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ON THE ORIGIN OF COMMENSURABILITIES IN THE SOLAR SYSTEM—II

The Orbital Period Relation

S. F. Dermott

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Summary

A systematic search for regularity in the major satellite systems has revealed that the orbital periods of the regular satellites are closely approximated by the relation, $T_n = T_0A^n$ where T_n is the orbital period of the nth satellite. It must be allowed though, that in any one system there are a small number of vacancies. For all systems A is the square root of a small integer and it is suggested that T_0 is related to the rotational period of the primary. The relation can be applied to the planetary system but there are some anomalies. It is suggested that this regularity, which is related to the preference for near-commensurability among pairs of mean motions in the solar system, is a condition of formation rather than the result of evolution and thus could be of considerable cosmogonic importance.

1. *Introduction.* Recent work by Roy & Ovenden (1), Goldreich (2), and the present author (3) indicates that in any discussion on the regularity of the solar system central importance should be given to the near-commensurabilities (—pairs of planets or satellites, the ratios of the mean motions, or orbital periods, of which can be closely approximated by ratios of small integers). There are many examples of near-commensurability in the solar system, the best-known being those found among the Galilean satellites of Jupiter. It has been shown (1)–(3) that the occurrence of near-commensurability among pairs of mean motions is in fact more frequent than in a chance distribution. From a different point of view, it has long been recognized that the distribution of planets and satellites is non-random. Laws of the form

$$R_n = R_0C^n, \tag{1}$$

i.e. 'Bode-type' laws, have been formulated for the planetary system and also for the various satellite systems. R_n is the semi-major axis of the orbit of the nth secondary and R_0 and C are constants, different for each system, these being found empirically. For example, Ter Haar & Cameron (4) consider that the planetary system is best represented by the relation

$$R_n = R_0(1\cdot 89)^n.$$

If the actual orbital radii are given by

$$R_n = R_0(1\cdot 89)^m$$

where the number $m = n+\delta n$, then the magnitudes of the deviations, δn are a measure of the adequateness of the relation. The deviations of the above relation are given in Table I

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TABLE I

Planet	m	n	δn
Mercury	0·92	1	−0·08
Venus	1·90	2	−0·10
Earth	2·41	—	—
Mars	3·07	3	+0·07
Ceres	3·94	4	−0·06
Jupiter	5·00	5	0·00
Saturn	5·95	6	−0·05
Uranus	7·05	7	+0·05
Neptune	7·76	8	−0·24
Pluto	8·18	—	—

It is, perhaps, not so well-known that not only does this relation break down for Pluto but also for the Earth.

In the geometric series, equation (1), C, the geometric ratio, is a dimensionless constant and it is the constant R_0, of dimensions L, which determines the basic scale of the system. Most modern theories of the origin of the solar system, including those of Berlage (**5**), von Weizsäcker (**6**) and Kuiper (**7**), do not discuss the latter factor. In these theories the Bode-type law is written

$$\frac{R_{n+1}}{R_n} = \text{constant, i.e. } C$$

and no attempt is made, explicitly, to relate R_0 to some fundamental parameter of the system, say, the radius of the primary. The constant C of any one system is also considered to be arbitrary. No attempt has ever been made, as far as I am aware, to find a rational relation between C and any other fundamental parameter of the system. Finally, no Bode-type law has ever been formulated for any system which incorporates the near-commensurabilities in that system and accounts for the observed preference for commensurability among pairs of mean motions.

The aim of this paper is to determine whether the near-commensurabilities in any one system are part of a more general regularity.

2. *The orbital period relation.* The observed preference for commensurability among pairs of mean motions suggests that in seeking some general relation to describe some subsystem of the solar system it may be fruitful to consider a relation of the form

$$T_n = T_c A^n \tag{2}$$

i.e. a relation between orbital periods rather than orbital radii. Accordingly a systematic investigation is made to determine the constants T_c and A in equation (2) such that this relation then gives the closest approximation to the orbital periods of the secondaries of the system. The systems considered are the three major satellite systems, i.e. the Jovian, Saturnian and Uranian systems. Alfvén (**8**), in a recent lecture, has argued that cosmogonists should concentrate their efforts on the major, well-developed, satellite systems rather than the planetary system as conclusions from a general theory of the formation of these bodies can be confronted with three systems rather than one. It can also be argued that the orbits of the regular satellites (this term is defined below) in these systems exhibit greater regularity than the planetary orbits, being more coplanar and having smaller eccentricities. The Uranian system is remarkable in this respect, the uniformly circular orbits of the satellites lying exactly (within the limits of observational accuracy) in the equatorial plane of the planet. Of the satellites in any one system only those which are regular are considered. Before proceeding it is necessary to define the term regular.

By and large satellites can be quite unambiguously classified as being either regular or irregular. The inner satellites of a system possessing near-circular, near-equatorial orbits being classified as regular, the others, including those with retrograde orbits, as irregular. There are, however, a few satellites which do not belong clearly to either of the two groups and the classification of these is a matter of dispute. For example, Kuiper, who has given a full discussion of this problem (**9**), contrary to Öpik (**10**), considers Iapetus to be irregular. In this paper the satellites of a planet are classified by their mode of origin. By considering the perturbations of a satellite's orbit by the oblateness of the planet and by solar tidal effects, it can be shown (a discussion of this well-known problem has recently been given by Goldreich (**11**), (**12**)) that for each planet there is a critical distance, R_c such that a satellite orbit lying well within this distance will maintain a nearly constant inclination to the planet's equatorial plane. This nearly constant inclination is maintained in spite of the precessional motions of both the satellite orbit plane and the equatorial plane of the planet. For orbits much larger than the

TABLE II

Classification of satellites

	Regular			Irregular		
	Satellites	R/A	$i^\star$	Satellites	R/A	$i^\star$
Jupiter						
	1. Io	5·90	0·0 *P*	6.	160·7	28·5 *B*
	2. Europa	9·40	0·0 *P*	7.	164·4	28·0 *B*
	3. Ganymede	14·99	0·0 *P*	8. †	326	33 *B*
	4. Callisto	26·36	0·0 *P*	9. †	332	24 *B*
	(5.	2·54	0·0 *P*)	10.	164	28·3 *B*
	(R_c/A = 38)			11. †	313	16·6 *B*
				12. †	290	—
Saturn						
	1. Mimas	3·11	1·5 *P*	8. Iapetus	59·67	14·7 *P*
	2. Enceladus	3·99	0·0 *P*	9. Phoebe	216·8	30 *P*
	3. Tethys	4·94	1·1 *P*			
	4. Dione	6·33	0·0 *P*			
	5. Rhea	8·84	0·3 *P*			
	6. Titan	20·48	0·3 *P*			
	7. Hyperion	24·83	0·6 *P*			
	(10. Janus	2·66	0 *P*)			
	(R_c/A = 57)					
Uranus						
	1. Ariel	8·08	0 *P*			
	2. Umbriel	11·25	0 *P*			
	3. Titania	18·46	0 *P*			
	4. Oberon	24·69	0 *P*			
	5. Miranda	5·49	?			
	(R_c/A) = 84)					

⋆ Values quoted (in degrees) are only approximate. $P \equiv$ inclination measured from planet's equator, $B \equiv$ inclination measured from planet's orbit.

† ≡ retrograde orbit with respect to planet's equatorial plane.

critical one, the satellite orbit plane no longer maintains constant inclination to the planet's equatorial plane, but rather it holds a nearly constant inclination to the plane of the planet's orbit around the Sun. This critical distance, R_c is the distance at which the torques on the satellite orbit caused by the planet and by the Sun are equal. If A is the radius of the planet then for Jupiter $R_c/A = 32$ (the effects of Ganymede increase this to 38), for Saturn $R_c/A = 43$ (the effects of Titan increase this to 57), and for Uranus $R_c/A = 84$ (see (**12**)). Values of R/A and i, R being the orbital radius and i the inclination, for the satellites of the major planets are shown in Table II. (All data in this paper is from Allen (**13**).) Without exception all satellites with $R < R_c$ are near-equatorial ($i < 2°$) and Goldreich (**11**), (**12**) has argued that these small inclinations are original rather than the result of evolution, the satellites having been formed from a thin equatorial disc of particles. I define these satellites to be regular and confine the search for regularity to these alone. The regular satellites J V and Janus (the recently discovered satellite of Saturn) are, however, not included. Because of their extremely small mass, as compared with the masses of the other regular satellites in their respective systems, I consider them to be unimportant.

If in any satellite system the regular satellites are numbered 1, 2, 3, ..., s, where s is the number of satellites considered then the constants T_c and A in equation (2) can be determined approximately by a simple graphical method. Application of such a simple method to the satellite system of, say, Saturn produces completely uninteresting results. Such a simple method though does not take into consideration the possibility that some of the satellite orbits may be vacant, i.e. the numbers to be associated with the satellites may not form an arithmetic progression of common difference unity. If this possibility is admitted, then as there is no way of predetermining the positions of these vacancies the problem can no longer be thought of as simple. It is, however, a problem which a high speed computer can handle with ease.

If the orbital period of some satellite is T_i then we write,

$$T_i = T_c A^{m_i} \tag{3}$$

where m_i is some number, not necessarily an integer but associated with some integer, n_i given by,

$$m_i - n_i = \delta n_i \text{ where } |\delta n_i| \leqslant 0{\cdot}5.$$

If a relation of the form (2) can be used to usefully describe the system then the deviations, δn_i will in general be small compared with unity. For any given value of A the best value of T_c can be found by varying T_c until the sum of the squares of the deviations is a minimum. A can then be changed by some small amount (the magnitude of which must be such that the change in the sum of the squares of the deviations produced is always small compared with the sum itself) and the procedure repeated. In this way all values of T_c and A can be compared and the values of T_c and A such that the sum of the squares of the deviations is a minimum can be found. This method, however, suffers from two defects, these being:

(i) as $A \to 1$ then $n_s \to \infty$ and $\sum_{i=1}^{s} \delta n_i^2 \to 0$, i.e. the greater the number of allowed vacancies then the easier it is to find a good fit;

(ii) it may be possible to find a relation of the form (2) to represent, say, $s-1$ of the s satellites considered, the deviation of the other satellite being large

($\delta n \sim 0{\cdot}5$). Relations of this nature which may well have some significance would not necessarily be revealed by the above method.

To overcome these difficulties the following method was adopted.

(a) Certain restrictions were placed on the values of n_i, these being:

(i) $n_1 = 1$;

(ii) $n_i \neq n_j$, i.e. no two satellites may occupy the same orbital;

(iii) $n_s \leqslant 20$, i.e. the number of allowed vacancies was limited to $20-s$.

These restrictions imply that T_c and A must lie between certain limits, these limits being given by,

$$\frac{\log T_s - \log T_1}{20{\cdot}5 - 0{\cdot}5} \leqslant \log A \leqslant \frac{\log T_s - \log T_1}{(s-0{\cdot}5) - 1{\cdot}5},$$

and

$$\frac{(s-0{\cdot}5)\log T_1 - 1{\cdot}5 \log T_s}{(s-0{\cdot}5)-1{\cdot}5} \leqslant \log T_c \leqslant \frac{20{\cdot}5 \log T_1 - 0{\cdot}5 \log T_s}{20{\cdot}5 - 0{\cdot}5}.$$

(b) For a given T_c and A the number of satellites, s_1 with deviations $\leqslant 0{\cdot}15$ and the sum of the squares of these deviations, Σ_1 were found. Similarly, s_2 and Σ_2 were found for satellites with deviations $\leqslant 0{\cdot}10$.

(c) A was varied, in steps of suitable magnitude, between the limits imposed by (a). For each value of A the value of T_c was found such that Σ_1 was a minimum, subject to the condition that s_1 was a maximum. Similarly, the value of T_c was found such that Σ_2 was a minimum, subject to the condition that s_2 was a maximum. These two values of T_c are not necessarily the same.

In this way all possible relations of the form (2) which could be used to represent approximately a given system of s satellites and an arbitrary number of vacancies ($\leqslant 20-s$) were systematically compared.

For all three major satellite systems values of T_c and A can be found such that s_2 is large (equal to $s-1$ in each case) and Σ_2 small (the root mean square deviation being less than $0{\cdot}08$), the number of vacancies in the system being small (zero for the systems of Jupiter and Uranus). The value of A such that Σ_2 is a minimum being; $2{\cdot}000$ for Jupiter, $1{\cdot}417$ (or $2{\cdot}007^{1/2}$) for Saturn and $1{\cdot}753$ (or $3{\cdot}07^{1/2}$) for Uranus. As it has been proved that there is a preference for commensurability among pairs of orbital periods, it is now hypothesized that for all regular systems A^2 is the ratio of two small integers or possibly the square of such a ratio. Thus for Jupiter A^2 is $4/1$, for Saturn $2/1$ and for Uranus $3/1$ or possibly $(7/4)^2$. Using the latter values of A^2, values of m, n and δn are calculated for a given value of T_c (different for each system)—see Table III. The values of T_c are chosen not so much as to make Σ_2 a minimum (a least square fit has no particular significance) but with reference to the commensurabilities with configurational regularity in that system. Thus, for the Jovian system T_c is chosen with reference to the satellites Io, Europa and Ganymede. These satellites are related by the following,

$$\frac{T_2}{T_1} = \frac{2}{1} + 0{\cdot}0073$$

$$\frac{T_3}{T_2} = \frac{2}{1} + 0{\cdot}0147$$

and

$$T_1^{-1} - 3\,T_2^{-1} + 2\,T_3^{-1} = 0$$

where T_1, T_2 and T_3 are the orbital periods of Io, Europa and Ganymede respectively. For the Saturnian system T_c is chosen with reference to the inner group of satellites, i.e. Mimas, Enceladus, Tethys and Dione, T_c being chosen such that the sum of the squares of the deviations of these satellites is a minimum. This group of satellites is particularly striking. Mimas and Tethys, and Enceladus and Dione have orbital periods which are nearly commensurate in the ratio 2 to 1, the commensurabilities possessing configurational regularity. This analysis now reveals that the satellites are also so distributed that the orbital periods can be quite accurately represented by a relation of the form (2), A^2 being 2. For the Uranian system no particular satellite or group of satellites can be singled out as being more significant than the others, there being no commensurabilities in this system and hence T_c cannot be specified to the same number of significant figures as T_c for the other systems. By choosing T_c such that the deviation of Ariel is a minimum a reasonable fit is obtained. As the deviation of Titania is large it is inappropriate to apply a least squares fit.

TABLE III

Regular satellites

Calculated values of *m*, *n* and δn

Primary	T_c (days)	A^2	Secondary	m	n	δn
Jupiter	0·8894	4	Io	0·992	1	−0·008
			Europa	1·997	2	−0·003
			Ganymede	3·008	3	+0·008
			Callisto	4·230	4	+0·230
Saturn	0·6758	2	Mimas	0·960	1	−0·040
			Enceladus	2·039	2	+0·039
			Tethys	2·965	3	−0·035
			Dione	4·036	4	+0·036
			Rhea	5·483	5	+0·483
			Titan	9·122	9	+0·122
			Hyperion	9·953	10	−0·047
Uranus	0·8400	3	Miranda	0·947	1	−0·053
			Ariel	2·000	2	0·000
			Umbriel	2·906	3	−0·094
			Titania	4·257	4	+0·257
			Oberon	5·050	5	+0·050
Uranus	0·8213	$(7/4)^2$	Miranda	0·969	1	−0·031
			Ariel	2·003	2	+0·003
			Umbriel	2·892	3	−0·108
			Titania	4·218	4	+0·218
			Oberon	4·997	5	−0·003

To some extent the value of T_c in equation (2) is arbitrary as any number of vacancies can be included in the system if they are placed before the first satellite. From equation (3) we have,

$$\begin{aligned}\log T_i &= \log T_c + m_i \log A \\ &= \log T_c + (n_i + \delta n_i) \log A \\ &= \log T_c - v \log A + (v + n_i + \delta n_i) \log A \\ &= \log T_0 + m_i' \log A\end{aligned}$$

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Thus the effect of including an integral number of vacancies, v before the first satellite is to increase n_i by v, δn_i being unchanged ($i = 1, 2, \ldots, s$). A further regularity in the solar system is revealed by the fact that for all three major satellite systems if $v = 1$ then the constant T_0 given by

$$\log T_0 = \log T_c - \log A$$

is closely associated with the rotational period (non-equatorial) of the primary, T_p; the ratio T_0/T_p being slightly greater than unity in all cases—see Table IV.

TABLE IV

*The relation between T_0 and the rotational period of the primary, T_p**

Planet	T_c (days)	A^2	T_p (days)	T_0/T_p
Jupiter	0·8894	4	0·4135	1·076
Saturn	0·6758	2	0·4430	1·079
Uranus	0·8400	3	0·4507	1·076
Uranus	0·8213	$(7/4)^2$	0·4507	1·041

* The rotational periods of observed surfaces on both Jupiter and Saturn depend on the latitudes of the surfaces. The only rotational period of cosmogonic significance is, of course, the rotational period of the planet as a whole. For Jupiter this is taken to be the period derived from observations of the radio-frequency emissions. This period is approximately equal to that of surfaces in the non-equatorial zones (latitudes $>45°$ or $<-45°$) and so, by analogy, the rotational period of surfaces in the temperate zones of Saturn is taken to be closest to that of Saturn as a whole. The rotational period of Uranus as a whole must be considered to be somewhat uncertain.

The fact, that for the Uranian system T_0/T_p is the same as that for the other planets when A^2 is a small integer, i.e. 3, is taken to indicate that the latter value of A^2 is of significance rather than $(7/4)^2$.

It has long been recognized that the major satellite systems are highly regular. Each primary possesses a number of regular secondaries moving around it, in the same sense as the rotation of the primary itself, in near-circular, near-equatorial orbits. The above analysis has now shown that the distribution of orbital periods in any one system is by no means random, as these orbital periods can be represented approximately by a Bode-type relation of the form

$$T_n = T_0 A^n \tag{4}$$

where T_n is the orbital period of the nth secondary. The constants in this relation are not arbitrary. T_0 is related to the rotational period of the primary and A is the square root of a small integer. It must be allowed though that in any one system there are a small number of vacancies. The distribution of these vacancies, however, is to some extent regular. From equation (4) we have,

$$\log T_n = \log T_0 + n \log A.$$

Plots of log T_n against n are shown in Figs 1–3. Also shown are the rotational periods of the primaries, these being in all cases the non-equatorial values. (Note, that in these figures the values of n are greater than those given in Table I. This is due, in all cases, to the insertion of a vacancy in the first orbital.)

The main features of these graphs can be summarized as follows.

(i) In all cases the orbital $n = 1$ is vacant (or occupied by a satellite of extremely small mass, viz. the recently discovered satellite of Saturn, Janus), all other inner orbitals being occupied.

(ii) In all cases the rotational period of the primary is slightly less than T_0. We have $T_0 \simeq 1 \cdot 077\ T_p$.

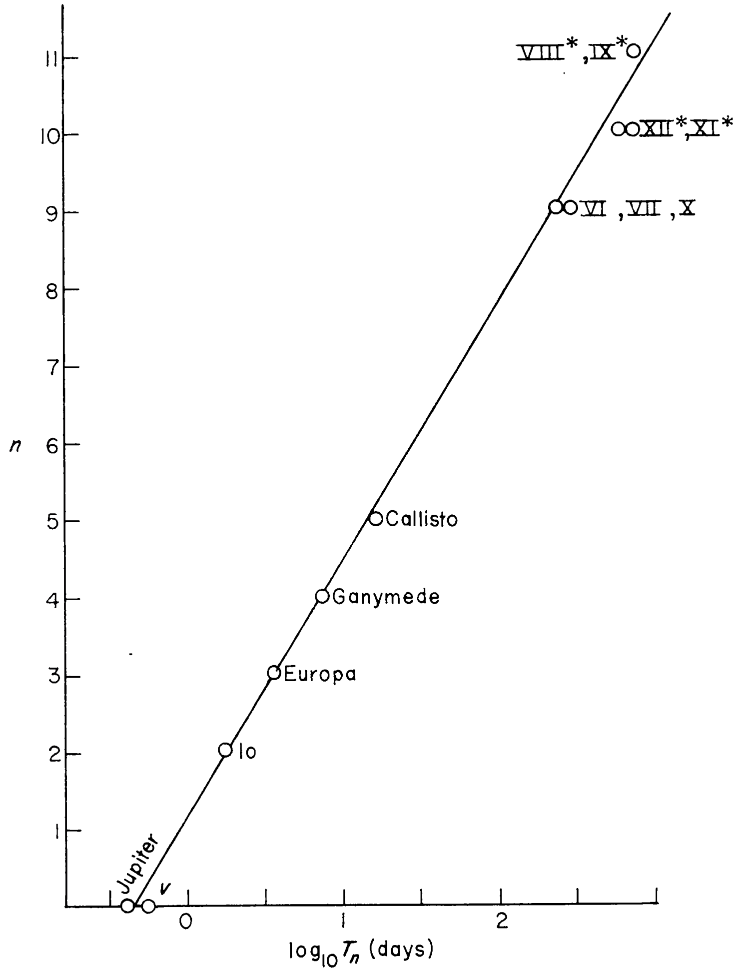

FIG. 1. *The orbital period relation,* $T_n = T_0 A^n$ *applied to the satellite system of Jupiter.* $A^2 = 4$ *and* $T_0/T_p = 1 \cdot 076$, T_p *being the non-equatorial rotational period of Jupiter (this is also shown on the graph). The four Galilean satellites occupy consecutive inner orbitals and lie near the line, the irregular satellites are randomly distributed.*

(iii) The value of A^2 is small being $\leqslant 4$.

(iv) The outer direct satellites of Jupiter, J VI, J VII and J X and the retrograde satellites (marked with an asterisk), i.e. the irregular satellites, do not fit on the line. This fact is important as it serves to emphasize the regularity of the other satellites.

(v) In the Saturnian system there are three vacancies between Rhea and Titan, Rhea being clearly off the line. It is to be noted that Titan is an unusually massive satellite, having a mass $\sim 10^2$ that of the other regular satellites. The outer, irregular satellites of Saturn, Iapetus (direct) and Phoebe (retrograde) do not fit on the line.

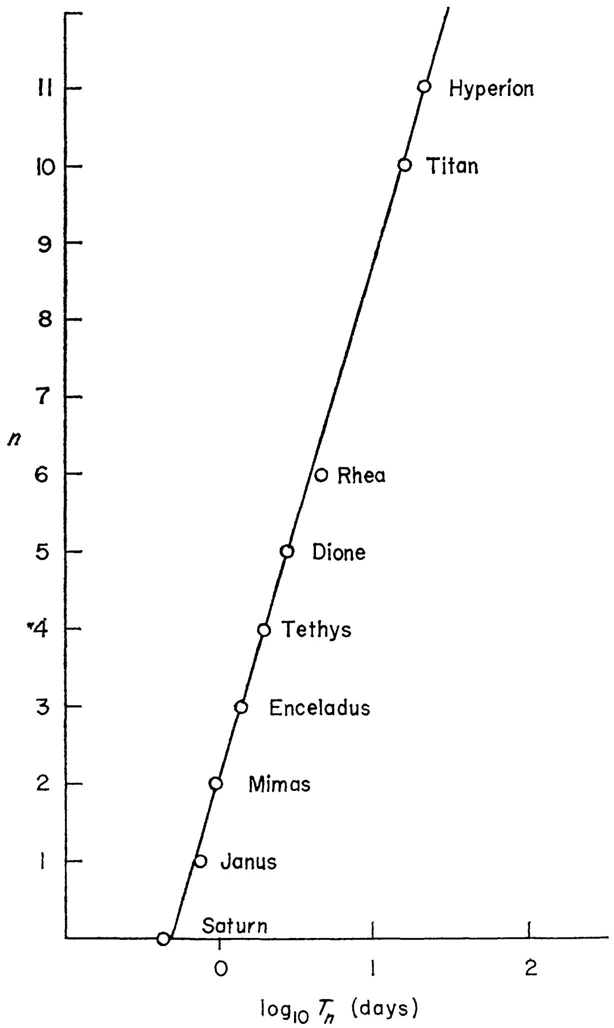

FIG. 2. *The orbital period relation,* $T_n = T_0 A^n$ *applied to the satellite system of Saturn.* $A^2 = 2$ *and* $T_0/T_p = 1{\cdot}079$, T_p *being the non-equatorial rotational period of Saturn (this is also shown on the graph). The irregular satellites, Iapetus and Phoebe occupy outer orbitals and are randomly distributed* ($n = 14{\cdot}75$ *and* $20{\cdot}27$ *respectively). These satellites cannot be conveniently shown on the above graph.*

TABLE V

Irregular satellites

Calculated values of m, n and δn

Primary	T_c (days)	A^2	Secondary	m	n	δn
Jupiter	0·8894	4	V	0·164	0	+0·164
			VI	9·138	9	+0·138
			VII	9·190	9	+0·190
			X	9·211	9	+0·211
			XII*	10·470	10	+0·470
			XI*	10·640	11	−0·396
			VIII*	10·698	11	−0·302
			IX*	10·735	11	−0·256
Saturn	0·6758	2	Janus	1·300	1	+0·300
			Iapetus	14·751	15	−0·249
			Phoebe*	20·271	20	+0·271

* Retrograde satellites.

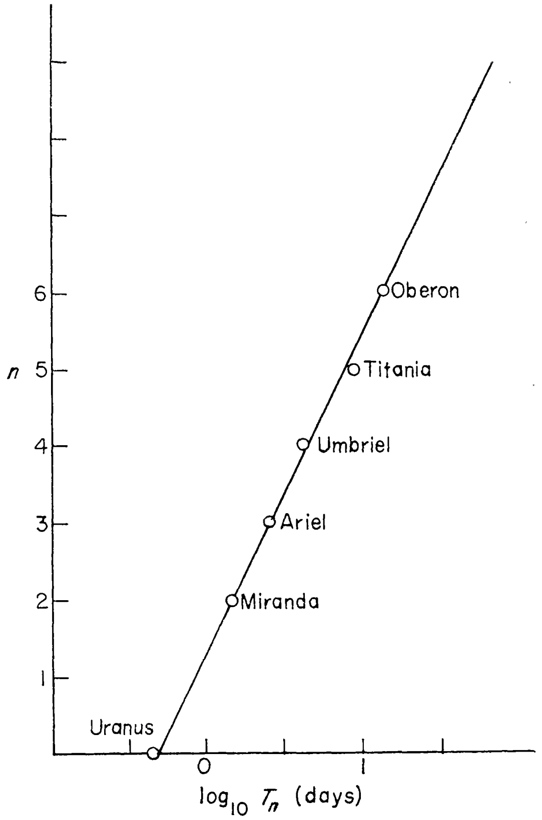

FIG. 3. *The orbital period relation,* $T_n = T_0 A^n$ *applied to the satellite system of Uranus.* $A^2 = 3$ *and* $T_0/T_p = 1 \cdot 076$, T_p *being the rotational period of Uranus (this is also shown on the graph). Satellites occupy consecutive inner orbitals and lie near the line, there are no known irregular satellites in this system.*

(vi) For Saturn the orbital $n = 0$ falls within Cassini's division ($T_0 = 11 \cdot 47$ hours, a particle with this period would have an orbital radius of $117 \cdot 9 \times 10^3$ km, the inner and outer radii of Cassini's division are 116×10^3 km and 120×10^3 km respectively). For Jupiter the orbital $n = 0$ is approximately occupied by J V.

Values of m, n and δn for the irregular satellites, corresponding to the above graphs, are shown in Table V.

TABLE VI

The planetary system

Calculated values of m, n and δn

Primary	T_0 (years)	A^2	Secondary	m	n	δn
Sun	0·1343	6	Mercury	0·648	1	−0·352
			Venus	1·707	2	−0·293
			Earth	2·241	2	+0·241
			Mars	2·944	3	−0·056
			Ceres	3·942	4	−0·058
			Pallas	3·945	4	−0·055
			Jupiter	5·000	5	0·000
			Saturn	6·015	6	+0·015
			Uranus	7·185	7	+0·185
			Neptune	7·937	8	−0·063
			Pluto	8·395	8	+0·395

The orbital period relation (4), derived for the regular satellites of the major planets can also be used to describe the planetary system but the cases are not exactly analogous. A plot of log T_n against n is given in Fig. 4 and the corresponding values of m and δn in Table VI. The representation is quite good, A again being the square root of a small integer. In particular, note that Ceres and Pallas are well represented. This is remarkable as these two asteroids, the most massive of all the asteroids, are by no means coplanar, the inclination to the ecliptic of Ceres being 10·6° and that of Pallas 34·8°. The orbit of Pallas is also quite eccentric, e being 0·235. The ratio $T_0/T_p = 1 \cdot 442$ and thus is of the order unity but is different from that for the major satellite systems. This discrepancy is discussed in a later paper. The position of Pluto is anomalous as is, perhaps, to be expected as it has been argued that this body is an escaped satellite of Neptune (14), (15), but also in this system there is the anomaly of two bodies, Venus and Earth, occupying the same orbital. Thus, it can be concluded that the planetary and satellite systems are, by and large, analogous but the former is not quite as regular as are the latter.

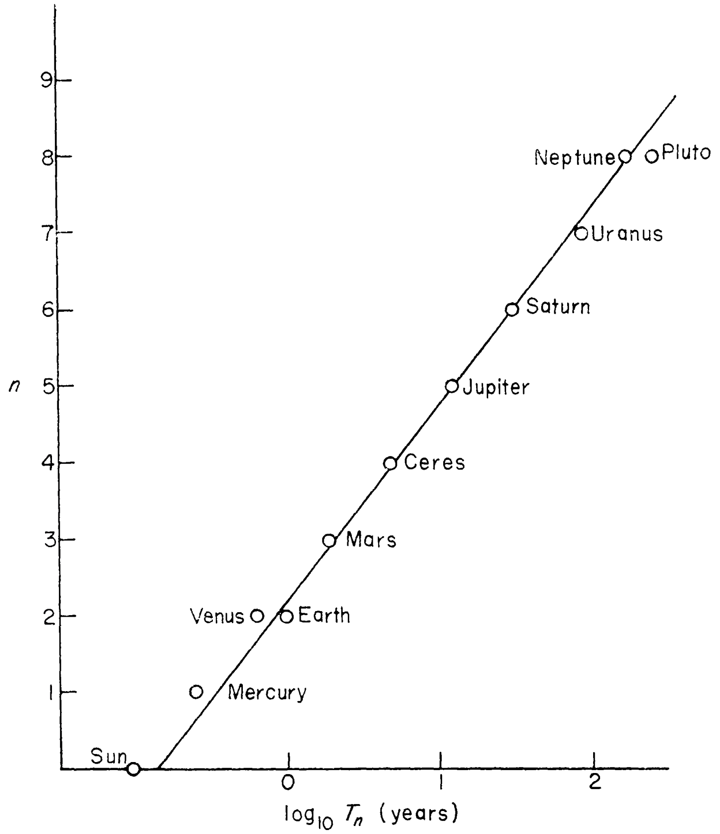

FIG. 4. *The orbital period relation,* $T_n = T_0 A^n$ *applied to the planetary system.* $A^2 = 6$ *and* $T_0/T_p = 1 \cdot 442$. *The line has been drawn to pass through Jupiter, this being the most massive planet.*

3. *Fictitious systems of randomly distributed satellites.* In this section the distribution of satellites in the Uranian system is compared with analogous systems of regular satellites the orbital periods of which are derived from random numbers. A regular satellite must have an orbital radius greater than the radius of the

primary and less than, by definition, the critical distance, R_c. For the Uranian system these restrictions imply that the orbital period, T of a satellite satisfies the inequality,

$$0\cdot 11088 \text{ days} < T < 85\cdot 36227 \text{ days}.$$

Orbital periods, randomly distributed in the above range, are derived from random numbers, $x(0 \leqslant x < 1)$ by the equation

$$T = x(85\cdot 36227 - 0\cdot 11088) + 0\cdot 11088.$$

The random numbers are taken from tables by Fisher & Yates (**16**), the first four lines of their tables being used to construct four fictitious systems of five satellites—see Table VII.

TABLE VII

Satellite periods derived from random numbers

System		Random number, $x(0 \leqslant x < 1)$ and corresponding satellite period, $T(0\cdot 11088 \text{ days} \leqslant T < 85\cdot 36227 \text{ days})$				
A	x	0·0347437386	0·3696473661	0·4698637162	0·3326168045	0·6011141095
	T	3·07285	31·62402	40·16765	28·46709	51·35699
B	x	0·9747266762	0·4281145720	0·4253323732	0·2707360751	0·2451798973
	T	83·20817	36·60846	36·37127	23·19164	21·01293
C	x	0·1676622766	0·5650267107	0·3290797853	0·1355385859	0·8897541410
	T	14·40441	48·28047	28·16555	11·66580	75·96410
D	x	0·1256859926	0·9696682731	0·0503729315	0·5712101421	0·8826498176
	T	10·82585	82·77693	4·40527	48·80762	75·35845

It may be objected that as in the Uranian system all satellites have orbital periods < 14 days the definition of regular used in this paper, which must to same extent be considered arbitrary, is unrealistic. The implication being that with this definition of regular the distribution of the Uranian satellites is in one sense obviously non-random, the orbital periods being small rather than large. I can see no justification for restricting the orbital periods to small values, indeed an aim of this paper is to point out that they are small, but to overcome this objection the random satellite systems are treated as follows.

The computer method outlined in this paper is applied to the systems A, C and D. The system B is rejected as it contains two satellites with approximately equal periods ($36\cdot 6 \simeq 36\cdot 4$). The same restrictions on the n_i values are used with the exception that the restriction $n_i \neq n_j$ is dropped. In a random distribution of five satellites the probability that $n_i = n_j$ for any one pair of satellites is $> 0\cdot 4186$, thus dropping the latter restriction gives a much wider choice of possible systems.

Table VIII gives the optimum values of m, n and δn found by the above method. As the range 0·11088 to 85·36227 is approximately equal to the range 0 to 85·36227 then the distribution of the satellites, i.e. the characteristics of the set of m, n and δn values, is to all intents independent of the scale of the system. If all the periods are multiplied by an arbitrary scaling factor the distribution still remains essentially random and the computer method predicts exactly the same optimum n-values. Thus the deductions from Table VIII are largely independent of the definition of a regular satellite. The characteristics of random systems of satellites are:

(i) a large number of randomly distributed vacancies;
(ii) orbitals containing more than one satellite;
(iii) deviations, δn which are small compared with unity.

TABLE VIII

Random satellites

Optimum values of m, n and δn as found by computer

System	A^2	Secondary	m	n	δn
A	1·59	(Primary	−7·308	−7	−0·308)
		1	0·947	1	−0·053
		2	10·518	11	−0·482
		3	10·970	11	−0·030
		4	11·999	12	−0·001
		5	13·055	13	+0·055
C	1·41	(Primary	−18·028	−18	−0·028)
		1	0·812	1	−0·188
		2	2·033	2	+0·033
		3	5·916	6	−0·084
		4	9·037	9	+0·037
		5	11·661	12	−0·339
D	1·54	(Primary	−9·671	−10	+0·329)
		1	0·862	1	−0·138
		2	5·016	5	+0·016
		3	11·973	12	−0·027
		4	13·980	14	−0·020
		5	14·414	14	+0·414

Thus although it is possible to find a reasonable ' fit ' in the sense that the deviations, δn of some of the satellites are small, systems of randomly distributed satellites tend to have large numbers of vacancies and some orbitals containing more than one satellite. Also, the optimum values of A^2 are not usually approximately equal to integers. With the definition of regular adopted in this paper, i.e. with the scales of the satellite systems fixed, it can be seen that in general the orbital periods are not related to the rotational period of the primary. The contrast between these random systems and the actual Uranian system is quite striking.

4. *Conclusion.* It has been shown that to a good approximation the distribution of the regular satellites of the major planets (Jupiter, Saturn and Uranus) can be represented by an orbital period relation of the form $T_n = 1\cdot077\, T_pA^n$ where T_p is the non-equatorial rotational period of the primary and A is the square root of a small integer. The relation can be applied to the planetary system but in this case $T_n = 1\cdot44\, T_pA^n$. It would appear that the rotational periods of primaries determine the basic scales of the various subsystems of the solar system. The fact that A^2 is an integer accounts for the preference for commensurability found among pairs of orbital periods. Thus, the near-commensurabilities with configurational regularity are part of a more general regularity. (Europa and Ganymede, and Titan and Hyperion, are perhaps exceptions but a discussion of this is reserved for a later paper.) The other striking regularity revealed by the above analysis is that all inter-orbitals of all sub-systems, except the first orbitals of the major satellite systems, are occupied by regular bodies of appreciable mass. Perhaps the recent discovery of the very small satellite of Saturn, Janus indicates that other very small satellites will be found in the corresponding orbitals of the other planets.

Discussions of the deductions which can be made from the orbital period relation with respect to the evolution of the solar system (including the origin of the Moon), tidal evolution in the solar system and the variation with time of the universal gravitational constant, G will be given in later papers. In these papers it is argued that as the rotational periods of the major planets and the orbital elements of their satellites have probably not changed by great amounts since their time of formation (**8**) then the orbital period relations reflect a condition of formation rather than the results of evolution.

Finally, it is remarked that Pythagoras would have, perhaps, interpreted the above quite differently. Introducing Kepler's third law into equation (4) we have

$$R_n^3 = R_0^3(A^2)^n$$

where R_0 is a constant and A^2 a small integer. Thus, the orbital radii of the planets and satellites can be said to define ' celestial spheres ' the volumes of which are in simple numerical ratio and therefore in ' harmony '.

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Physics Department,
Royal Military College of Science,
Shrivenham,
Wilts.
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