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RELATIVISTIC, SPHERICALLY SYMMETRIC STAR CLUSTERS
I. STABILITY THEORY FOR RADIAL PERTURBATIONS*

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ABSTRACT

There is some indication that very dense star clusters might play important roles in quasi-stellar sources and in the nuclei of certain galaxies The roles of such star clusters should be strongly influenced by a relativistic instability, which sets in when a cluster surpasses a certain critical density. In this paper the groundwork is laid for the study of that instability: the theory of small, radial perturbations of a spherically symmetric star cluster is developed within the framework of general relativity. The cluster is idealized as a solution to the collisionless Boltzmann-Liouville equation (an idealization which should be valid on the short time scale associated with the relativistic instability). The equation of motion governing small radial perturbations is derived and is shown to be self-conjugate. From the equation of motion follows a variational principle for the normal modes, which provides a necessary and sufficient condition for the stability of the cluster. Also presented are (1) the corresponding Newtonian analysis, much of which has been developed previously by Antonov and by Lynden-Bell, (2) the relationship between the Newtonian and relativistic analyses, and (3) necessary and sufficient conditions for the existence of a zero-frequency mode of radial motion.

I. MOTIVATION

Between 1964 and 1967 it was generally believed that the redshifts of quasi-stellar sources (QSSs) could not possibly be gravitational in origin. (One of us—K. S. T.—was a particularly firm proponent of this view.) Not only are there difficulties with the sharpness of the spectral lines in a gravitational redshift model (Greenstein and Schmidt 1964); there is also an absolute upper limit of $z < 0.63$ on the redshift of light from the surface of any non-rotating equilibrium configuration of perfect fluid with reasonable equation of state and density distribution (Bondi 1964), and this limit probably cannot be changed much by angular velocities which are compatible with the sharpness of the emission lines. (Large angular velocities are not permitted because of Doppler broadening.)

However, in early 1967 Hoyle and Fowler (1967) revived the gravitational redshift hypothesis by introducing a new model which may circumvent both of the above difficulties: they suggested that each QSS might rest at the center of a very massive relativistic star cluster and might derive its redshift from the gravitational field of the cluster. One can, indeed, construct star-cluster models in which sharp spectral lines and large gravitational redshifts are produced. However, one does not know today whether such star clusters are stable against gravitational collapse.

There is good reason to fear that star clusters with central redshifts as large $z \approx 2$ might be unstable against collapse. In Newtonian theory the stabilities of collisionless, spherical star clusters and of gas spheres are somewhat related (Lynden-Bell 1966; see also § III*g* of this paper), and a similar relationship seems likely in general relativity. Chandrasekhar (1964) showed that when a gas sphere (star) of given mass contracts beyond a certain critical point, it becomes unstable against gravitational collapse. This

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instability, which is catalyzed by general-relativistic effects, has been studied in great detail since it was first discovered (see Thorne 1967 for a review). An examination of a variety of relativistic gas spheres as computed, e.g., by Tooper (1965 and private communication) and by us (unpublished) reveals this: a contracting gas sphere becomes unstable against collapse when the redshift from its center exceeds a limit which is typically $z_{\text{central}} \sim 1$. It is not unlikely that the critical redshifts for spherical star clusters will be similar in magnitude, but we cannot know until the theory of pulsating star clusters has been developed fully.

There is another motivation for studying the relativistic instability in star clusters: two independent lines of investigation have suggested recently that, when a *Newtonian* star cluster contracts beyond a certain critical density—one far less than the density for relativistic instabilities—it may become unstable against a "thermal runaway." In this thermal runaway the cluster gradually develops a dense core and a diffuse envelope. For some clusters of astrophysical interest (e.g., the compact nuclei of certain galaxies), the core evolves toward ever higher densities on a time scale which may be short compared with 10^{10} years but which is very long compared with the time scale for the relativistic instability ($\lesssim 1$ year). The evidence which suggests that a thermal runaway may occur comes (1) from dynamical computer experiments on the many-body gravitational problem (Arseth 1963; see also Hénon 1961, 1965) and (2) from analytic studies of the configurations of maximal entropy for a star cluster inclosed in a spherical cavity (Antonov 1962; Lynden-Bell and Wood 1968).

One is invited to speculate that the star densities in the nuclei of some galaxies (and in potential QSSs) may exceed the critical density for thermal runaway, that runaway may occur, and that the nuclei may thereby evolve in times $t < 10^{10}$ years to such high densities that relativistic effects become important and collapse sets in. Indeed, the outbursts which occur in the nuclei of galaxies might conceivably be associated with the onset of collapse or with encounters between an already collapsed nucleus and surrounding stars.

The above discussion of models for QSSs and of outbursts in galaxies is necessarily very speculative. Before these speculations can be analyzed with confidence, we must understand, among many other things, the onset of the relativistic instability in star clusters. This paper is the first of several in which we shall attempt to delineate the theory of the stability of relativistic star clusters and thereby contribute to the tools needed for studying dense galactic nuclei and QSS models.

II. SUMMARY

In relativistic gas spheres the time scale for the growth of the relativistic instability is roughly the sound travel time across the sphere. Similarly, in star clusters one expects the time scale to be roughly the star travel time across the cluster, which—for the clusters that interest us—is short compared with the mean time between close stellar encounters. Consequently, in discussing cluster stability, we shall idealize the cluster as a statistical distribution of mass points, which interact only through the smoothed-out gravitational field of the entire cluster. The mathematical formalism used in such a treatment is relativistic kinetic theory (Synge 1934; Walker 1936; Tauber and Weinberg 1961; Lindquist 1966). The cluster is described by a density in phase space and by a metric for the curvature of spacetime. The density in phase space determines a stress-energy tensor, which generates the metric through the Einstein field equations; the metric in turn determines the density in phase space via the collisionless Boltzmann-Liouville equation.

This type of statistical treatment of star clusters has been used in Newtonian theory for about fifty years. However, only very recently (Antonov 1960; Lynden-Bell 1966; Milder 1967) has the collisionless stability of Newtonian clusters been investigated, and those Newtonian investigations of stability have all been of a formal nature: no applications have been made yet to specific models for clusters. In general relativity the structures of spherical star clusters have been investigated recently by Zel'dovich and Podurets (1965) and by Fackerell (1966, 1968*a*–*c*), but no treatment of their stability has been attempted.[1]

In this paper we shall treat the stability of star clusters by means of a relativistic analysis which is patterned after the Newtonian analysis of Antonov (1960). However, Antonov made two restrictions in his Newtonian analysis which we do not wish to make: he assumed that the stars in his cluster all had identical masses, and he assumed that the number density in phase space for the equilibrium configuration depends only on energy. Before presenting our relativistic treatment of stability, we shall redo the Newtonian treatment, dropping Antonov's restrictions but imposing in their place the demand that both the equilibrium and the perturbed configurations be spherically symmetric. We shall also extend the Newtonian analysis somewhat beyond that of Antonov: in addition to obtaining his stability criterion, we shall derive a variational principle (action principle) for the pulsation of the cluster; we shall obtain from our variational principle a conserved quantity for arbitrary radial pulsations; and we shall derive an elegant, new criterion for the existence of a zero-frequency mode of motion. All of this Newtonian discussion is found in § III.

In § IV we shall use the Newtonian analysis as a guide in developing the corresponding relativistic analysis. All the Newtonian results will be generalized to relativity theory except Lynden-Bell's relationship between the stabilities of star clusters and gas spheres: we shall obtain (1) a self-conjugate equation of motion for the small spherical pulsations of a spherical cluster, (2) an action principle for the pulsations, (3) a variational principle for the normal modes, which is also a necessary and sufficient condition for stability, (4) a conserved quantity analogous to pulsational energy, and (5) an elegant criterion for the existence of a zero-frequency mode.

Throughout this paper we adopt the mathematical conventions of Thorne (1967), including the use of "geometrized units" in which the speed of light, c, Newton's gravitational constant, G, and Boltzmann's constant, k, are equal to unity. Also, we number the equations in a manner designed to bring out the close relationship between the Newtonian and relativistic analyses; for example, the relativistic equation (12;R) has as its Newtonian limit equation (12;N).

III. NEWTONIAN THEORY OF STABILITY

a) *Equations of Stellar Dynamics*

In Newtonian theory the density of stars in phase space, which we denote by $\mathfrak{N}$, is defined as follows: At a particular time t an observer concentrates his attention on a particular volume $d\mathcal{V}_x$ in physical space and a particular volume $d\mathcal{V}_p$ in momentum space. In a Cartesian coordinate system these volumes are

$$d\mathcal{V}_x = dxdydz\ , \qquad d\mathcal{V}_p = dp^x dp^y dp^z dm\ , \tag{1;N}$$

where m is the rest mass of a star and $p^j = m dx^j/dt$. If the observer sees dN stars in the volume $d\mathcal{V}_x d\mathcal{V}_p$ at time t, then the number density in phase space ("distribution function") is

$$\mathfrak{N} \equiv dN/d\mathcal{V}_x d\mathcal{V}_p = dN/(dxdydzdp^x dp^y dp^z dm)\ . \tag{2;N}$$

The density $\mathfrak{N}$ is a function of time, t, and of location (x^j, p^j) in the seven-dimensional phase space.

[1] Zel'dovich and Podurets (1965) and Zel'dovich and Novikov (1967, § 11.19) argued without proof that one should be able to diagnose the stability of isothermal, relativistic star clusters from binding-energy considerations; but the discussion presented in § IV*f* of this paper makes that seem highly improbable.

The smoothed-out gravitational field of the star cluster is described by the Newtonian gravitational potential, $\Phi(t,x,y,z)$. The distribution function determines Φ through the source equation

$$\nabla^2\Phi = 4\pi\rho \;, \qquad \rho = \int m\mathfrak{N}d\mathcal{V}_p \;, \tag{3a;N}$$

which has the solution

$$\Phi(t,x) = -\int \frac{m'\mathfrak{N}(t,x',p',m')}{|x-x'|}\, d\mathcal{V}_{x'}d\mathcal{V}_{p'} \;. \tag{3b;N}$$

The gravitational field determines the distribution function through the collisionless Boltzmann-Liouville equation (or simply "Liouville equation")

$$\mathfrak{D}\mathfrak{N} = 0 \;. \tag{4;N}$$

Here $\mathfrak{D}$, the Liouville operator, is differentiation with respect to time along the path of a star in phase space. In a Cartesian coordinate system, $\mathfrak{D}$ is given by

$$\mathfrak{D} = \frac{\partial}{\partial t} + \frac{dx^j}{dt}\frac{\partial}{dx^j} + \frac{dp^j}{dt}\frac{\partial}{\partial p^j} + \frac{dm}{dt}\frac{\partial}{\partial m} = \frac{\partial}{\partial t} + \frac{p^j}{m}\frac{\partial}{\partial x^j} - m\frac{\partial\Phi}{\partial x^j}\frac{\partial}{\partial p^j} \,. \tag{5;N}$$

(We sum over repeated indices unless otherwise indicated.)

Equations (3;N) and (4;N), which couple Φ and $\mathfrak{N}$, are the fundamental equations of Newtonian stellar dynamics.

b) Spherical Equilibrium Configurations

In stellar dynamics an equilibrium configuration is one for which the distribution function and the gravitational field are independent of time. From the Liouville equation (4;N) one readily verifies that a Newtonian star cluster is in equilibrium if and only if $\mathfrak{N}$ is a function of the integrals of the motion of the stars ("Jeans's theorem"; see, e.g., Ogorodnikov 1965). For spherically symmetric equilibrium configurations there are five independent integrals of the motion: the stellar mass m, the energy E, the total angular momentum J, and the two angles which determine the (conserved) plane of the orbit. Of these, $\mathfrak{N}$ can depend only on m, E, and J, since a dependence on the plane of the orbit would lead to a non-spherical mass density and thence to a non-spherical gravitational field (cf. eq. [3a;N]). Consequently, the distribution function and the gravitational potential have the form

$$\mathfrak{N} = F(m,E,J) \;, \qquad \Phi = \Phi(r) = \Phi[(x^2+y^2+z^2)^{1/2}] \;, \tag{6a;N}$$

where

$$E = (p^j)^2/2m + m\Phi(r) \;, \qquad J = |x \times p| \;. \tag{6b;N}$$

When F is independent of J, the cluster has an isotropic velocity distribution at each point in space.

Equilibrium configurations for spherical star clusters have been studied extensively during the last fifty years. (See Ogorodnikov 1965 for references.) However, none of the models constructed have ever been tested for collisionless stability.

c) Equations of Motion for a Perturbed Spherical Cluster

Consider a particular spherically symmetric equilibrium configuration described by the distribution function $\mathfrak{N} = F(m,E,J)$ and by the gravitational potential $\Phi = \Phi_A(r)$. Perturb the equilibrium configuration slightly without destroying its spherical symmetry. The perturbed configuration can be described by a gravitational potential and a distribution function of the forms

$$\Phi(t,x^j) = \Phi_A(r) + \Phi_B(t,r) \,, \tag{7;N}$$

$$\mathfrak{N}(t,x^j,p^j,m) = F(m,E_A,J) + f(t,x^j,p^j,m) \,, \tag{8;N}$$

where

$$E_A = (p^j)^2/2m + m\Phi_A \,, \quad J = |\boldsymbol{x} \times \boldsymbol{p}| \,. \tag{9;N}$$

Notice that f is the perturbation in the distribution function at a fixed point in space $\boldsymbol{x}$, for mixed momentum $\boldsymbol{p}$, and for fixed rest mass m; i.e., it is an "Eulerian perturbation" in phase space.

Throughout this paper, as above, the subscript A will refer to quantities in the unperturbed cluster, and a subscript B will refer to perturbations in those quantities accurate to first order in the amplitude of the motion. Our treatment of stability will not be carried beyond the first order.

The distribution function, $\mathfrak{N}$, for the perturbed cluster must satisfy the Liouville equation (4;N). When the Liouville equation is linearized in the perturbation functions Φ_B and f, it takes the form

$$\partial f/\partial t + \mathfrak{D}_A f - F_E p^r \partial\Phi_B/\partial r = 0 \,. \tag{10;N}$$

Here F_E stands for

$$F_E \equiv (\partial F/\partial E_A)_{m\,J} \,, \tag{11;N}$$

and $\mathfrak{D}_A$ is the Liouville operator of the unperturbed cluster,

$$\mathfrak{D}_A = \frac{p^j}{m}\frac{\partial}{\partial x^j} - m\frac{\partial\Phi_A}{\partial x^j}\frac{\partial}{\partial p^j} \,. \tag{12;N}$$

The derivation of the perturbed Liouville equation (10;N) follows.

The full Liouville equation states:

$$\mathfrak{D}\mathfrak{N} = [\partial/\partial t + \mathfrak{D}_A - m(\partial\Phi_B/\partial x^j)(\partial/\partial p^j)][F + f] = 0 \,. \tag{13a;N}$$

Linearizing in f and Φ_B, and subtracting the zero-order Liouville equation, we obtain

$$\partial f/\partial t + \mathfrak{D}_A f - m(\partial F/\partial p^j)_{x^j,p^j,m}(\partial\Phi_B/\partial x^j) = 0 \,. \tag{13b;N}$$

Since Φ_B depends only on t and $r = |\boldsymbol{x}|$, we have

$$\begin{aligned} m(\partial F/\partial p^j)_{x^j,p^j,m}\partial\Phi_B/\partial x^j &= m(\partial F/\partial p^r)_{x^j,p^j,m}\partial\Phi_B/\partial r \\ &= mF_E(p^r/m)\partial\Phi_B/\partial r \,. \end{aligned} \tag{13c;N}$$

By combining equations (13b,c;N), we obtain equation (10;N). Q.E.D.

The perturbed Liouville equation (10;N) must be supplemented by an equation for Φ_B in terms of f. From the linearity of equations (3;N) one readily sees that the required relation is

$$\nabla^2\Phi_B = 4\pi \int mf d\mathcal{V}_p \,, \tag{14a;N}$$

which has the solution

$$\Phi_B(t,\boldsymbol{x}) = -\int \frac{m'f(t,\boldsymbol{x}',\boldsymbol{p}',m')}{|\boldsymbol{x} - \boldsymbol{x}'|} d\mathcal{V}_{x'} d\mathcal{V}_{p'} \,. \tag{14b;N}$$

Equations (10;N) and (14b;N)—or their analogues for his version of this analysis—are taken by Antonov (1960) to be the equations of motion of the perturbed cluster.[2] Unfortunately, one cannot readily obtain an analogue of equation (14b;N) in general relativity. In order to produce a Newtonian analysis which parallels so far as possible the relativistic analysis, we shall use in place of equation (14b;N) the relation

$$\begin{aligned}\frac{\partial^2\Phi_B}{\partial t\partial r} &= \frac{1}{r^2}\frac{\partial}{\partial t}\begin{pmatrix}\text{mass inside}\\ \text{radius } r\end{pmatrix} = \frac{1}{r^2}(-4\pi r^2)\begin{pmatrix}\text{mass flux in}\\ \text{radial direction}\end{pmatrix}\\ &= -4\pi\int p^r f d\mathcal{V}_p\,,\end{aligned}$$

so that our version of the equations of motion is

$$\partial f/\partial t + \mathfrak{D}_A f - F_E p^r \partial\Phi_B/\partial r = 0\,, \tag{15a;N}$$

$$\partial^2\Phi_B/\partial t\partial r = -4\pi\int p^r f d\mathcal{V}_p\,. \tag{15b;N}$$

d) Equation of Motion for the Odd Part of f

It may seem surprising that the equation of motion (15a;N) is of first order rather than second order. Physical intuition suggests that a perturbed cluster should pulsate, collapse, or explode, and such motions are usually described by hyperbolic second-order differential equations. Actually, a hyperbolic second-order differential equation is hidden in equation (15a;N) and can be extracted by a method due to Antonov (1960):

i) Split the function $f(t,x,p,m)$ into "even" and "odd" parts:

$$\begin{aligned}f_+(t,x,p,m) &= \tfrac{1}{2}[f(t,x,p,m) + f(t,x,-p,m)]\,,\\ f_-(t,x,p,m) &= \tfrac{1}{2}[f(t,x,p,m) - f(t,x,-p,m)]\,.\end{aligned} \tag{16;N}$$

The even part, f_+, is that part which is unaffected by reflections in momentum space ("even parity" in momentum space); the odd part, f_-, is that which changes sign under reflections in momentum space ("odd parity" in momentum space):

$$\begin{gathered}f_+(t,x,-p,m) = f_+(t,x,p,m)\,,\quad f_-(t,x,-p,m) = -f_-(t,x,p,m)\,;\\ f = f_+ + f_-\,.\end{gathered} \tag{17;N}$$

Notice that the even part of f, f_+, determines the star density, the mass density, and the stresses inside the star cluster (these are even moments of f in phase space), while the odd part, f_-, determines the flow of stars, the flow of mass, and the flow of energy (odd moments of f).

ii) Similarly, split equations (15;N) into even and odd parts, noticing in the process that $\mathfrak{D}_A$ is an odd operator (it changes the parity of a function) and that only the odd part of f contributes to the integral in equation (15b;N)

$$\partial f_+/\partial t + \mathfrak{D}_A f_- = 0\,, \tag{18a;N}$$

$$\partial f_-/\partial t + \mathfrak{D}_A f_+ - F_E p^r \partial\Phi_B/\partial r = 0\,, \tag{18b;N}$$

$$\partial^2\Phi_B/\partial t\partial r = -4\pi\int p^r f_- d\mathcal{V}_p\,. \tag{18c;N}$$

[2] One can readily verify that Antonov's equations of motion and all other results of his analysis are valid, not only when F depends on E alone (the case he considered) and not only for spherical perturbations of spherical clusters (the case presented here), but also for those perturbations of any cluster which do not destroy the space symmetries of the equilibrium configuration. For example, his results are valid for all axially symmetric perturbations of a rotating, axially symmetric equilibrium configuration. We do not present the more general treatment here because our motivation is to obtain a Newtonian guide for the relativistic analysis, and in relativity only spherical motions of spherical clusters are free of the difficulties of gravitational radiation.

iii) Differentiate equation (18b;N) with respect to t and combine with equations (18a,c;N), to get

$$(1/F_E)(\partial^2 f_-/\partial t^2) = \mathfrak{J} f_- , \tag{19;N}$$

where $\mathfrak{J}$ is the operator

$$\mathfrak{J} f_- = \frac{\mathfrak{D}_A \mathfrak{D}_A f_-}{F_E} - 4\pi p^r \int p^r f_- d\mathfrak{V}_p . \tag{20;N}$$

Equation (19;N) is the fundamental dynamical equation which governs the pulsation of Newtonian star clusters. Once equation (19;N) has been integrated to give the odd part of f, equation (18a;N) can be solved for the even part, and equation (14b;N) or (18c;N) can be integrated to give Φ_B.

e) Properties of the Equation of Motion; Variational Principles

The dynamical equation (19;N) has a key property which simplifies considerably the study of its solutions: *the operator $\mathfrak{J}$ is self-conjugate for functions which are bounded in phase space.* That is, if h and k are functions which are zero outside some finite region of phase space, then they satisfy

$$\int h \mathfrak{J} k d\mathfrak{V}_p d\mathfrak{V}_x = \int k \mathfrak{J} h d\mathfrak{V}_p d\mathfrak{V}_x = \int \frac{(\mathfrak{D}_A h)(\mathfrak{D}_A k)}{-F_E} d\mathfrak{V}_x d\mathfrak{V}_p - 4\pi \int \left(\int p^r h d\mathfrak{V}_p\right)\left(\int p^r k d\mathfrak{V}_p\right) d\mathfrak{V}_x . \tag{21;N}$$

Proof of equation (21;N): The second term on the right-hand side of equation (21;N) follows trivially from equation (20;N). The first term follows from the fact that $\mathfrak{D}_A$ is anti-self-conjugate for bounded functions u and v

$$\int u \mathfrak{D}_A v d\mathfrak{V}_x d\mathfrak{V}_p = -\int v \mathfrak{D}_A u d\mathfrak{V}_x d\mathfrak{V}_p \tag{22;N}$$

and from the fact that F_E is a function of the integrals of motion of the equilibrium configuration, so that $\mathfrak{D}_A F_E = 0$. That $\mathfrak{D}_A$ is anti-self-conjugate for bounded functions (eq. [22;N]) follows from simple integrations by parts (cf. eq. [12;N]). Q.E.D.

Since $\mathfrak{J}$ is self-conjugate for bounded functions, the dynamical equation (19;N) has a number of well-known and useful properties, *provided only that the star cluster is bounded.*

Property 1: The dynamical equation (19;N) follows from the action principle

$$\delta \int \left[\frac{(\partial f_-/\partial t)^2}{-F_E} - f_- \mathfrak{J} f_-\right] d\mathfrak{V}_p d\mathfrak{V}_x dt = 0 . \tag{23;N}$$

Property 2: Associated with the action principle (23;N) there is a dynamically conserved quantity analogous to pulsational energy:[3]

$$H = \int \left[\frac{(\partial f_-/\partial t)^2}{-F_E} + f_- \mathfrak{J} f_-\right] d\mathfrak{V}_p d\mathfrak{V}_x = \text{constant} . \tag{24a;N}$$

[3] Lynden-Bell (1966, eq. [17]) has previously discussed a conserved quantity similar to expressions (24;N). Rewritten in our notation, his conserved quantity is

$$\epsilon = \tfrac{1}{2} \int \frac{f^2}{-F_E} d\mathfrak{V}_p d\mathfrak{V}_x - \tfrac{1}{2} \int \int \frac{m m' f f'}{|x - x'|} d\mathfrak{V}_p d\mathfrak{V}_x d\mathfrak{V}_{p'} d\mathfrak{V}_{x'} .$$

His conserved quantity, ϵ, can be obtained from ours, H, as follows: Re-express the second term of expression (24b;N) for H in terms of $\partial f/\partial t$ by using eqs. (15b;N) and (14;N). Then simply replace $\partial f/\partial t$ in H by f and divide by 2. The resultant quantity is ϵ. By splitting f into its normal modes and using their

With the help of equations (17;N), (18a;N), and (21;N), we can rewrite this conserved quantity in terms of the full perturbation $f = f_+ + f_-$:

$$H = \int (-1/F_E)(\partial f/\partial t)^2 d\mathcal{V}_p d\mathcal{V}_x - 4\pi \int \left(\int p^r f d\mathcal{V}_p\right)^2 d\mathcal{V}_x \,. \qquad (24b;N)$$

Property 3: If f_- is split up into normal modes

$$f_- = \mathfrak{f}(x^j, p^j, m)e^{i\omega t}\,, \quad f_+ = (i/\omega)\mathfrak{D}_A \mathfrak{f} e^{i\omega t}\,, \qquad (25;N)$$

then the eigenfunctions $\mathfrak{f}$ satisfy the self-conjugate eigenequation

$$(-\omega^2/F_E)\mathfrak{f} = \mathfrak{J}\mathfrak{f}\,, \qquad (26;N)$$

for which there is a variational principle

$$\omega^2 = \frac{\int \mathfrak{f}\,\mathfrak{J}\mathfrak{f} d\mathcal{V}_p d\mathcal{V}_x}{\int (-1/F_E)\mathfrak{f}^2 d\mathcal{V}_p d\mathcal{V}_x}\,. \qquad (27;N)$$

The stationary values of the right-hand side of this equation are the squared eigenfrequencies, ω^2; and the functions $\mathfrak{f}$ which produce those stationary values are the corresponding eigenfunctions.

Property 4: If F_E is negative or zero throughout the phase space of the equilibrium configuration, then the squared eigenfrequencies, ω^2, are all real; i.e , each eigenfrequency is real (stable mode) or imaginary (unstable mode).

Property 5: The eigenfunctions belonging to different eigenfrequencies satisfy the orthogonality relation

$$\int (-1/F_E)\mathfrak{f}_m \mathfrak{f}_n d\mathcal{V}_p d\mathcal{V}_x = 0\,. \qquad (28;N)$$

Property 6: If F_E is negative or zero throughout the phase space of the equilibrium configuration, then that configuration is stable against spherical perturbations if and only if $\mathfrak{J}$ is a positive-definite operator for spherical functions bounded in phase space—i.e., if and only if

$$\int h\,\mathfrak{J} h d\mathcal{V}_p d\mathcal{V}_x > 0 \qquad (29;N)$$

for all non-zero, bounded h. (*Note:* The condition $F_E \leq 0$ will be satisfied by most if not all equilibrium configurations of physical interest, since it states that there are fewer high-energy stars than low-energy stars.)

Most of these properties have been discussed previously by Antonov (1960) for clusters with F a function of E only. However, he did not mention properties 1 and 2 or the variational principle (27;N).

f) Criterion for the Existence of a Zero-Frequency Mode

From the equations of motion in the form (18;N) one can derive an elegant criterion for the existence of a zero-frequency mode: *In a spherically symmetric Newtonian star*

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orthogonality (eq. [28;N]), one can show that the conservation of H implies the conservation of ϵ. Neither Lynden-Bell's conserved quantity nor ours appears to be the pulsational energy of the cluster. Lynden-Bell claims that there is an intimate relation between pulsational energy and his conserved quantity, but his analysis proves only the trivial result that his conserved quantity differs from pulsational energy by a constant. Milder (1967) has also discussed the relation between pulsational energy and the conserved quantity, ϵ, but the physical meaning of his formal mathematical result is unclear to us.

cluster for which $F_E \leq 0$,[4] there exists a zero-frequency mode of spherical, collisionless motion if and only if the following holds: there exists another, slightly different equilibrium configuration such that the difference in distribution functions between the two configurations,

$$\Delta\mathfrak{N}(x^j,p^j,m) = \mathfrak{N}_2(x^j,p^j,m) - \mathfrak{N}_1(x^j,p^j,m)\ , \tag{30;N}$$

satisfies the relation

$$\Delta\mathfrak{N} = \mathfrak{D}_A G \tag{31;N}$$

for some function, G, in phase space. Equivalently, it is necessary and sufficient that

i) *When $\Delta\mathfrak{N}$ is integrated around any closed stellar orbit, $\mathfrak{C}$, in the phase space of the equilibrium configuration, the result is zero:*

$$\int_{\mathfrak{C}} \Delta\mathfrak{N}\, dt = 0\ ; \tag{32a;N}$$

and

ii) *When $\Delta\mathfrak{N}$ is integrated along any possible stellar orbit in phase space which originates outside the cluster and terminates outside the cluster, the result is also zero:*

$$\int_{\mathfrak{C}'} \Delta\mathfrak{N}\, dt = 0\ . \tag{32b;N}$$

Moreover, when a zero-frequency mode is present, it has the form

$$f_+ = (t/\tau)\Delta\mathfrak{N}\ , \quad f_- = -G/\tau\ , \quad \Phi_B = (t/\tau)\Delta\Phi\ , \tag{33;N}$$

where τ is a constant. Hence the zero-frequency mode carries the cluster from one of its two equilibrium configurations to the other during the lapse of time τ.

The significance of this theorem will be discussed in the relativistic section (§ IV*f*).

Proof of the theorem: We first determine the general form for a zero-frequency mode. Any zero-frequency mode must be a finite power series in time, t, for which f_+ vanishes at time $t = 0$:

$$\begin{aligned} f_- &= \alpha_-^{(0)} + \alpha_-^{(1)}t + \ldots + \alpha_-^{(n)}t^n\ , \\ f_+ &= \qquad\quad\ \alpha_+^{(1)}t + \ldots + \alpha_+^{(n)}t^n\ . \end{aligned} \tag{34a;N}$$

The exponent n must be 1 for the following reason. The equations of motion (19;N) and (18a;N) demand that

$$\mathfrak{J}\alpha_-^{(n)} = \mathfrak{J}\alpha_-^{(n-1)} = 0\ , \tag{34b;N}$$

$$n(n-1)\alpha_-^{(n)}/F_E = \mathfrak{J}\alpha_-^{(n-2)}\ , \quad (n-1)(n-2)\alpha_-^{(n-1)}/F_E = \mathfrak{J}\alpha_-^{(n-3)}\ , \tag{34c;N}$$

$$n\alpha_+^{(n)} = -\mathfrak{D}_A\alpha_-^{(n-1)}\ . \tag{34d;N}$$

Multiplying equations (34c;N) by $\alpha_-^{(n)}$ and $\alpha_-^{(n-1)}$, integrating over phase space, and using equations (21;N) and (34b;N), we obtain

$$n(n-1)\int \frac{[\alpha_-^{(n)}]^2}{F_E}\, d\mathcal{V}_x d\mathcal{V}_p = (n-1)(n-2)\int \frac{[\alpha_-^{(n-1)}]^2}{F_E}\, d\mathcal{V}_x d\mathcal{V}_p = 0\ . \tag{34e;N}$$

[4] If we define a zero-frequency mode to be one for which f has the form

$$f_+ = \beta(x^j,p^j)t\ ; \quad f_- = \gamma(x^j,p^j)\ ,$$

then we can drop from the theorem the demand that $F_E \leq 0$.

Suppose $n > 2$. Then equations (34e;N) together with the condition $F_E \leq 0$ and equation (34d;N) tell us that

$$\alpha_+^{(n)} = \alpha_-^{(n)} = \alpha_-^{(n-1)} = 0 \; . \tag{34f;N}$$

Hence n must be ≤ 2. When $n = 2$, the above argument tells us only that $\alpha_-^{(2)} = 0$, but equations (34a;N) and (18b;N) allow us to conclude that $\alpha_-^{(1)} = 0$ as well; and equation (34d;N) then reveals that $\alpha_+^{(2)} = 0$. Consequently, n can only be equal to 1; and the general zero-frequency mode is of the form (33;N).[5]

Next we verify that expression (33;N) represents a zero-frequency motion if and only if $\Delta\mathfrak{N}$ and G satisfy conditions (30;N) and (31;N). Equation (30;N) is equivalent to the statement that $\Delta\mathfrak{N}$ satisfies the perturbed Liouville equation

$$\mathfrak{D}_A \Delta\mathfrak{N} = F_E p^r \partial\Delta\Phi/\partial r \; , \quad \nabla^2(\Delta\Phi) = 4\pi \int m \Delta\mathfrak{N} d\mathcal{V}_p \tag{34g;N}$$

(cf. eq. [10;N] or eqs. [4;N] and [5;N]). Hence equations (30;N) and (31;N) are equivalent to equations (34g;N) and (31;N). On the other hand, expression (33;N) represents a zero-frequency mode if and only if it satisfies the equations of motion (18a,b;N) and (14a;N), which become identical with equations (34g;N) and (31;N) upon manipulation. Q.E.D.

Only condition (32;N) remains to be verified. Equations (32;N) are nothing more than the integrability conditions for the existence of the potential function, G, of equation (31;N). This is because $\mathfrak{D}_A$ is the derivative with respect to time along the unique stellar orbit that goes through a given point in the phase space of the equilibrium configuration. Q.E.D.

g) Relation between Stabilities of Clusters and of Gas Spheres

The variational principles and stability criterion derived in § III*e* will be much more difficult to apply than the corresponding results in the theory of gas spheres For a gas sphere the variational principles and eigenequations involve only one coordinate, r, whereas for clusters the radius r, radial momentum p^r, angular momentum J, and mass m all enter non-trivally. In certain circumstances one may be able to handle the effects of J and m analytically (recall that J and m are conserved along a stellar orbit in the pulsating cluster), but typically one may have to analyze numerically a two-dimensional problem in (r,p^r).

Recently Lynden-Bell (1966) has partially saved us from the pain of two-dimensional numerical analyses by devising a simple one-dimensional criterion for the stability of certain star clusters. Lynden-Bell's criterion has one drawback: it is a sufficient condition for stability but not (so far as we know) a necessary condition. Nevertheless, it should prove extremely useful for many problems.

Lynden-Bell's criterion for the special case of spherical clusters with isotropic velocity distributions (F independent of J)[6] says this: *Consider a bounded, spherically symmetric Newtonian cluster with isotropic velocity distribution and with $F_E \leq 0$. Such a cluster is stable against collisionless, spherical perturbations if the gas sphere with the same radial distributions of density,*

$$\rho = \int m F d\mathcal{V}_p = 4\pi \int m^2 [2m(E_A - m\Phi_A)]^{1/2} F dE_A dm \; , \tag{35a;N}$$

and of pressure,

$$\begin{aligned} P &= \int (p^r p^r/m) F d\mathcal{V}_p = \tfrac{1}{2} \int (J^2/mr^2) F d\mathcal{V}_p \\ &= (4\pi/3) \int [2m(E_A - m\Phi_A)]^{3/2} F dE_A dm \; , \end{aligned} \tag{35b;N}$$

[5] Antonov (1960) concluded incorrectly that zero-frequency modes with $n = 2$ are possible.

[6] Lynden-Bell proves his theorem in a somewhat more general context, but here we are concerned only with spherical clusters.

is stable against radial perturbations for which the "adiabatic index" is

$$\Gamma_1 = \frac{\rho}{P}\frac{dP/dr}{d\rho/dr} . \tag{35c;N}$$

Since it is a simple one-dimensional problem to determine whether a gas sphere is stable, this theorem gives us a simple, one-dimensional, sufficient criterion for the stability of a spherical cluster.

Paragraph added July 3, 1968.—Recent discussions between Donald Lynden-Bell and James R. Ipser, motivated in part by remarks of Edward Lee, have revealed that Lynden-Bell's (1966) proof of this theorem was incorrect.[7] However, a new, corrected proof of the theorem has been devised by Lynden-Bell (paper in preparation), and a relativistic version of the theorem has been proved by Ipser (to be published in Paper II of this series).

IV RELATIVISTIC THEORY OF STABILITY

We now develop the relativistic generalization of our Newtonian discussion of stability. Our treatment follows as closely as possible the corresponding Newtonian treatment, with the corresponding equations being given similar numbers (e.g , eq [1;R] corresponds to eq. [1;N]).

a) Equations of Stellar Dynamics

In general relativity the density of stars in phase space, which we denote by $\mathfrak{N}$, is defined as follows: we concentrate attention on those stars near a particular event, x, in spacetime with 4-momenta near a particular value, p. As seen in the rest frame of these stars, they occupy a particular three-dimensional volume, $d\mathcal{V}_x$, in physical space and a particular four-dimensional volume, $d\mathcal{V}_p$, in momentum space In terms of a general curvilinear coordinate system, $d\mathcal{V}_x$ and $d\mathcal{V}_p$ are given by

$$d\mathcal{V}_x = (p^0/m)\sqrt{(-g)}dx^1dx^2dx^3 ; \quad d\mathcal{V}_p = -dp_0dp_1dp_2dp_3/\sqrt{(-g)} . \tag{1;R}$$

Here p^α and p_α are the contravariant and covariant components of the 4-momentum, g is the determinant of the metric tensor, and $m = (p_\alpha p^\alpha)^{1/2}$ is the rest mass of a star with 4-momentum p. If there are dN stars in the volume $d\mathcal{V}_x d\mathcal{V}_p$, then the number density in phase space ("distribution function") is given by

$$\mathfrak{N} \equiv dN/d\mathcal{V}_x d\mathcal{V}_p = dN/(-dx^1dx^2dx^3dp_1dp_2dp_3dm) . \tag{2;R}$$

The density $\mathfrak{N}$ is a function of location (x,p) in eight-dimensional phase space. Through part of our discussion we shall use as coordinates in phase space general curvilinear spacetime coordinates, x^α, and the "conjugate" covariant components of the 4-momentum, p_α. However, we shall sometimes employ other sets of coordinates, for example, (x^α,p_j,m)[8] and coordinates specially adapted to spherical symmetry.

The smoothed-out gravitational field of the star cluster is described by the metric

[7] The error lies in the argument showing that positive-definiteness of the Lynden-Bell operator

$$S \equiv -\frac{\nabla^2}{4\pi} - \rho\frac{d\rho/dr}{dP/dr}$$

is a *necessary* condition for stability of the gas sphere. It is *not* necessary for stability.

[8] Greek indices run from 0 to 3; Latin indices, from 1 to 3.

tensor, $g_{\alpha\beta}(x)$. The distribution function determines a smoothed-out stress-energy tensor through the equations

$$T_\alpha{}^\beta = \int p_\alpha p^\beta (\mathfrak{N}/m) d\mathcal{V}_p \,, \tag{3a;R}$$

and that stress-energy tensor determines the metric, $g_{\alpha\beta}$, through Einstein's equations,

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \,. \tag{3b;R}$$

The gravitational field in turn determines the distribution function through the collisionless Boltzmann-Liouville equation (or "Liouville equation")

$$\mathfrak{D}\mathfrak{N} = 0 \,. \tag{4;R}$$

Here $\mathfrak{D}$, the Liouville operator, is differentiation with respect to proper time along the path of a star in phase space:

$$\mathfrak{D} = \frac{dx^\alpha}{ds}\frac{\partial}{\partial x^\alpha} + \frac{dp_\alpha}{ds}\frac{\partial}{\partial p_\alpha} = \frac{p^\alpha}{m}\frac{\partial}{\partial x^\alpha} - \frac{1}{2m}\frac{\partial g^{\mu\nu}}{\partial x^\alpha} p_\mu p_\nu \frac{\partial}{\partial p_\alpha} \,. \tag{5;R}$$

Equations (3;R) and (4;R), which couple $g_{\alpha\beta}$ and $\mathfrak{N}$, are the fundamental equations of relativistic stellar dynamics.

b) Spherical Equilibrium Configurations

In general relativity, as in Newtonian theory, the distribution function for an equilibrium configuration depends only on the integrals of the motion. For spherical symmetry the relevant integrals of the motion are the rest mass m, the "energy at infinity," $E \equiv p_0$, and the total angular momentum J; hence we have

$$\mathfrak{N} = F(m,E,J) \,. \tag{6a;R}$$

When F is independent of J, the cluster has an isotropic velocity distribution at each point in space We shall use the "Schwarzschild coordinate system" (t,r,θ,ϕ) to describe spherical equilibrium configurations. In this coordinate system the gravitational field is described by

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \,, \tag{6b;R}$$

where ν and λ are functions of r, and the angular momentum and "energy at infinity" are given by

$$J = [p_\theta{}^2 + (p_\phi/\sin\theta)^2]^{1/2} \,, \quad E = p_0 \,. \tag{6c;R}$$

The theory of spherically symmetric equilibrium configurations has been developed in great detail by Fackerell (1966; 1968*a–c*). Independently Zel'dovich and Podurets (1965) have treated the restricted problem of a cluster of identical stars with a truncated, isotropic Maxwell-Boltzmann velocity distribution—i.e., a cluster with

$$\begin{aligned} F &= Ae^{-E/T}\delta(m - m_0) && \text{if } E < E_0 \\ &= 0 && \text{if } E > E_0 \,. \end{aligned}$$

c) Equation of Motion for a Perturbed Spherical Star Cluster

If a spherically symmetric equilibrium configuration is perturbed in a spherical manner, and if Schwarzschild coordinates are adopted for the perturbed configuration as for the unperturbed configuration, then the perturbed gravitational field is described by

$$ds^2 = e^{\nu_A(r)+\nu_B(t,r)}dt^2 - e^{\lambda_A(r)+\lambda_B(t,r)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \,. \tag{7;R}$$

As in the Newtonian case, so also here, we work only to first order in the perturbation quantities ν_B and λ_B.

The radial coordinate, r, in equation (7;R) is defined uniquely by the demand that $4\pi r^2$ be the area of an invariant sphere about the center of symmetry. The time coordinate, t, is also defined uniquely if we insist that the perturbed metric (7;R) become the static Schwarzschild metric outside the cluster (Birkhoff's theorem). Consequently, there is no coordinate arbitrariness in the functions ν_A, ν_B, λ_A, and λ_B.

In defining the perturbation of the distribution function, we must decide how to identify points in the phase space of the perturbed cluster with points in the phase space of the unperturbed cluster. There is a variety of possibilities: We could identify points with the same Schwarzschild coordinates, x^α, and with the same covariant components of the momentum, p_α, so that

$$\mathfrak{N} = \mathfrak{N}_A(x^\alpha,p_\alpha) + \mathfrak{N}_B(x^\alpha,p_\alpha) .$$

Alternatively, we could use contravariant components of the momentum, p^α, in making the identification:

$$\mathfrak{N} = \mathfrak{N}_A(x^\alpha,p^\alpha) + \mathfrak{N}_B(x^\alpha,p^\alpha) .$$

Either of these choices is reasonable on mathematical grounds, but from a physical standpoint it is preferable to identify points with the same Schwarzschild coordinates, x^α, and the same *physical components* of the momentum, $p_{(\alpha)} = |g^{\alpha\alpha}|^{1/2}p_\alpha$. (See Table 1.)

TABLE 1

PHYSICAL COMPONENTS OF THE 4-MOMENTUM*

Component	Value in Equilibrium Configuration	Value in Perturbed Configuration
$p_{(0)}$ . .	$p_0 \exp(-\nu_A/2) = p^0 \exp(\nu_A/2)$	$p_0 \exp[-(\nu_A+\nu_B)/2] = p^0 \exp[(\nu_A+\nu_B)/2]$
$p_{(r)}$ . . .	$p_r \exp(-\lambda_A/2) = -p^r \exp(\lambda_A/2)$	$p_r \exp[-(\lambda_A+\lambda_B)/2] = p^r \exp[(\lambda_A+\lambda_B)/2]$
$p_{(\theta)}$	$p_\theta r^{-1} = -p^\theta r$	$p_\theta r^{-1} = -p^\theta r$
$p_{(\phi)}$	$p_\phi(r \sin\theta)^{-1} = -p^\phi(r\sin\theta)$	$p_\phi(r\sin\theta)^{-1} = -p^\phi(r\sin\theta)$

* The physical components are the projections of p on an orthonormal tetrad with legs in the t, r, θ, and ϕ directions (See, e.g , chap ii of Thorne 1967)

This is the type of identification which observers using proper reference frames or locally inertial reference frames would make, and it leads to a formalism which is considerably simpler than the other choices. With this choice of identification of points in phase space, the distribution function of the perturbed cluster takes the form

$$\mathfrak{N} = F(m,E_A,J) + f(x^\alpha,p_{(\alpha)}) , \tag{8;R}$$

where

$$m = [p_{(0)}^2 - p_{(r)}^2 - p_{(\theta)}^2 - p_{(\phi)}^2]^{1/2} , \quad E_A = p_{(0)}e^{\nu_A/2}, \quad J = r[p_{(\theta)}^2 + p_{(\phi)}^2]^{1/2} . \tag{9;R}$$

The Liouville equation which this distribution function obeys takes the following form when linearized in the perturbations ν_B, λ_B, and f:

$$\frac{p^0}{m}\frac{\partial f}{\partial t} + \mathfrak{D}_A f - \tfrac{1}{2}\frac{p_0}{m}F_E p^r\frac{\partial \nu_B}{\partial r} + \frac{1}{2m}F_E p_r p^r \frac{\partial\lambda_B}{\partial t} = 0 . \tag{10;R}$$

Here F_E stands for

$$F_E \equiv (\partial F/\partial E_A)_{m,J} , \tag{11;R}$$

and $\mathfrak{D}_A$ is the Liouville operator of the unperturbed cluster, which has the form

$$\mathfrak{D}_A = \frac{p^j}{m}\frac{\partial}{\partial x^j} - \frac{1}{2m}\frac{\partial g_A{}^{\mu\nu}}{\partial x^j} p_\mu p_\nu \frac{\partial}{\partial p_j} \tag{12;R}$$

when (x^α, p_α) are used as coordinates in phase space. (Note that we must use care in the choice of coordinates only while defining f. Now that f has been defined explicitly, we are free to use whatever coordinates we wish in manipulating it, except that *we shall demand that our coordinates leave the equilibrium configuration explicitly static.*)

The derivation of the perturbed Liouville equation (10;R) follows.

For the purpose of the derivation we shall use as coordinates in phase space the Schwarzschild space coordinates (t,r,θ,ϕ), the physical zero component of the momentum, $p_{(0)}$, the angular momentum, J, and the rest mass of a star, $m = [p_{(0)}{}^2 - (J/r)^2 - p_{(r)}{}^2]^{1/2}$. The rest mass, m, is used in place of $p_{(r)}$; and J is used in place of *both* $p_{(\theta)}$ and $p_{(\phi)}$. (This is possible because spherical symmetry guarantees that $\mathfrak{N}$ can depend on $p_{(\theta)}$ and $p_{(\phi)}$ only through J.) The full Liouville equation for a dynamical, spherically symmetric cluster is

$$\frac{p^\alpha}{m}\frac{\partial \mathfrak{N}}{\partial x^\alpha} + \frac{dp_{(0)}}{ds}\frac{\partial \mathfrak{N}}{\partial p_{(0)}} + \frac{dJ}{ds}\frac{\partial \mathfrak{N}}{\partial J} + \frac{dm}{ds}\frac{\partial \mathfrak{N}}{\partial m} = 0\,. \tag{13a;R}$$

Because J and m are integrals of the motion in a dynamical, spherical cluster, we have

$$dJ/ds = dm/ds = 0\,. \tag{13b;R}$$

The change in $p_{(0)}$ along a star's world line, as calculated from the geodesic equation, is

$$dp_{(0)}/ds = (1/2m)e^{-\nu/2}[p_r p^r(\partial\lambda/\partial t) - p_0 p^r(\partial\nu/\partial r)]\,. \tag{13c;R}$$

Consequently, the Liouville equation (13a;R) reads

$$\begin{aligned}\frac{p_{(0)}e^{-\nu/2}}{m}\frac{\partial \mathfrak{N}}{\partial t} - \frac{p_{(r)}e^{-\lambda/2}}{m}\frac{\partial \mathfrak{N}}{\partial r} + \frac{1}{2m}\Big[&- e^{-\nu/2}p_{(r)}p_{(r)}\frac{\partial\lambda}{\partial t} \\ &+ e^{-\lambda/2}p_{(0)}p_{(r)}\frac{\partial\nu}{\partial r}\Big]\frac{\partial \mathfrak{N}}{\partial p_{(0)}} = 0\,.\end{aligned} \tag{13d;R}$$

When we split into unperturbed and perturbed parts,

$$\mathfrak{N} = F + f\,, \quad \nu = \nu_A + \nu_B\,, \quad \lambda = \lambda_A + \lambda_B\,, \tag{13e;R}$$

and linearize in the perturbation, this becomes

$$\begin{aligned}&\frac{p^0}{m}\frac{\partial f}{\partial t} + \mathfrak{D}_A f - \frac{p^r}{m}\frac{\lambda_B}{2}\left(\frac{\partial F}{\partial r}\right)_{t,p_{(0)}\,J\,m} \\ &+ \frac{e^{-\nu_A/2}}{2m}\left[p_r p^r\frac{\partial\lambda_B}{\partial t} + p_0 p^r\left(\tfrac{1}{2}\frac{\partial\nu_A}{\partial r}\lambda_B - \frac{\partial\nu_B}{\partial r}\right)\right]\left(\frac{\partial F}{\partial p_{(0)}}\right)_{t\,r\,J\,m} = 0\,.\end{aligned} \tag{13f;R}$$

The value of F depends on $p_{(0)}$ and r only through $E_A = p_{(0)}e^{\nu_A/2}$ when J and m are held fixed. Consequently,

$$\left(\frac{\partial F}{\partial r}\right)_{t\,p_{(0)},J,m} = \tfrac{1}{2}p_0\frac{\partial\nu_A}{\partial r}F_E\,, \quad \left[\frac{\partial F}{\partial p_{(0)}}\right]_{t\,r\,J\,m} = e^{\nu_A/2}F_E\,. \tag{13g;R}$$

When equations (13g;R) are combined with equation (13f;R), the perturbed Liouville equation (10;R) results. Q.E.D.

The perturbed Liouville equation (10;R) must be supplemented by equations for ν_A and ν_B in terms of f. The required relations are the perturbations of Einstein's field equations (3;R). The perturbation in the stress-energy tensor, which enters into the field equations, is

$$T_{B\alpha}{}^{\beta} = \int p_\alpha p^\beta (f/m) d\mathcal{V}_p ,$$

and the perturbation in the Einstein tensor is the same as that used by Chandrasekhar (1964) in studying the radial pulsation of gas spheres. By combining the perturbed Einstein and stress-energy tensors, one obtains three useful field equations:

$$\partial\lambda_B/\partial t = -8\pi r e^{\lambda_A} \int p_0 p^r (f/m) d\mathcal{V}_p , \tag{14a;R}$$

$$(\partial/\partial r)(r e^{-\lambda_A}\lambda_B) = 8\pi r^2 \int p_0 p^0 (f/m) d\mathcal{V}_p , \tag{14b;R}$$

$$\partial\nu_B/\partial r = (\partial\nu_A/\partial r + 1/r)\lambda_B - 8\pi r e^{\lambda_A} \int p_r p^r (f/m) d\mathcal{V}_p . \tag{14c;R}$$

Equations (10;R) and (14;R) are the equations of motion for the perturbed cluster. These four equations for f, ν_B, and λ_B are not all independent. The Liouville equation (10;R), when combined with equation (14a;R), can be made to yield (14b;R); when combined with (14b;R), it yields (14a;R).

Equations (10;R) and (14;R) are not the most useful forms for the equations of motion. Rather, it is convenient to remove ν_B and $\partial\lambda_B/\partial t$ from equation (10;R) by use of (14a,c;R), and to take the resultant equation along with (14a;R) as coupled equations for f and λ_B:

$$\frac{\partial f}{\partial t} + \mathcal{B} f - \frac{p_0}{p^0}\left(1 + r\frac{\partial \nu_A}{\partial r}\right) F_E p^r \frac{\lambda_B}{2r} = 0 , \tag{15a;R}$$

$$\frac{\partial}{\partial t}\left(\frac{\lambda_B}{2r}\right) = -4\pi e^{\lambda_A} \int \frac{p_0}{m} p^r f d\mathcal{V}_p . \tag{15b;R}$$

Here $\mathcal{B}$ is the operator in phase space,

$$\mathcal{B}\psi \equiv \frac{m}{p^0}\mathfrak{D}_A\psi + 4\pi r e^{\lambda_A}\frac{m}{p^0} F_E\left(\frac{p_0 p^r}{m}\int \frac{p_r p^r}{m}\psi d\mathcal{V}_p - \frac{p_r p^r}{m}\int\frac{p_0 p^r}{m}\psi d\mathcal{V}_p\right) . \tag{15c;R}$$

d) Equation of Motion for the Odd Part of f

In order to convert the equations of motion (15;R) into self-adjoint, hyperbolic, second-order form, we follow the Newtonian procedure of splitting them into even and odd parts. Such a split in general is not Lorentz-invariant in momentum space, because the parity is defined in terms of inversions of the *space* part of the 4-momentum ($p_{(0)}$ is not inverted); and the space part of p is not a Lorentz-invariant entity. Fortunately, this need not disturb us. The static nature of the unperturbed geometry provides us with preferred time directions in both physical space and momentum space. In the pulsating cluster, the preferred time directions are well defined to zero order in the perturbations—which is sufficiently well defined for our purposes—and they are automatically embodied in the coordinate system $(t,r,\theta,\phi,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)})$ which we are using.

Consequently, without any loss of generality, and without any introduction of arbitrariness into the analysis, we can define the even and odd parts of f as

$$\begin{aligned} f_+(x,p) &= \tfrac{1}{2}[f(x,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)}) + f(x,p_{(0)},-p_{(r)},-p_{(\theta)},-p_{(\phi)})] , \\ f_-(x,p) &= \tfrac{1}{2}[f(x,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)}) - f(x,p_{(0)},-p_{(r)},-p_{(\theta)},-p_{(\phi)})] . \end{aligned} \tag{16;R}$$

As in Newtonian theory, f_+ and f_- have even and odd parities in the spatial part of momentum space, and their sum is f:

$$f_+(x,p_{(0)},-p_{(r)},-p_{(\theta)},-p_{(\phi)}) = f_+(x,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)}) \,,$$
$$f_-(x,p_{(0)},-p_{(r)},-p_{(\theta)},-p_{(\phi)}) = -f_-(x,p_{(0)},p_{(r)},p_{(\theta)},p_{(\phi)}) \,, \qquad (17;\mathrm{R})$$
$$f = f_+ + f_- \,.$$

If we split equations (15;R) into even and odd parts, noticing in the process that $\mathcal{B}$ is an odd operator and that only the odd part of f contributes to the integral in equation (15b;R), we obtain the equations

$$(\partial f_+/\partial t) + \mathcal{B}f_- = 0 \,, \qquad (18a;\mathrm{R})$$

$$\frac{\partial f_-}{\partial t} + \mathcal{B}f_+ - \frac{p_0}{p^0}\left(1 + r\frac{\partial \nu_A}{\partial r}\right)F_E p^r \frac{\lambda_B}{2r} = 0 \,, \qquad (18b;\mathrm{R})$$

$$\frac{\partial}{\partial t}\left(\frac{\lambda_B}{2r}\right) = -4\pi e^{\lambda_A}\int \frac{p_0}{m}\, p^r f_- d\mathcal{V}_p \,. \qquad (18c;\mathrm{R})$$

Finally, if we differentiate equation (18b;R) with respect to t and combine it with equations (18a,c;R), we obtain the desired hyperbolic second-order differential equation

$$(1/F_E)(\partial^2 f_-/\partial t^2) = \mathfrak{J} f_- \,. \qquad (19;\mathrm{R})$$

Here $\mathfrak{J}$ is the operator

$$\mathfrak{J} f_- = \frac{1}{F_E}\mathcal{B}\mathcal{B}f_- - 4\pi\left(1 + r\frac{\partial \nu_A}{\partial r}\right)e^{\lambda_A}\frac{m}{p^0}\frac{p_0}{m}\, p^r \int \frac{p_0}{m}\, p^r f_- d\mathcal{V}_p \,. \qquad (20;\mathrm{R})$$

Equation (19;R) is the fundamental dynamical equation which governs the pulsation of relativistic star clusters. Once it has been integrated to give f_-, equation (18a;R) can be solved for f_+, and the field equations (14;R) can be solved for λ_B and ν_B.

e) Properties of the Equation of Motion; Variational Principles

The operator $\mathfrak{J}$, like its Newtonian counterpart, is self-conjugate for functions which are bounded in phase space. That is, if h and k are spherically symmetric functions which are zero outside some finite region of phase space, then they satisfy

$$\int h\,\mathfrak{J}k\, d\mathcal{V}_p d\mathcal{V}_x = \int k\,\mathfrak{J}h\, d\mathcal{V}_p d\mathcal{V}_x = \int \frac{(\mathcal{B}h)(\mathcal{B}k)}{(-F_E)}\, d\mathcal{V}_x d\mathcal{V}_p$$
$$- 4\pi\int\left(1 + r\frac{\partial \nu_A}{\partial r}\right)e^{\lambda_A}\left(\int \frac{p_0}{m}\, p^r h\, d\mathcal{V}_p\right)\left(\int \frac{p_0}{m}\, p^r k\, d\mathcal{V}_p\right)\frac{m}{p^0}\, d\mathcal{V}_x \,. \qquad (21;\mathrm{R})$$

Proof of equation (21;R): the second term of the right-hand side follows directly from definition (20;R) of $\mathfrak{J}$ and from expression (1;R) for $d\mathcal{V}_x$ and $d\mathcal{V}_p$. The first term follows, once we have verified that $\mathcal{B}$ is anti-self-conjugate with the weighting function $1/F_E$, i.e., once we have shown that, for bounded, spherical u and v,

$$\int (1/F_E)\, u\mathcal{B}v\, d\mathcal{V}_p d\mathcal{V}_x = -\int (1/F_E)\, v\mathcal{B}u\, d\mathcal{V}_p d\mathcal{V}_x \,. \qquad (22a;\mathrm{R})$$

Equation (22a;R) is readily verified from definition (15c;R) of $\mathcal{B}$, once it is recognized that $\mathfrak{D}_A$ is also anti-self-conjugate, but with the weighting function (m/p^0),

$$\int \frac{m}{p^0}\, u\mathfrak{D}_A v\, d\mathcal{V}_p d\mathcal{V}_x = -\int \frac{m}{p^0}\, v\mathfrak{D}_A u\, d\mathcal{V}_p d\mathcal{V}_x \,, \qquad (22b;\mathrm{R})$$

and that $\mathfrak{D}_A F_E = 0$. Relation (22b;R) follows from integration by parts plus simple manipulations, if (x^α, p_a) are used as the coordinates in phase space. Note that with this choice of coordinates $\mathfrak{D}_A$ has the form (12;R), $d\mathcal{V}_x$ and $d\mathcal{V}_p$ have the form (1;R), and the relation

$$\frac{\partial}{\partial x^j}\left(\frac{p^j}{m}\right) - \frac{\partial}{\partial p_j}\left(\frac{1}{2m}\frac{\partial g_A{}^{\mu\nu}}{\partial x^j}\, p_\mu p_\nu\right) = 0 \tag{22c;R}$$

is satisfied. Q.E.D.

Since $\mathfrak{Z}$ is self-conjugate for bounded functions, the dynamical equation (19;R) has the same types of well-known and useful properties as its Newtonian analogue (19;N), *provided only that the star cluster is bounded.*

Property 1: The dynamical equation (19;R) follows from the action principle

$$\delta \int \left[\frac{(\partial f_-/\partial t)^2}{-F_E} - f_- \mathfrak{Z} f_-\right] d\mathcal{V}_p d\mathcal{V}_x dt = 0 . \tag{23;R}$$

Property 2: Associated with the action principle (23;R) there is a dynamically conserved quantity

$$H = \int \left[\frac{(\partial f_-/\partial t)^2}{-F_E} + f_- \mathfrak{Z} f_-\right] d\mathcal{V}_p d\mathcal{V}_x = \text{constant} . \tag{24a;R}$$

With the help of equations (17;R), (18a;R), and (21;R), we can rewrite this conserved quantity in terms of the full perturbation $f = f_+ + f_-$:

$$H = \int \frac{(\partial f/\partial t)^2}{-F_E} d\mathcal{V}_p d\mathcal{V}_x - 4\pi \int \left(1 + r\frac{\partial \nu_A}{\partial r}\right) e^{\lambda_A} \left(\int \frac{p_0}{m} p^r f d\mathcal{V}_p\right)^2 \frac{m}{p^0} d\mathcal{V}_x . \tag{24b;R}$$

Property 3: If f_- is split up into normal modes,

$$f_- = \mathfrak{f}(x^j, p_a) e^{i\omega t} , \quad f_+ = (i/\omega)\mathfrak{B}\mathfrak{f} e^{i\omega t} , \tag{25;R}$$

then the eigenfunctions $\mathfrak{f}$ satisfy the self-conjugate eigenequation

$$(-\omega^2/F_E)\mathfrak{f} = \mathfrak{Z}\mathfrak{f} , \tag{26;R}$$

for which there is a variational principle

$$\omega^2 = \frac{\int \mathfrak{f}\, \mathfrak{Z}\mathfrak{f} d\mathcal{V}_p d\mathcal{V}_x}{\int (-1/F_E)\mathfrak{f}^2 d\mathcal{V}_p d\mathcal{V}_x} \tag{27;R}$$

analogous to the Newtonian variational principle (27;N).

Property 4: If F_E is negative or zero throughout the phase space of the equilibrium configuration, then the squared eigenfrequencies, ω^2, are all real; i.e., each eigenfrequency is real (stable mode) or imaginary (unstable mode).

Property 5: The eigenfunctions belonging to different eigenfrequencies satisfy the orthogonality relation

$$\int (-1/F_E)\mathfrak{f}_m \mathfrak{f}_n d\mathcal{V}_p d\mathcal{V}_x = 0 . \tag{28;R}$$

Property 6: If F_E is negative or zero throughout the phase space of the equilibrium configuration, then that configuration is stable against spherical perturbations if and only if $\mathfrak{Z}$ is a positive-definite operator for spherical functions bounded in phase space:

$$\int h \mathfrak{Z} h d\mathcal{V}_p d\mathcal{V}_x > 0 . \tag{29;R}$$

f) Criterion for the Existence of a Zero-Frequency Mode

Zel'dovich and Podurets (1965) and Zel'dovich and Novikov (1967) have argued that one should be able to diagnose the stability of isothermal star clusters from binding-energy curves, in much the same way as one diagnoses the stability of isentropic stellar models from such curves (Fowler 1964; Bardeen 1965; Thorne 1967, § 4.1.4). This seems highly unlikely to us, because in isothermal clusters one must, in some arbitrary manner, introduce a cutoff at high energies in the distribution function, and this cutoff must be chosen uniquely for each central density in order to produce a one-parameter binding-energy curve (cf. end of § IV*b*). Only a very special choice of the cutoff—which choice is not yet known—could lead to a binding-energy criterion for stability, and perhaps no choice will work.

That the situation in star clusters is much more complicated than that in stars is indicated also by the following theorem, which is the direct generalization of our Newtonian theorem of § III*f:*

In a spherically symmetric, relativistic star cluster for which $F_E \leq 0$ (see n. 4, p. 259), *there exists a zero-frequency mode of spherical, collisionless motion if and only if the following holds: there exists another, slightly different equilibrium configuration such that the difference in distribution functions* (at fixed *physical* components of the momentum) *between the two configurations,*

$$\Delta\mathfrak{N} = \mathfrak{N}_2(r,\theta,\phi,p_{(r)},p_{(\theta)},p_{(\phi)},p_{(0)}) - \mathfrak{N}_1(r,\theta,\phi,p_{(r)},p_{(\theta)},p_{(\phi)},p_{(0)}) \,, \tag{30;R}$$

satisfies the relation

$$\Delta\mathfrak{N} = \mathcal{B}G \tag{31;R}$$

for some function, G, in phase space. Equivalently (*integrability condition for eq.* [*31;R*]), *it is necessary and sufficient that*

i) *When one integrates the following quantity around any closed stellar orbit,* $\mathcal{C}$, *in the phase space of the equilibrium configuration, the result is zero:*

$$\oint_{\mathcal{C}} [\Delta\mathfrak{N} + \tfrac{1}{2}(F_E/p^0)p_r p^r \Delta\lambda]dt = 0 \,; \tag{32a;R}$$

and

ii) *When the same quantity is integrated along any possible stellar orbit in phase space which originates outside the cluster and terminates outside the cluster, the result is also zero:*

$$\int_{\mathcal{C}'} [\Delta\mathfrak{N} + \tfrac{1}{2}(F_E/p^0)p_r p^r \Delta\lambda]dt = 0 \,. \tag{32b;R}$$

Moreover, when a zero-frequency mode is present, it has the form

$$f_+ = (t/\tau)\Delta\mathfrak{N} \,, \quad f_- = -G/\tau \,, \quad \nu_B = (t/\tau)\Delta\nu \,, \quad \lambda_B = (t/\tau)\Delta\lambda \,, \tag{33;R}$$

where τ *is a constant. Hence the zero-frequency mode carries the cluster from one of its two equilibrium configurations to the other during the lapse of time* τ.

Proof of the theorem: We first verify that any zero-frequency mode must have the general form (i.e., time dependence) of expressions (33;R). This is done by precisely the same procedure as was used in the Newtonian analysis (eqs. [34a–f;N]).

Next we verify that expression (33;R) represents a zero-frequency motion if and only if $\Delta\mathfrak{N}$ and G satisfy conditions (30;R) and (31;R). Equation (30;R) is equivalent to the statement that $\Delta\mathfrak{N}$ satisfies the perturbed Liouville equation

$$\mathcal{B}\Delta\mathfrak{N} - \frac{p_0}{p^0}\left(1 + r\frac{\partial\nu}{\partial r}\right)F_E p^r \frac{\Delta\lambda}{2r} = 0 \,. \tag{34a;R}$$

(cf. eq. [15a;R]). Hence equations (30;R) and (31;R) are equivalent to equations (34a;R) and (31;R). On the other hand, expression (33;R) represents a zero-frequency mode if and only if it satisfies the equations of motion (18a,b;R) and (14b,c;R), which upon manipulation become identical with equations (34a;R), (31;R), and the perturbed field equations for $\Delta\lambda$ and $\Delta\nu$. Q.E.D.

Condition (32;R) remains to be verified. Equation (31;R), when combined with definition (15c;R) of $\mathcal{B}$, with the form (33;R) of the zero-frequency mode, and with the field equation (15b;R), becomes

$$(m/p^0)\mathfrak{D}_A G = \Delta\mathfrak{N} + \tfrac{1}{2}(F_E/p^0)p_r p^r \Delta\lambda\,. \qquad \text{(34b;R)}$$

The operator $(m/p^0)\mathfrak{D}_A$ is the derivative with respect to coordinate time along the unique stellar orbit that goes through a given point in the phase space of the equilibrium configuration. Consequently, equations (32) are the integrability conditions for the potential function G. Q.E.D.

The criteria for zero-frequency modes provided by this theorem are quite elegant conceptually, but without some sort of extension they are useless for numerical calculations. This theorem can be compared to the statement that a hot stellar model possesses a zero-frequency mode if and only if there exists another, slightly different model with identically the same chemical composition, rest mass, and binding energy and with the same distribution of entropy. In the stellar case, the demand for identical entropy distributions provides an infinity of constraints analogous to the constraints (31;R) or (32;R) for clusters. In the stellar case, we know how to simplify the stability criterion by looking only at isentropic configurations (Bardeen 1965; Thorne 1967, § 4.1.4). Perhaps future thought will reveal an analogous simplification for star clusters.

V. CONCLUSION

In this paper we have reviewed and extended the tools available for analyzing the collisionless stability of Newtonian star clusters, and we have derived a number of analogous tools for studying relativistic star clusters. One of the authors (J. R. I.) is now using these tools to study numerically the onset of the relativistic instability in spherical star clusters. It is hoped that the numerical analyses (which will be reported in a sequel to this paper) will yield improved understanding of possible processes in the nuclei of galaxies and of the Fowler-Hoyle star-cluster model for QSSs.

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