-\pi/\omega_B}^{\pi/\omega_B} \frac{\boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{\beta}') \times (d\boldsymbol{\beta}'/dt')}{(1 - \boldsymbol{\beta}'\cdot\boldsymbol{n})^2} \\ \times \exp\left[in\frac{\omega_B}{a}\left(t' - \boldsymbol{n}\cdot\frac{\boldsymbol{r}_1'}{c}\right)\right]dt' \,, \end{aligned} \tag{3}$$

where $\boldsymbol{n}$ is the direction from the source at time t' to the field point at time t. The terminals of the period are chosen so as to include the effective interval of emission, in the neighborhood of $t' = 0$. The integral was resolved approximately, under the ultra-relativistic condition $\xi = \sqrt{(1 - \beta'^2)} \ll 1$. Only the first approximation was retained. The numerator of the first term was found to be $O(\xi^2)\omega_B$ and the denominator $O(\xi^4)$, for directions of motion within an angular distance $O(\xi)$ of $\boldsymbol{n}$, so that the bulk of the emission in the direction $\boldsymbol{n}$ occurs within a time interval $O(\xi)/\omega_B$ about $t' = 0$. Again, the bulk of the emission was found to occur in the harmonics of large order $n = O(\xi^{-3})$ and, during the emission in the direction $\boldsymbol{n}$, $\omega_B(t' - \boldsymbol{n}\cdot\boldsymbol{r}_1'/c) = O(\xi^3)$, so that the exponent of the second factor is $O(1)$.

It follows that the next approximation to $\boldsymbol{E}_n$ will result if the first factor is expanded to terms in $O(\xi^{-1})$ and the exponent to terms in $O(\xi)$. On integration by parts, noting that the contribution from the integrated term at the terminals is negligible, we obtain the formula

$$\boldsymbol{E}_n = \frac{\mu e c}{8\pi^2 r}\, in \frac{\omega_B{}^2}{a^2} \exp\left(in\frac{\omega_B}{a}\frac{r}{c}\right)\int_{-\infty}^{\infty} \boldsymbol{\beta}'_T \exp\left[in\frac{\omega_B}{a}\left(t' - \boldsymbol{n}\cdot\frac{\boldsymbol{r}_1'}{c}\right)\right]dt' \,, \tag{4}$$

where $\boldsymbol{\beta}_T'$ is the component of $\boldsymbol{\beta}'$ transverse to $\boldsymbol{n}$. In this form it will be necessary to evaluate $\boldsymbol{\beta}_T'$ to terms in $O(\xi^2)$ and the exponent again to terms in $O(\xi)$.

We choose base vectors appropriate to electromagnetic radiation traveling in the direction $\boldsymbol{n}$, viz., $\boldsymbol{i}_3 = \boldsymbol{n}$, $\boldsymbol{i}_1$ along the projection of $\boldsymbol{k}$ transverse to $\boldsymbol{n}$ (Fig. 1), and $\boldsymbol{i}_2 = \boldsymbol{i}_3 \times \boldsymbol{i}_1$. Then, as before, taking ψ as the angle between the initial direction of motion $\boldsymbol{\tau}(0)$ and $\boldsymbol{n}$, α as the pitch angle between $\boldsymbol{\tau}$ and $\boldsymbol{k}$, and $\chi = \omega_B t'$, we find

$$\boldsymbol{\beta}'_T = -\boldsymbol{i}_1(\psi - \tfrac{1}{2}\chi^2 \sin\alpha\cos\alpha) - \boldsymbol{i}_2\chi\sin\alpha + O(\xi^3)\,, \tag{5}$$

and, for the appropriate approximation[2] to a,

$$a = \sin^2\alpha(1 - \psi\cot\alpha) + O(\xi^2)\,. \tag{6}$$

Thus, in the exponent, we have

$$\frac{\omega_B}{a}\left(t' - \boldsymbol{n}\cdot\frac{\boldsymbol{r}_1'}{c}\right) = \frac{(\xi^2 + \psi^2)\chi}{2\sin^2\alpha}(1 + \psi\cot\alpha) + \tfrac{1}{6}\chi^3 + O(\xi^5)\,. \tag{7}$$

In order to preserve the form of the Airy integral, which was found to be basic in Paper I, we write[3]

$$\gamma = \frac{n}{2\sin^3\alpha}\,, \qquad \eta^2 = (\xi^2 + \psi^2)(1 + \psi\cot\alpha) \tag{8}$$

and change the variable of integration to $u = \chi\sin\alpha$. Then we find

$$\boldsymbol{E}_n = -\frac{\mu e c\omega_B}{4\pi^2 r\sin^2\alpha}\frac{i\gamma}{1 - 2\psi\cot\alpha}\exp\left(in\frac{\omega_B}{a}\frac{r}{c}\right)\int_{-\infty}^{\infty}\exp\left[i\gamma(\eta^2 u + \tfrac{1}{3}u^3)\right] \times [\boldsymbol{i}_1(\psi - \tfrac{1}{2}u^2\cot\alpha) + \boldsymbol{i}_2 u]du\,. \tag{9}$$

We express this in terms of the function

$$F_\gamma(\eta) = \int_{-\infty}^{\infty}\exp\left[i\gamma(\eta^2 u + \tfrac{1}{3}u^3)\right]du$$

of Paper I, which is related to the Airy function of the first kind and the modified Bessel function of order $\frac{1}{3}$. $F_\gamma(\eta)$ is an even function which, for $\eta \geq 0$, decreases monotonically from the value $2.23\gamma^{-1/3}\eta$ at $\eta = 0$ to zero in an exponential manner. Its derivative $F_\gamma'(\eta)$ is an odd function approximating $-3.25\gamma^{1/3}\eta$ for small η and decreasing to a minimum value of about -2.1 at about $\gamma\eta^3 = 0.6$, from which it increases monotonically to zero.

[2] The first approximation $a = \sin^2\alpha$ is appropriate to Paper I.

[3] This value of γ should replace that given in Paper I.

Both functions become relatively negligible for values of $\gamma\eta^3$ greater than a few multiples of unity. Within this range $F_\gamma(\eta) = O(\xi)$ and $F_\gamma'(\eta) = O(1)$. They are expressed in terms of tabulated functions in § IV. Since

$$\int_{-\infty}^{\infty} i\gamma(\eta^2 + u^2) \exp [i\gamma(\eta^2 u + \tfrac{1}{3}u^3)]du = \exp [i\gamma(\eta^2 u + \tfrac{1}{3}u^3)]\big|_{-\infty}^{\infty} ,$$

and physically there can be no contribution at the terminals, the $\boldsymbol{i}_1$ component of the integral in equation (9) is equal to $(\psi + \frac{1}{2}\eta^2 \cot \alpha)F_\gamma(\eta)$; the $\boldsymbol{i}_2$ component is easily seen to be $(1/2i\gamma\eta)F_\gamma'(\eta)$. Thus we have the required result

$$\boldsymbol{E}_n = -\frac{\mu e c \omega_B}{4\pi^2 r \sin^2 \alpha} \exp\left(in\frac{\omega_B}{\alpha}\frac{r}{c}\right) \times \frac{\boldsymbol{i}_1 i\gamma(\psi + \frac{1}{2}\eta^2 \cot \alpha)F_\gamma(\eta) + (\boldsymbol{i}_2/2\eta)F_\gamma'(\eta)}{1 - 2\psi \cot \alpha} , \tag{10}$$

which, in the first approximation, represents a correction to equation (18) of Paper I. The complex polarization is now

$$Q_n(\psi) = \frac{F_\gamma'(\eta)/F_\gamma(\eta)}{2i\gamma\eta(\psi + \frac{1}{2}\eta^2 \cot \alpha)} , \tag{11}$$

which, with equations (8), shows that the axes of the polarization ellipse remain parallel to $\boldsymbol{i}_1,\boldsymbol{i}_2$, with the major axis parallel to $\boldsymbol{i}_2$ for ψ small and parallel to $\boldsymbol{i}_1$ for larger values, and that its sense of description is RH or LH according as $\psi \gtrless 0$.

The approximations that we have obtained are obviously invalid in the neighborhood of $\alpha = 0,\pi$. We do not pursue this case in the present paper, since the corresponding electron motions are then practically unaccelerated along the field lines.

III. THE EMISSION-POLARIZATION TENSOR AND STOKES PARAMETERS FOR A SINGLE ELECTRON

Although the complex polarization Q_n completely specifies the polarization characteristics of a monochromatic field from a single source, it is not suited to the specification of the polarization of the radiation resulting from the superposition of incoherent fields from different sources. In this case an appropriate formulation is in terms of a polarization tensor (Westfold 1959*b*) the components of which consist of linear combinations of the Stokes parameters. The addition of such tensors then corresponds to the superposition of different radiation fields.

It was shown in Paper I that if $\langle P_n(\boldsymbol{n})\rangle d\Omega(\boldsymbol{n})$ is the average power in the nth harmonic within the solid angle $d\Omega(\boldsymbol{n})$, as reckoned at a field point distant r from the source,

$$\langle P_n(\boldsymbol{n}) \rangle = \frac{2}{\mu c} |E_n|^2 r^2 .$$

This will consist of contributions from the components of $\boldsymbol{E}_n$ in any two orthogonal directions $\boldsymbol{e}_1,\boldsymbol{e}_2$ (which may be complex) transverse to $\boldsymbol{n}$. In particular, the contribution to $|E_n|^2$ from the component parallel to $\boldsymbol{e}_1$ is

$$|E_n^1|^2 = \boldsymbol{E}_n \cdot \boldsymbol{e}_1^* \boldsymbol{E}_n^* \cdot \boldsymbol{e}_1 = \boldsymbol{E}_n \boldsymbol{E}_n^* : \boldsymbol{e}_1 \boldsymbol{e}_1^* .$$

We are led to define the emission-polarization tensor for a single electron,

$$\langle \boldsymbol{P}_n(\boldsymbol{n}) \rangle = \frac{2}{\mu c} \boldsymbol{E}_n \boldsymbol{E}_n^* r^2 \tag{12}$$

and, on substitution from equation (10), we obtain its representation in the base $\boldsymbol{i}_1, \boldsymbol{i}_2$. Since $F_\gamma(\eta)$ is real,

$$\langle \boldsymbol{P}_n(\boldsymbol{n}) \rangle = \frac{\mu e^2 c \omega_B{}^2}{8\pi^4 \sin^4 \alpha} (1 + 4\psi \cot \alpha) \{ \gamma^2(\psi^2 + \eta^2 \psi \cot \alpha) F_\gamma{}^2(\eta) \boldsymbol{i}_1 \boldsymbol{i}_1 \\ + i\gamma(\psi + \tfrac{1}{2}\eta^2 \cot \alpha)[F_\gamma(\eta) F_\gamma{}'(\eta)/2\eta](\boldsymbol{i}_1 \boldsymbol{i}_2 - \boldsymbol{i}_2 \boldsymbol{i}_1) + [F_\gamma{}'^2(\eta)/4\eta^2] \boldsymbol{i}_2 \boldsymbol{i}_2 \} , \tag{13}$$

where, consistently with our approximation, we have retained only terms in $O(\xi^{-2})$ and $O(\xi^{-1})$ within the braces.

The components of the polarization tensor (13) can be expressed in terms of the Stokes parameters I, M, C, S associated with the directions $\boldsymbol{i}_1, \boldsymbol{i}_2$. These are (cf. Born and Wolf 1959)

$$\langle P_n(\boldsymbol{n}) \rangle_{11} = \tfrac{1}{2}(I_n + M_n) , \quad \langle P_n(\boldsymbol{n}) \rangle_{12} = \tfrac{1}{2}(C_n - iS_n) ,$$
$$\langle P_n(\boldsymbol{n}) \rangle_{21} = \tfrac{1}{2}(C_n + iS_n) , \quad \langle P_n(\boldsymbol{n}) \rangle_{22} = \tfrac{1}{2}(I_n - M_n) .$$

The principal-diagonal terms $\langle P_n(\boldsymbol{n}) \rangle_{11}$, $\langle P_n(\boldsymbol{n}) \rangle_{22}$ are the intensities $\langle P_n{}^{(1)}(\boldsymbol{n}) \rangle$, $\langle P_n{}^{(2)}(\boldsymbol{n}) \rangle$ of the $\boldsymbol{i}_1, \boldsymbol{i}_2$ components of the emission. Since in equation (13) $C_n = 0$, the axes of the polarization ellipse are along the directions of $\boldsymbol{i}_1, \boldsymbol{i}_2$, as has already been inferred. In this case we have the relation, similar to that of Roberts and Komesaroff (1965),

$$S_n = 2 \operatorname{sgn} \psi [\langle P_n{}^{(1)}(\boldsymbol{n}) \rangle \langle P_n{}^{(2)}(\boldsymbol{n}) \rangle]^{1/2} .$$

The products of F_γ, $F_\gamma{}'$ that occur in equation (13) can be expressed as single integrals, using the same transformations as in Paper I. The result not given there is

$$\frac{i\gamma}{2\eta} F_\gamma(\eta) F_\gamma{}'(\eta) = -\tfrac{1}{2}\sqrt{\pi}(2\gamma)^{3/2} e^{\pi i/4} \int_{-\infty}^{\infty} \exp[2i\gamma(\eta^2 x + \tfrac{1}{3}x^3)] x^{1/2} dx .$$

We are interested in summing contributions to the radiation in a fixed direction $\boldsymbol{n}$, at an angle $\theta = \alpha - \psi$ with $\boldsymbol{k}$, from a distribution of particles having different pitch angles α. It is therefore appropriate to express $\langle \boldsymbol{P}_n(\boldsymbol{n}) \rangle$ in terms of θ and ψ rather than α and ψ. Moreover, for large-order harmonics $n = O(\xi^{-3})$, the closely spaced line spectrum becomes quasi-continuous. If $\langle P_f(\boldsymbol{n}) \rangle \, df d\Omega(\boldsymbol{n})$ is the average power in the frequency band $(f, f + df)$, where $f = n\omega_B/2\pi a$, received within the solid angle $d\Omega(\boldsymbol{n})$, we have

$$\langle P_f(\boldsymbol{n}) \rangle = \langle P_n(\boldsymbol{n}) \rangle a / f_B , \quad f_B = \omega_B / 2\pi .$$

For the corresponding polarization tensor we then find, to the required order of approximation,

$$\langle \boldsymbol{P}_f(\boldsymbol{n}) \rangle = \frac{\mu e^2 c f_B}{4\pi^{3/2} \sin^2 \theta} \left(\frac{f}{f_B \sin \theta} \right)^{3/2} e^{\pi i/4} (1 - 2\psi \cot \theta) \\ \times \int_{-\infty}^{\infty} \exp\left\{ \frac{if}{f_B \sin \theta} [(\xi^2 + \psi^2)x + \tfrac{1}{3}x^3] \right\} \left\{ 1 - \frac{if}{f_B \sin \theta} [(\xi^2 + \psi^2)x \\ + \tfrac{2}{3}x^3] \psi \cot \theta \right\} \left\{ [\psi^2 + (\xi^2 + \psi^2)\psi \cot \theta] x^{-1/2} \boldsymbol{i}_1 \boldsymbol{i}_1 \\ - [\psi + \tfrac{1}{2}(\xi^2 + \psi^2) \cot \theta] x^{1/2} (\boldsymbol{i}_1 \boldsymbol{i}_2 - \boldsymbol{i}_2 \boldsymbol{i}_1) \\ - \left[x^{3/2} + \frac{f_B \sin \theta}{2if} (1 + 2\psi \cot \theta) x^{-3/2} \right] \boldsymbol{i}_2 \boldsymbol{i}_2 \right\} dx , \tag{14}$$

in which the integrals are to be interpreted as Cauchy principal values in respect to the singularities at $x = 0$.

An inspection of this result indicates that among the next-order terms in the fourth Stokes parameter are some that are even in ψ, so that proviso *a* of §I is satisfied as well as proviso *b*. We proceed to calculate the resultant effect on the emission from a distribution of electrons.

IV. THE EMISSIVITY-POLARIZATION TENSOR FOR A DISTRIBUTION OF ELECTRONS

As in Paper I, we here consider a distribution of gyrating electrons such that $N(\mathfrak{E}/\mathfrak{E}_0)d\mathfrak{E}/\mathfrak{E}_0$ is the number density of those whose energies lie in the range $(\mathfrak{E},\mathfrak{E} + d\mathfrak{E})$, where $\mathfrak{E}_0 = mc^2$ is the rest energy. We also take account of a possible non-isotropic velocity distribution having axial symmetry about the direction of $\boldsymbol{B}_0$ by assuming that the proportion of electrons having pitch angles within the range $(\alpha,\alpha + d\alpha)$ in the solid angle $d\Omega(\boldsymbol{\tau}) = 2\pi \sin \alpha \, d\alpha$ is $\phi(\alpha)d\Omega(\boldsymbol{\tau})$.

It has recently been pointed out by Scheuer (1968) and elucidated in greater detail by Ginzburg *et al.* (1968), that in calculating the emission from a fixed volume element dV in space we must allow for the circumstance that an electron source (as specified at the beginning of § II) that remains within dV for a time interval dt' emits radiation which is received at a field point over an interval adt'. It follows that the average power within the band $(f,f + df)$ *emitted* by an electron source within dV into the solid angle $d\Omega(\boldsymbol{n})$ is $a\langle P_f(\boldsymbol{n})\rangle df d\Omega(\boldsymbol{n})$. The quantity $a\langle P_f(\boldsymbol{n})\rangle$ is the same as $\langle P_f(\boldsymbol{n})\rangle$ in Paper I, validating the results of the emissivity calculations in that paper.

The emissivity-polarization tensor, which specifies the average power radiated per unit volume and per unit frequency band width and solid angle, is then

$$\mathfrak{n}_f(\boldsymbol{n}) = 2\pi \int_0^\infty N\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right) \int_0^\pi \phi(\alpha) \, (\sin \alpha) \, a\langle \boldsymbol{P}_f(\boldsymbol{n}) \rangle d\alpha d\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right) . \tag{15}$$

The first integration proceeds by writing $\alpha = \theta + \psi$, replacing $\phi(\alpha) \sin \alpha$ by its second approximation $\phi(\theta)(\sin \theta)[1 + g(\theta)\psi \cot \theta]$, where

$$g(\theta) = 1 + \frac{\phi'(\theta)}{\phi(\theta)} \tan \theta , \tag{16}$$

and integrating with respect to ψ over its range, which is conveniently taken as between $+\infty$ and $-\infty$. Then the terms that are odd in ψ do not survive the integration, while those that are even are readily integrated to give

$$\begin{aligned}\mathfrak{n}_f(\boldsymbol{n}) = -\tfrac{1}{4}\mu e^2 c\phi(\theta) \sin \theta \int_0^\infty f_B N\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right) \int_{-\infty}^\infty \exp \left[\frac{if}{f_B \sin \theta} (\xi^2 x + \tfrac{1}{3}x^3)\right] \\ \times \left[\!\left[x^{-2}\boldsymbol{i}_1\boldsymbol{i}_1 + \left(\frac{2if}{f_B \sin \theta} x + x^{-2}\right)\boldsymbol{i}_2\boldsymbol{i}_2 \right.\right. \\ \left.\left. + \left\{\frac{2if}{f_B \sin \theta} (\xi^2 + \tfrac{1}{3}x^2) - [1 + g(\theta)]x^{-1}\right\} (\cot \theta)(\boldsymbol{i}_1\boldsymbol{i}_2 - \boldsymbol{i}_2\boldsymbol{i}_1) \right]\!\right] dx d\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right) ,\end{aligned} \tag{17}$$

in which it will be recalled that the gyrofrequency f_B is inversely proportional to the electron energy,

$$f_B = f_{B_0}\xi = f_{B_0}\frac{\mathfrak{E}_0}{\mathfrak{E}} , \quad f_{B_0} = \frac{eB_0}{2\pi m} . \tag{18}$$

In this result the principal-diagonal terms consist only of the contributions from the first approximation while, as expected, the off-diagonal terms consist only of contributions from the second approximation. We note that, for an isotropic velocity distribution, $g(\theta) = 1$.

The integration with respect to x involves evaluation of integrals of the type

$$F_{\gamma,s}(\xi) = \int_{-\infty}^{\infty} \exp\left[i\gamma(\xi^2 x + \tfrac{1}{3}x^3)\right] x^s dx \, , \tag{19}$$

where $s = -2, -1, 0, 1, 2$. These can be expressed in terms of modified Bessel functions as follows:

$$F_{\gamma,-2}(\xi) = \frac{\sqrt{3}}{\xi}\left[F_p(\tfrac{2}{3}\gamma\xi^3) - F(\tfrac{2}{3}\gamma\xi^3)\right] , \tag{20}$$

$$F_{\gamma,-1}(\xi) = -\frac{i\sqrt{3}}{\gamma\xi^3}\left[2F_p(\tfrac{2}{3}\gamma\xi^3) - F(\tfrac{2}{3}\gamma\xi^3)\right] , \tag{21}$$

$$F_{\gamma,0}(\xi) = F_\gamma(\xi) = \frac{\sqrt{3}}{\gamma\xi^2} F_s(\tfrac{2}{3}\gamma\xi^3) \, , \tag{22}$$

$$F_{\gamma\, 1}(\xi) = \frac{1}{2i\gamma\xi} F_\gamma'(\xi) = \frac{i\sqrt{3}}{\gamma\xi} F_p(\tfrac{2}{3}\gamma\xi^3) \, , \tag{23}$$

$$F_{\gamma,2}(\xi) = -\frac{\sqrt{3}}{\gamma} F_s(\tfrac{2}{3}\gamma\xi^3) \, , \tag{24}$$

where

$$F(x) = x\int_x^{\infty} K_{5/3}(\eta)d\eta \, , \quad F_p(x) = xK_{2/3}(x) \, , \quad F_s(x) = xK_{1/3}(x) \, . \tag{25}$$

The functions $F(x)$ and $F_p(x)$ are such that $F_p(x) < F(x) < 2F_p(x)$ for $x > 0$. They are tabulated together with $F_s(x)$ in Table 1, which incorporates a revision of Table 1 of Paper I. Asymptotic series for $F(x)$, $F_p(x)$ are given in equations (28′) and (29′) of Paper I; the series for $F_s(x)$, which is equal to $x^{2/3}G_p(x)$, can be obtained directly from equation (50′) of paper I.

In terms of these functions we get

$$\begin{aligned} \mathbf{n}_f(\mathbf{n}) = {} & \frac{\sqrt{3}}{4}\mu e^2 c\phi(\theta) f_{B_0} \sin\theta \int_0^{\infty} N\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right) \\ & \times \left[\!\left[\left[F\left(\frac{f}{f_c}\right) - F_p\left(\frac{f}{f_c}\right)\right]\mathbf{i}_1\mathbf{i}_1 + \left[F\left(\frac{f}{f_c}\right) + F_p\left(\frac{f}{f_c}\right)\right]\mathbf{i}_2\mathbf{i}_2 \right.\right. \\ & - \tfrac{4}{3}i\left(\frac{2f}{3f_{B_0}\sin\theta}\right)^{-1/2} \cot\theta \left\{\left(\frac{f}{f_c}\right)^{1/2} F_s\left(\frac{f}{f_c}\right) + [1 + g(\theta)]\left(\frac{f}{f_c}\right)^{-1/2} \right. \\ & \left.\left.\left. \times \left[F_p\left(\frac{f}{f_c}\right) - \tfrac{1}{2}F\left(\frac{f}{f_c}\right)\right]\right\}(\mathbf{i}_1\mathbf{i}_2 - \mathbf{i}_2\mathbf{i}_1) \right]\!\right] d\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right) , \end{aligned} \tag{26}$$

where

$$f_c = \frac{3}{2}\frac{f_{B_0}\sin\theta}{\xi^2} = \tfrac{3}{2}f_{B_0}\sin\theta\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right)^2 , \tag{27}$$

the critical frequency.

As for $\langle \boldsymbol{P}_n(\boldsymbol{n}) \rangle$ in equation (13), the Stokes parameters corresponding to the emissivity tensor $\mathbf{n}_f(\boldsymbol{n})$ and the base $\boldsymbol{i}_1, \boldsymbol{i}_2$ are, in order,

$$\eta_f = \eta_{f11} + \eta_{f22}\,, \qquad \eta_f^{(p)} \cos 2\lambda_f \cos 2\psi_f = \eta_{f11} - \eta_{f22}\,,$$
$$\eta_f^{(p)} \cos 2\lambda_f \sin 2\psi_f = \eta_{f12} + \eta_{f21}\,, \qquad \eta_f^{(p)} \sin 2\lambda_f = i(\eta_{f12} - \eta_{f21})\,, \tag{28}$$

where[4] (Westfold 1959*b*) $\eta_f^{(p)}$ is the polarized part of the total emissivity η_f, ψ_f is the angle made by the major axis of the polarization ellipse with the direction $\boldsymbol{i}_1$, $|\tan \lambda_f|$ is the ratio of the principal axes ($|\lambda_f| \leq \frac{1}{4}\pi$), and the sense of description of the ellipse is RH for $\lambda_f > 0$ and LH for $\lambda_f < 0$. The corresponding degree of polarization,

$$p_f = \frac{\eta_f^{(p)}}{\eta_f}\,, \tag{29}$$

can be derived from the four Stokes parameters. The simple ratio of the fourth to the first parameter, $p_f \sin 2\lambda_f$, is the so-called degree of circular polarization.

TABLE 1

$$F(x) = x\int_x^\infty K_{5/3}(\eta)d\eta\,, \quad F_p(x) = xK_{2/3}(x)\,, \quad F_s(x) = xK_{1/3}(x)\,.$$

x	$F(x)$	$F_p(x)$	$F_s(x)$	x	$F(x)$	$F_p(x)$	$F_s(x)$
0 ...	0	0	0	1 00	0 6514	0 4945	0 4384
0 001..	0 2131	0 1075	0 0167	1 20	.5653	.4394	3959
005	3585	.1836	0480	1 40	4867	.3859	3519
010	4450	2310	0749	1 60	.4167	.3359	.3092
025	.5832	.3117	.1325	1 80	.3552	.2904	.2694
.050	.7016	3881	.1996	2 00	3016	.2497	.2331
.075	.7714	.4383	.2497	2 50	.1981	.1681	.1589
100	.8182	4753	.2900	3 00	.1286	.1112	1059
.150	8747	.5269	.3514	3 50	.0827	07257	06957
.200	9034	.5604	.3959	4 00	05282	04692	.04520
.250	9160	5822	4286	4 50	03357	03012	02912
.300	.9177	.5960	4527	5 00	02124	01922	01864
400	9019	6069	4823	6 00	00842	00772	00753
500	8708	6030	4945	7 00	00331	00306	00300
600	8315	5897	4951	8 00	00129	00120	00118
700	7879	5703	4876	9 00	000498	000469	.00461
800	7424	5471	4745	10 00	0 000192	0 000182	0 00179
0 900	0 6966	0 5214	0 4577				

Since, in equation (26), $\eta_{f12} = -\eta_{f21}$, the third Stokes parameter is zero; moreover, since $\eta_{f22} > \eta_{f11}$, we immediately have $\psi_f = \frac{1}{2}\pi$, as for the first approximation, i.e., the major axis of the ellipse remains perpendicular to the projection of $\boldsymbol{B}_0$ transverse to $\boldsymbol{n}$; further, since $i\eta_{f12}/\cot\theta > 0$, the sense of description of the ellipse is RH or LH according as θ is in the first or second quadrant. In accordance with our expectation, the degree of circular polarization is determined by the second approximation.

Case (*i*): *monoenergetic electrons*.—If all the electrons have the same energy $\mathfrak{E}_1$, we have

$$N\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right) = \mathfrak{N}\delta\left(\frac{\mathfrak{E}}{\mathfrak{E}_0} - \frac{\mathfrak{E}_1}{\mathfrak{E}_0}\right), \tag{30}$$

where $\mathfrak{N}$ is their number density. On substitution in equation (26), we get for the first, second, and fourth Stokes parameters, which determine the emissivity η_f in the direction

[4] Here ψ_f should not be confused with the angle ψ previously defined; λ_f should not be confused with wavelength.

$\boldsymbol{n}$ and the quantity tan λ_f, which itself determines the ratio of the principal axes of the polarization ellipse and its sense of description,

$$
\begin{aligned}
\eta_f &= \frac{\sqrt{3}}{2}\,\mathfrak{N}\mu e^2 c\phi(\theta) f_{B_0}\,(\sin\theta) F\left(\frac{f}{f_{c1}}\right), \\
\eta_f^{(p)}\cos 2\lambda_f &= \frac{\sqrt{3}}{2}\,\mathfrak{N}\mu e^2 c\phi(\theta) f_{B_0}\,(\sin\theta) F_p\left(\frac{f}{f_{c1}}\right), \\
\eta_f^{(p)}\sin 2\lambda_f &= \sqrt{2}\,\mathfrak{N}\mu e^2 c\phi(\theta)\cot\theta\,(f_{B_0}\sin\theta)^{3/2} f^{-1/2}\left\{\left(\frac{f}{f_{c1}}\right)^{1/2} F_s\left(\frac{f}{f_{c1}}\right)\right. \\
&\quad \left. + \left[2 + \frac{\phi'(\theta)}{\phi(\theta)}\tan\theta\right]\left(\frac{f}{f_{c1}}\right)^{-1/2}\left[F_p\left(\frac{f}{f_{c1}}\right) - \tfrac{1}{2}F\left(\frac{f}{f_{c1}}\right)\right]\right\},
\end{aligned}
\tag{31}
$$

where f_{c1} is the critical frequency corresponding to the electron energy $\mathfrak{E}_1$. The functions of f/f_{c1} that occur in this expression for the fourth Stokes parameter are tabulated in Table 2, together with the sum contained in the braces for the isotropic case $\phi'(\theta) = 0$.

TABLE 2

(*a*) $x^{1/2}F_s(x)$, (*b*) $x^{-1/2}[F_p(x) - \frac{1}{2}F(x)]$, (*c*) $x^{1/2}F_s(x) + 2x^{-1/2}[F_p(x) - \frac{1}{2}F(x)]$

x	(*a*)	(*b*)	(*c*)	x	(*a*)	(*b*)	(*c*)
0 ..	0	0	0	1 00	0 4384	0 1688	0 7760
0 001	0 000529	0 02828	0 05709	1 20	.4336	.1431	7199
005	.003392	06155	1265	1 40	4163	1204	6572
010	007486	.08495	.1774	1 60	3911	.1008	5927
.025	02096	1270	2749	1 80	3614	08407	5295
050	04462	.1669	3783	2 00	3296	06990	4694
.075	06839	1919	4522	2 50	2512	04368	3385
.100	09170	2094	5104	3 00	1835	02707	2376
.150	1361	2312	5986	3 50	.1302	01669	.1635
.200	.1770	2430	6630	4 00	09040	01025	.1109
.250	2143	.2487	.7116	4 50	.06178	.006285	.07435
.300	2480	.2504	.7487	5 00	.04169	003845	.04938
.400	.3050	.2465	.7980	6 00	.01844	.001434	.02131
.500	.3497	.2370	8236	7 00	.007926	.000533	.008991
600	.3835	2246	8326	8 00	.003336	.000198	.003731
.700	4079	.2109	8296	9 00	001382	000073	001528
800	4244	1967	8177	10 00	0 000565	0 000027	0 000619
0 900	0 4342	0 1825	0 7993				

Apart from the anisotropy factor $2 + (\tan\theta)\phi'(\theta)/\phi(\theta)$, it follows from equations (31) that for a given energy $\mathfrak{E}_1$ the degree of circular polarization $p_f \sin 2\lambda_f$ depends on the magnitude of the magnetic field $\boldsymbol{B}_0$ and the angle θ between its direction and the direction of observation $\boldsymbol{n}$, according to the factor $[(f_{B_0}\sin\theta)/f]^{1/2}\cot\theta$. The singular behavior of this factor in the directions $\theta = 0, \pi$ along the field need not concern us, since the approximations that have been made in obtaining these formulae depend on the assumption that cot α and cot θ are $O(1)$. Moreover, our results give $\eta_f = 0$ in these directions, along which the electrons are unaccelerated. Both $p_f \cos 2\lambda_f$ and the residual $p_f \sin 2\lambda_f/[(f_{B_0} \sin \theta/f)^{1/2}\cot\theta]$ with $\phi'(\theta) = 0$ are represented in Figure 2 as functions of f/f_{c1}. Their values at $f/f_{c1} = 0$ are $\frac{1}{2}$ and 0, respectively, and for large arguments they approach 1 and 1.63 $(f/f_{c1})^{1/2}$.

For an electron in a field $\boldsymbol{B}_0$ we have $f_{B_0} = 2.80B_0$ Mc/s when B_0 is in gauss. In the case of galactic objects we may take the order of magnitude as $B_0 = 10^{-4}$ gauss, so that

$p_f \sin 2\lambda_f$ is clearly negligible for synchrotron emission; then the polarization remains linear in the direction of $\boldsymbol{i}_2$, perpendicular to the projection of $\boldsymbol{B}_0$. For the Jovian atmosphere, however, the appropriate order of magnitude is 10 gauss, which for $\cot\theta = 1$ and $f = 1000$ Mc/s gives $[(f_{B_0} \sin\theta)/f]^{1/2} \cot\theta = 0.14$. In this case the measurement of $\eta_f^{(p)} \sin 2\lambda_f$ becomes feasible.

Case (ii): Power-law energy spectrum.—The differential energy distribution of electrons emitting synchrotron radiation is usually found to correspond to a power law of the form[5]

$$N\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right) = A\left(\frac{\mathfrak{E}}{\mathfrak{E}_0}\right)^{-\gamma}, \quad \mathfrak{E}_1 < \mathfrak{E} < \mathfrak{E}_2 , \tag{32}$$

with $\gamma > 0$ and the energies $\mathfrak{E}$ lying well within the cutoff values $\mathfrak{E}_1$, $\mathfrak{E}_2$, so that these specific quantities do not appear in any derived formula. As in Paper I, we transform the variable of integration in equation (26) to $x = f/f_c$ and substitute from equation (32). Corresponding to the energy range $(\mathfrak{E}_1, \mathfrak{E}_2)$, we have the range $(f/f_{c2}, f/f_{c1})$ of x, so that if the emissivity tensor is to be sensibly independent of these values it is necessary to

[5] The index γ is not to be confused with the quantity defined in eq. (8).

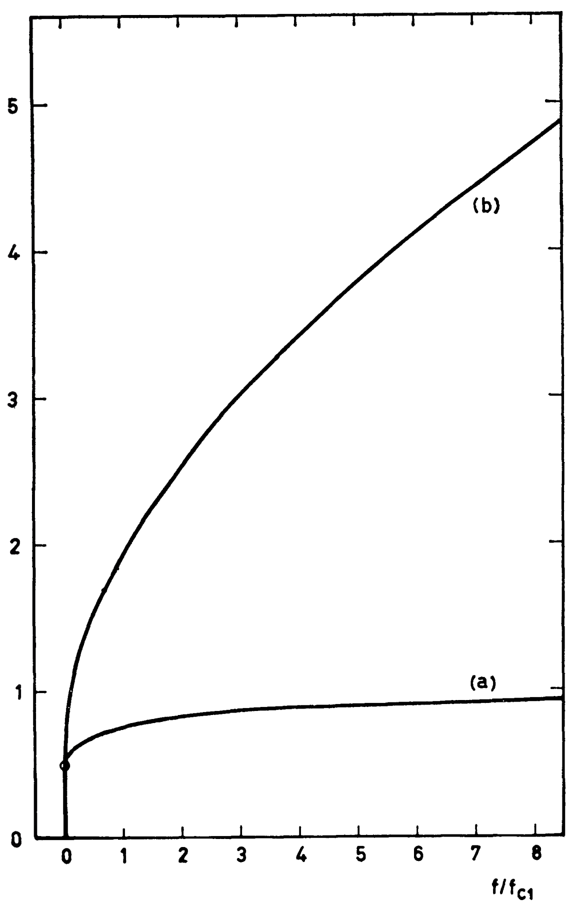

FIG 2

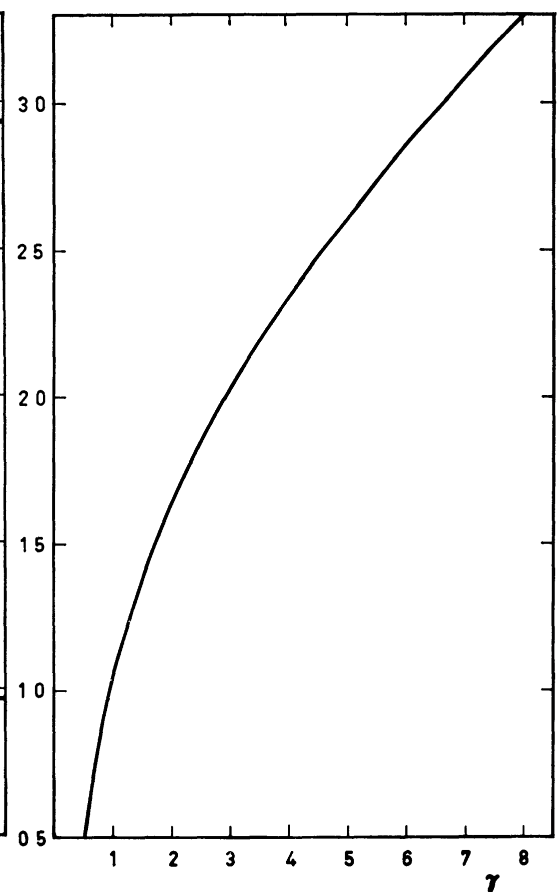

FIG 3

FIG. 2.—Parameters that determine the degree of polarization of the emissivity for an isotropic velocity distribution of monoenergetic electrons: (*a*) $p_f \cos 2\lambda_f$; (*b*) $p_f \sin 2\lambda_f/[(f_{B_0} \sin\theta)/f]^{1/2} \cot\theta$.

FIG. 3.—Parameter $p_f \sin 2\lambda_f/[(f_{B_0} \sin\theta)/f]^{1/2} \cot\theta$ for electrons having an isotropic velocity distribution with energies distributed according to a power law.

have $f_{c1} \ll f \ll f_{c2}$. Then the terminals of integration may be taken as zero and infinity, and we have

$$\begin{aligned}
\eta_f &= \frac{A\mu e^2 c}{2\sqrt{2}} (\tfrac{3}{2})^{\gamma/2}\phi(\theta)(f_{B_0} \sin\theta)^{(\gamma+1)/2} f^{-(\gamma-1)/2} \mathfrak{J}_{(\gamma+1)/2} \\
\eta_f^{(p)} \cos 2\lambda_f &= \frac{A\mu e^2 c}{2\sqrt{2}} (\tfrac{3}{2})^{\gamma/2}\phi(\theta)(f_{B_0} \sin\theta)^{(\gamma+1)/2} f^{-(\gamma-1)/2} \mathfrak{L}_{(\gamma+1)/2} , \\
\eta_f^{(p)} \sin 2\lambda_f &= \frac{A\mu e^2 c}{\sqrt{3}} (\tfrac{3}{2})^{\gamma/2}\phi(\theta) \cot\theta (f_{B_0} \sin\theta)^{\gamma/2+1} f^{-\gamma/2} \\
&\quad \times \{\mathfrak{K}_{\gamma/2+1} + [1 + g(\theta)](\mathfrak{L}_{\gamma/2} - \tfrac{1}{2}\mathfrak{J}_{\gamma/2})\} ,
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
\mathfrak{J}_n &= \int_0^\infty x^{n-2} F(x) dx = \frac{\frac{2}{3} + n}{n} \mathfrak{L}_n , && n > \tfrac{2}{3} , \\
\mathfrak{L}_n &= \int_0^\infty x^{n-2} F_p(x) dx = 2^{n-2}\Gamma(\tfrac{1}{2}n - \tfrac{1}{3})\Gamma(\tfrac{1}{2}n + \tfrac{1}{3}) , && n > \tfrac{2}{3} , \\
\mathfrak{K}_n &= \int_0^\infty x^{n-2} F_s(x) dx = 2^{n-2}\Gamma(\tfrac{1}{2}n - \tfrac{1}{6})\Gamma(\tfrac{1}{2}n + \tfrac{1}{6}) , && n > \tfrac{1}{3} .
\end{aligned} \tag{34}$$

The evaluations have been carried out using equation (35) and the equation immediately preceding equation (48) of Paper I. It will be noted that the quantities $\mathfrak{J}_{(\gamma+1)/2}$, $\mathfrak{L}_{(\gamma+1)/2}$ are, respectively, the same as the quantities $G(0), G_p(0)$. On substitution from equations (34) in equations (32), we finally obtain, for $\gamma > \frac{1}{3}$,

$$\begin{aligned}
\eta_f &= \tfrac{1}{8} A\mu e^2 c 3^{\gamma/2} \Gamma\left(\frac{3\gamma - 1}{12}\right)\Gamma\left(\frac{3\gamma + 7}{12}\right) \frac{\gamma + 7/3}{\gamma + 1} \phi(\theta)(f_{B_0} \sin\theta)^{(\gamma+1)/2} f^{-(\gamma-1)/2} , \\
\eta_f^{(p)} \cos 2\lambda_f &= \frac{\gamma + 1}{\gamma + 7/3} \eta_f , \\
\eta_f^{(p)} \sin 2\lambda_f &= \tfrac{1}{2} A\mu e^2 c 3^{(\gamma-1)/2} \Gamma\left(\frac{3\gamma + 4}{12}\right)\Gamma\left(\frac{3\gamma + 8}{12}\right) \frac{\gamma + 2 + (\tan\theta)\phi'(\theta)/\phi(\theta)}{\gamma} \\
&\quad \times \phi(\theta) \cot\theta (f_{B_0} \sin\theta)^{\gamma/2+1} f^{-\gamma/2} ,
\end{aligned} \tag{35}$$

the first, second, and fourth Stokes parameters. In view of the range of values of the spectral index γ that have been assigned to various radio sources we provide in Table 3 values of products involving gamma functions that are necessary for the evaluation of η_f and $\eta_f^{(p)} \sin 2\lambda_f$, for values of γ in the range 0.4–9.0. We also have

$$p_f \cos 2\lambda_f = \frac{\gamma + 1}{\gamma + 7/3} ,$$

which is again effectively the degree of polarization p_f when $[(f_{B_0} \sin\theta)/f]^{1/2} \cot\theta$ is so small that the polarization is linear. The degree of circular polarization is determined by this factor and the quantity $p_f \sin 2\lambda_f/[(f_{B_0} \sin\theta(/f)]^{1/2} \cot\theta$, which is represented in Figure 3 as a function of γ for the isotropic case $\phi'(\theta) = 0$.

Over the range of ϕ here depicted, $p_f \sin 2\lambda_f$ is of the same order of magnitude as in the case of monoenergetic electrons. Again, it appears feasible to measure the quantity $\eta_f^{(p)} \sin 2\lambda_f$ for the planet Jupiter.

V. DISCUSSION

Since previous attempts to measure the degree of circular polarization have been interpreted in terms of the calculations of Roberts and Komesaroff (1965) for monoenergetic electrons, it is important to see how their results differ from those of our case (i). In fact, as was mentioned in § I, Roberts and Komesaroff took account of only proviso

TABLE 3

$$(a)\ \Gamma\left(\frac{3\gamma-1}{12}\right)\Gamma\left(\frac{3\gamma+7}{12}\right)\frac{\gamma+7/3}{\gamma+1}\,,$$

$$(b)\ \Gamma\left(\frac{3\gamma+4}{12}\right)\Gamma\left(\frac{3\gamma+8}{12}\right)\frac{\gamma+2}{\gamma}\,.$$

γ	(a)	(b)	γ	(a)	(b)
0 4	153.8	14 77	4 8	1 077	1 196
0 6	33 15	9 102	5 0	1 067	1 208
0 8	16 68	6 392	5 2	1 062	1 226
1 0	10 47	4 840	5 4	1 062	1.250
1 2	7 333	3 857	5 6	1 068	1 279
1 4	5 496	3 189	5 8	1 080	1 316
1 6	4 318	2 712	6 0	1 096	1.358
1 8	3 516	2 360	6 2	1 117	1 407
2 0	2 945	2 095	6 4.	1 143	1 462
2 2	2 524	1 889	6 6	1 174	1 525
2 4	2 205	1 728	6 8	1 211	1 596
2 6	1 959	1 600	7 0	1 254	1 676
2 8	1 766	1 497	7 2	1 303	1 766
3 0	1 612	1 416	7 4	1 358	1 865
3 2	1 489	1.351	7 6	1 420	1 977
3 4	1 390	1 300	7.8	1 490	2 101
3 6	1 309	1 260	8 0	1 569	2 224
3 8	1 244	1 230	8 2	1 656	2 393
4 0.	1.192	1 210	8 4	1 753	2.563
4 2	1 151	1 196	8 6	1 861	2 754
4 4	1 119	1.190	8 8	1 982	2.964
4 6	1 094	1.189	9 0	2 115	3.200

b, which is represented in our calculations by the terms having as a factor $g(\theta)$ given by equation (16), or, in terms of the notation of Roberts and Komesaroff, who define $N(\alpha)d\alpha$ as the number of electrons with pitch angles in the range $(\alpha,\alpha + d\alpha)$, $N'(\theta)/N(\theta) = g(\theta) \cot \theta$.

The contribution in question to the degree of circular polarization $p_f \sin 2\lambda_f$, given by equations (29) and (31), may then be written

$$\tfrac{2}{3}\sqrt{6}\,\frac{N'(\theta)}{N(\theta)}\left(\frac{f_{B_0}\sin\theta}{f}\right)^{1/2}\left(\frac{f}{f_{c1}}\right)^{-1/2}\left[F_p\left(\frac{f}{f_{c1}}\right)-\tfrac{1}{2}F\left(\frac{f}{f_{c1}}\right)\right]\Big/ F\left(\frac{f}{f_{c1}}\right).$$

For an isotropic velocity distribution the relative magnitudes of this contribution and the full expression is exhibited in columns b and c of Table 2 for a range of values of the argument $x = f/f_{c1}$. For the case $f/f_{c1} = \frac{1}{3}$ taken by Roberts and Komesaroff, the above contribution becomes $0.406\,[(N'(\theta)/N(\theta)][(f_{B_0}\sin\theta)/f]^{1/2}$, which is close to their result, for which the numerical factor was 0.43. For this value of f/f_{c1} the full isotropic expression gives

$$p_f \sin 2\lambda_f = 1.37[(f_{B_0}\sin\theta)/f]^{1/2}\cot\theta\ .$$

The additional factor of about 3 will have the effect of reducing estimates of the magnetic field made in terms of the calculations of Roberts and Komesaroff by a factor of about 9.

However, apart from numerical differences, the general point remains true: that measurements of the degree of circular polarization will provide estimates of the magnitude of magnetic field in the region of origin of synchrotron radiation, through the term $f_{B_0}\sin\theta$, which is proportional to the component of the magnetic field transverse to the line of sight.

In actual astronomical situations the power-law distribution of case (ii) is more relevant. However, from Figure 3 it can be seen that for γ ranging from 1 to 3, the factor replacing 1.37 in the last formula ranges only from about 1 to 2. Observations of the intensity of the radiation, from a synchrotron source over a range of frequencies, that is proportional to $f^{(\gamma-1)/2}$, provide an estimate of the index γ of the energy spectrum. Then, for any particular frequency, an observation of the degree of circular polarization will provide, through equation (15), an estimate of $B_0 \sin\theta$ as an average over the source region. Further, when the degree of circular polarization is measured over a range of frequencies, it may be expected to vary as $f^{-1/2}$; it if does not, the hypothesis that the source of the radiation is synchrotron emission must be questioned. In this connection it is worth emphasizing that both the intensity of the total radiation and the degree of circular polarization are independent of any Faraday depolarization that might be suffered by the radiation during its outward passage.

Attempts to measure the degree of circular polarization of radio sources outside the solar system have, on the whole, been no more successful than in the case of Jupiter. Mayer, Hollinger, and Allen (1963) have assigned upper limits to this quantity in the case of a number of sources observed at a wavelength of 3.15 cm. Of those thought to emit by the synchrotron process, they estimate for the extragalactic sources Cygnus A, Centaurus A, and Virgo A upper limits of 2, 5, and 5 per cent, and for both the galactic supernova remnants Taurus A (the Crab Nebula) and Cassiopeia A an upper limit of 1 per cent. A limit of 5 per cent had previously been placed by Ryle and Smith (1948) on the 3.75-m radiation from Cygnus A and Cassiopeia A and a limit of 4 per cent by Hanbury Brown, Palmer, and Thompson (1955) on the 1.9-m radiation from Cygnus A, Cassiopeia A, and Taurus A. For order-of-magnitude estimates we may take a value 2.6 for γ, whence, for $\cot\theta = 1$ and B_0 and f expressed in gauss and megacycles per second, Figure 3 gives a value of about 2.7 for the quantity $p_f(\sin 2\lambda_f)(B_0/f)^{-1/2}$. For the figure of 5 per cent at 100 Mc/s the upper limit on B_0 is about 3.4×10^{-2} gauss, and for 1 per cent it is reduced to 1.4×10^{-3} gauss. Since fields of such magnitude are somewhat greater than those of about 10^{-4} gauss, which are usually attributed to these objects, little significance can be attributed to the limits on the degree of circular polarization provided by these measurements. These conclusions remain unaltered by the more recent, but still unsuccessful, attempts to detect circular polarization at 10.6 m, reported by Seielstad (1967).

The observational situation with respect to the Crab Nebula has been changed by recent measurements reported by Andrew, Purton, and Terzian (1967). They find that at 38 Mc/s the degree of circular polarization (RH) for the whole source is 0.4 ± 0.5 per cent. They attribute this to the small, intense, compact source of low-frequency radiation within the nebula, which is estimated to provide about 20 per cent of the total radiation, corresponding to a degree of circular polarization of 2.0 ± 2.5 per cent. Although it appears that the characteristics of the radiation from the small source are not consistent with synchrotron emission, it is of interest to note that the over-all figure obtained for the whole nebula is consistent with our calculations and a magnetic field of the accepted magnitude of 10^{-4} gauss. For, taking the appropriate value $\gamma = 1.7$, we find, as before for $\cot \theta = 1$, that $p_f (\sin 2\lambda_f)(B_0/f)^{-1/2} = 2.1$ and that substitution of the values 10^{-4} for B_0 and 40 for f gives $p_f \sin 2\lambda_f = 0.3$ per cent. Thus it is possible that the difference between the measurements of the circularly polarized components is a characteristic of synchrotron emission from the nebula, exclusive of any contribution from the small source.

It might be inferred from the marginal character of the radio measurements that optical observations of the same sources would be unable to provide measurable figures for the degree of circular polarization. However, recent optical observations of the Crab Nebula have not confirmed such an inference. First, Oetken (1965, 1966*a*) reported a positive result in an attempt to measure elliptic polarization in the continuous spectrum at a wavelength of 5700 Å. In at least one area it appeared that the ellipticity $|\tan \lambda_f|$ of the polarization ellipse was about 0.15. Then Dzhakusheva and Mychelkin (1966) gave details of their measurements in the band 5300–6300 Å. They found[6] that, on the average, across the nebula the ellipticity of the polarization ellipse was of the order 0.2–0.3, reaching values beyond 0.4 in the peripheral regions of the nebula. They estimated the maximum and minimum values of the degree of polarization p_f as 25 and 19 per cent. Taking the smallest value $|\tan \lambda_f| = 0.2$, we get 9.6 and 7.3 per cent for the maximum and minimum values of the degree of circular polarization $p_f \sin 2\lambda_f$.

According to our previous figures, it would appear that, if the emission were due to the synchrotron process and if the energy distribution were of the form of equations (32), the magnetic field in the Crab Nebula would be of the order of 10^6 instead of 10^{-4} gauss. Such a figure is hardly credible; indeed, in a subsequent communication Oetken (1966*b*) reported that in repeating her measurements she had been unable to find positive evidence of any circular polarization in the nebula.

It is possible, however, that the predicted degree of circular polarization will be modified to some extent if the energy spectrum (32) is not maintained. In fact, a discussion by Kardashev (1962) indicates a "break" in the emission spectrum (35) within the optical range, which may be attributable to the more rapid energy losses from this comparatively young source. We propose to examine the effect of this on the Stokes parameters and the consequent inferences as to the magnitude of the magnetic field at a future time.

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[6] In interpreting these measurements as indicating values for the degree of circular polarization of close to 100 per cent, Wolstencroft (1966) appears not to have taken account of the fact that the radiation from a distribution of electrons is only partially polarized; the conclusions he draws apply to the polarized part of this radiation. His own optical measurements suggest an upper limit of less than 1 per cent on the degree of circular polarization

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