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INVERSION OF FAN-BEAM SCANS IN RADIO ASTRONOMY*

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ABSTRACT

A classical problem of data reduction in radio astronomy is how to reconstruct a two-dimensional brightness distribution over a source from fan-beam scans taken in various position angles. The integral equation underlying the problem arises also in interferometry and in the reduction of data from occultations. A new solution to the integral equation for the case of infinitely narrow fan beams has been obtained, which makes use of numerical operations on the observational data in the sky plane and does not require the computation of Fourier transforms.

I. INTRODUCTION

Let $f(x,y)$ describe the brightness distribution over a radio source which is supposed to be of such limited angular extent that there is no objection to using rectangular coordinates x,y on the patch of sky concerned.

A radio telescope having the form of a long, narrow aperture will receive radiation from a narrow, long strip of sky. Let us say that the axis L of this strip passes through or near the radio source, at a distance R from the origin of coordinates and in a direction described by the angle θ in Figure 1.

The amount of power received by the radio telescope is proportional to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, g(x,y)\, dx\, dy\,,$$

where $g(x,y)$ is the gain of the radio telescope in the direction (x,y) and the integrals are conventionally shown with infinite limits on the grounds that the integrand quickly becomes negligible with increasing x and y in the case of highly directive antennas.

Now let

$$g(x,y) = A(x\cos\theta + y\sin\theta - R)B(y\cos\theta - x\sin\theta)\,.$$

In this equation, A describes the cross-section of the reception pattern of the radio telescope beam in the narrow direction as a function of the distance $x\cos\theta + y\sin\theta - R$ of a point (x,y) from the axis L. The function B describes the cross-section in the broad direction, but it will be assumed in what follows that the source is so small in angular extent in comparison with the width of the reception pattern in the broad direction that the factor B may be taken as constant.

Then the received power is proportional to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, A(x\cos\theta + y\sin\theta - R)\, dx\, dy\,.$$

When, as a result of Earth's rotation or adjustments to the radio telescope, the distance R changes while θ remains fixed, one says that the source has been scanned and the record of received power obtained as a function of R is known as a strip scan or strip integral in position angle θ.

The problem of how to extract the original brightness distribution $f(x,y)$ from strip

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scans in various position angles θ arose quite early in radio astronomy with strip scans of the Sun (Christiansen and Warburton 1953, 1955). A discussion of the theory (Bracewell 1956) showed that (i) a finite number of position angles sufficed to obtain all the available information, (ii) that $f(x,y)$ could not be recovered but that a "principal solution" (Bracewell and Roberts 1954) existed, and (iii) that the resolution obtainable in two dimensions was the full resolution allowed in one dimension by the narrow cross-section A.

The present paper is concerned with a new method of reconstruction, but the question of restoration for smoothing caused by convolution with the cross-section A will be completely separated from the problem of reconstruction. The problem we consider first is: find $f(x,y)$, given

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x\cos\theta + y\sin\theta - R)\,dx\,dy \tag{1}$$

for all R and θ.

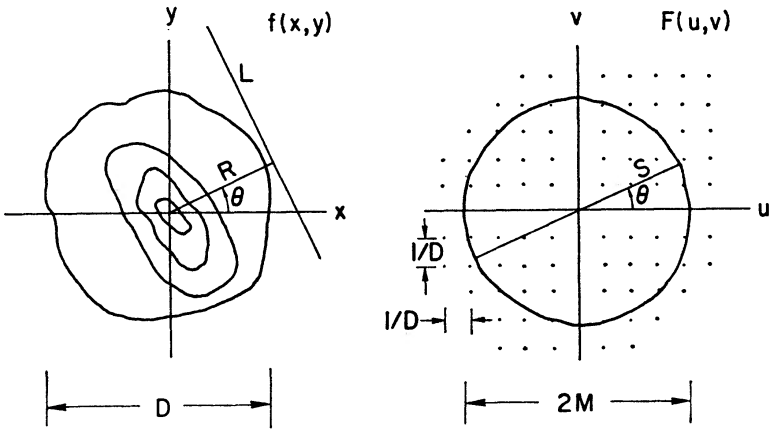


FIG. 1.—A source distribution $f(x,y)$ of outer diameter D and its two-dimensional Fourier transform $F(u,v)$.

This integral, in which the narrow cross-section A becomes a delta function, deals with strip scans in their limiting form as line scans. In this case it has been shown that the solution is unique (Bracewell 1956).

This limiting problem is referred to as the problem of reconstruction. Although it refers to infinite resolving power and an infinite number of position angles θ, it nevertheless provides a convenient theoretical basis for the treatment of actual observations.

The integral equation (1) may also arise in the reduction of observations made with multielement or variable-spacing interferometers (aperture supersynthesis) and in the preparation of a brightness distribution from records of lunar occultations.

II. RECONSTRUCTION THEOREM

The theorem for reconstruction of $f(x,y)$ from line scans is this; if

$$f_\theta(R) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x\cos\theta + y\sin\theta - R)\,dx\,dy,$$

where $f_\theta(R)$ denotes the line scan of $f(x,y)$ in position angle θ, then

$$f(x,y) = \int_0^{\pi} h_\theta(x\cos\theta + y\sin\theta)\,d\theta,$$

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where the function h_θ is derived from the corresponding line scan f_θ by the relation

$$h_\theta(w) = f_\theta(w) - \int_{-\infty}^{\infty} f_\theta(w-w')M \operatorname{sinc}^2(Mw')\,dw';$$

M is a spatial frequency which is not exceeded by any Fourier component of $f(x,y)$; and

$$\operatorname{sinc} w \triangleq \frac{\sin \pi w}{\pi w}.$$

This result may be derived as follows. Let the two-dimensional Fourier transform of $f(x,y)$ be $F(u,v)$, i.e.,

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}dx\,dy$$

and in terms of polar coordinates ρ,θ on the uv-plane let

$$F_\theta(\rho) \equiv F(u,v),$$

where

$$u = \rho\cos\theta \quad \text{and} \quad v = \rho\sin\theta.$$

As a preliminary step we show that f_θ and F_θ are a one-dimensional Fourier transform pair. (It is sufficient to show that f_0 and F_0 are a transform pair because of a property of the two-dimensional Fourier transformation that if $f(x,y)$ and $F(u,v)$ are a two-dimensional transform pair then $f(x',y')$ and $F(u',v')$ are also, where the primes indicate coordinates that are rotated equally in both planes.) Consider the Fourier transform of $f_0(R)$. Since

$$f_\theta(R) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x\cos\theta + y\sin\theta - R)\,dx\,dy,$$

$$f_0(R) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x-R)\,dx\,dy = \int_{-\infty}^{\infty} f(R,y)\,dy.$$

Taking the Fourier transform of $f_0(R)$,

$$\int_{-\infty}^{\infty} f_0(R)\, e^{-i2\pi\rho R}dR = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(R,y)\, e^{-i2\pi\rho R}dy\,dR.$$

But

$$F_\theta(\rho) = F(\rho\cos\theta,\rho\sin\theta)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(\rho x\cos\theta+\rho y\sin\theta)}dx\,dy.$$

Therefore

$$F_0(\rho) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi\rho x}dx\,dy,$$

which has just been shown to be the Fourier transform of $f_0(R)$.

We now assume that

$$F_\theta(\rho) = 0, \qquad |\rho| > M.$$

This is not a restrictive assumption, for with any given radio telescope there exists a critical spatial frequency such that Fourier components of the source distribution are unobservable if their spatial frequency exceeds the critical value. (The cutoff is quite sharp

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and has been discussed quantitatively by Bracewell 1962.) The limiting value M may be taken equal to or greater than the cutoff value pertaining to the available line-scan data. Then

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{i2\pi(ux+vy)}\, du\, dv$$

$$= \int_0^{\pi}\int_{-M}^{M} F_\theta(\rho)\, e^{i2\pi\rho(x\cos\theta + y\sin\theta)}\, |\rho|\, d\rho\, d\theta$$

$$= \int_0^{\pi}\int_{-\infty}^{\infty} F_\theta(\rho)\left[\Pi\left(\frac{\rho}{2M}\right) - \Lambda\left(\frac{\rho}{M}\right)\right] e^{i2\pi\rho(x\cos\theta + y\sin\theta)}\, d\rho\, d\theta$$

$$= \int_0^{\pi}\int_{-\infty}^{\infty} F_\theta(\rho)\, K(\rho)\, e^{i2\pi\rho(x\cos\theta + y\sin\theta)}\, d\rho\, d\theta,$$

where

$$\Pi(\rho) = \begin{cases} 1 & |\rho| < \frac{1}{2} \\ 0 & |\rho| > \frac{1}{2} \end{cases}$$

$$\Lambda(\rho) = \begin{cases} 1 - |\rho| & |\rho| < 1 \\ 0 & |\rho| > 1 \end{cases}$$

and

$$K(\rho) = \Pi\left(\frac{\rho}{2M}\right) - \Lambda\left(\frac{\rho}{M}\right).$$

Now the rectangle function $\Pi(\rho)$ is the Fourier transform of sinc R and the triangle function $\Lambda(\rho)$ is the Fourier transform of $\text{sinc}^2 R$:

$$\int_{-\infty}^{\infty} \Pi(\rho)\, e^{i2\pi\rho R}\, d\rho = \text{sinc}\, R,$$

$$\int_{-\infty}^{\infty} \Lambda(\rho)\, e^{i2\pi\rho R}\, d\rho = \text{sinc}^2 R,$$

as can be shown immediately by direct integration, and

$$\int_{-\infty}^{\infty} K(\rho)\, e^{i2\pi\rho R}\, d\rho = 2M\ \text{sinc}\ (2MR) - M\ \text{sinc}^2\ (MR) = k(R).$$

By the convolution theorem for Fourier transforms,

$$\int_{-\infty}^{\infty} F_\theta(\rho)\, K(\rho)\, e^{i2\pi\rho w}\, d\rho = \int_{-\infty}^{\infty} f_\theta(w-w')\, k(w')\, dw'$$

$$= \int_{-\infty}^{\infty} f_\theta(w-w')\, 2M\ \text{sinc}\ (2Mw')\, dw'$$

$$- \int_{-\infty}^{\infty} f_\theta(w-w')\, M\ \text{sinc}^2\ (Mw')\, dw'$$

$$= f_\theta(w) - \int_{-\infty}^{\infty} f_\theta(w-w')\, M\ \text{sinc}^2\ (Mw')\, dw'.$$

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The first term on the right-hand side assumes the simple form shown because convolution with $2M$ sinc (Mw) is a selection operation which removes Fourier components with spatial frequencies exceeding M. Now $f(x,y)$ has no such components, hence no line scan f_θ can have them; consequently removal of the non-existent from f_θ leaves it unchanged. Since we have shown above that

$$f(x,y) = \int_0^\pi \int_{-\infty}^{\infty} F_\theta(\rho) K(\rho) e^{i2\pi\rho(x\cos\theta + y\sin\theta)} d\rho \, d\theta ,$$

it now follows that

$$f(x,y) = \int_0^\pi h_\theta(x\cos\theta + y\sin\theta)\, d\theta , \tag{2}$$

where

$$h_\theta(w) = f_\theta(w) - \int_{-\infty}^{\infty} f_\theta(w - w') M \operatorname{sinc}^2(Mw')\, dw' ,$$

which was to be proved.

III. INTERPRETATION

According to equation (2) of the previous section, line scans f_θ are to be modified by subtracting a term derived from the scan by convolution with a sinc^2 function. The modified function is called h_θ. Then the value of $f(x,y)$ at each point (x,y) is obtained by superposition of values of $h_\theta(x\cos\theta + y\sin\theta)$, an expression which, for any given θ, may be pictured as representing a brightness distribution on the xy-plane whose isolines are all parallel to the line L in Figure 1 and whose transverse section is h_θ.

IV. FINITE NUMBER OF EQUALLY SPACED SCANS

The scans constituting the data were supposed to be available at infinitesimal intervals of θ, but in practice let there be N scans at an average interval π/N in position angle. Then we are led to consider

$$\sum_{i=1}^{N} a_i h_{\theta_i}(x\cos\theta_i + y\sin\theta_i) \tag{3}$$

as an approximate solution, where

$$\theta_i = \frac{i\pi}{N} \quad \text{and} \quad a_i = \frac{\pi M}{N} .$$

The number of line scans necessary to reconstruct a source of diameter not exceeding D, up to a limiting spatial frequency M, can be estimated as follows. The source distribution can be determined in full if its Fourier transform $F(u,v)$ is known on a square lattice of points spaced $1/D$ (Bracewell 1958). Since f_θ and F_θ are a Fourier transform pair it follows that a line scan in position angle θ is equivalent to knowledge of $F(u,v)$ along the line S which makes an angle θ with the u-axis (Fig. 1). Line scans in N equally spaced position angles will furnish information in the uv-plane along N equally spaced radial lines whose separation at a radius M reaches $\pi M/N$. If this spacing is not to exceed $1/D$, then we require that

$$N > \pi M D .$$

The principal question to be discussed is the degree of approximation obtained when the integral in the reconstruction theorem is approximated by a sum. To avoid details of source distributions we consider a source distribution $(M/r)J_1(2\pi Mr)$ which is a unit

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point source from which all spatial frequencies exceeding M have been removed, and which, if reconstructed by the theorem, will be recovered in full. When we reconstruct by the approximation sum we call the result the "equivalent reception pattern." We can then inquire how well the equivalent pattern agrees with the ideal.

Let

$$f(x,y) = \frac{MJ_1(2\pi Mr)}{r}.$$

Then

$$f_{\theta_i}(R) = 2M \text{ sinc } (2MR) \qquad \text{(all } i\text{)},$$

and

$$h_{\theta_i}(R) = 2M \text{ sinc } (2MR) - M \text{ sinc}^2 (MR) = k(R) \qquad \text{(all } i\text{)}.$$

Substituting in equation (3) for a_i, θ_i, and h_{θ_i}, we have for equivalent reception pattern

$$b(x,y) = \sum_{i=1}^{N} \frac{\pi M}{N} k(x \cos \theta_i + y \sin \theta_i).$$

If we fix MD at any value exceeding unity, then we find by numerical examination that the discrepancy

$$\left| b(x,y) - \frac{M}{r} J_1(2\pi Mr) \right|$$

increases roughly as $(r/D)^{2N}$ to a value at $r = D/2$ that is always less than 1 per cent of the central value of $(M/r)J_1(2\pi Mr)$ for all N greater than πMD. However, even if N exceeds only $\pi MD/2$, the discrepancy proves to be quite small ($< 130/(2N + 1)$ per cent).

V. UNEQUALLY SPACED SCANS

When the scans are unequally spaced in angle there is reason to expect from the one-dimensional analogue that the same number of scans will suffice as for equal spacing. However, when the spacing departs seriously from uniform spacing, deterioration of the solution arises through a phenomenon of magnification of errors (Bracewell 1965). The following proposal is expected to be satisfactory when the available scans do not leave gaps in position angle that are too wide, and carries with it its own evaluation of the error committed.

Let scans h_{θ_i} be available at $\theta = \theta_1, \theta_2, \ldots, \theta_N$. Put

$$a_i^{-1} \propto \sum_{n=1}^{N} \exp[-N^2(\theta_i - \theta_n)^2/6], \tag{4}$$

where $\theta_i - \theta_n$ is measured in such a sense that its absolute value is less than π and choose the constant of proportionality so that

$$\sum_i a_i = \pi M.$$

The right-hand side is simply an arbitrary measure of the density of scans per unit of position angle.

Now the proposed reconstruction is

$$\sum_{i=1}^{N} a_i h_{\theta_i}(x \cos \theta_i + y \sin \theta_i).$$

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The error is the same as that which would result if the same source distribution had been mapped by a pencil beam

$$\sum_{i=1}^{N} a_i k (x \cos \theta_i + y \sin \theta_i),$$

which can be calculated and recorded.

As an example we consider scans of the moon taken at the position angles permitted by the east-west and north-south arrays of the Stanford 9.1 cm interferometer (Bracewell and Swarup 1961) over the period March 4 to April 1, 1961 (Cudaback 1962).

As the peculiar interval determined by the array length is 1.28′ and the Moon's semidiameter is 15′, then we have $\pi MD = 2\pi(\frac{1}{2} \times 1.28) \times 15 = 37$. Thus thirty-seven scans will suffice. However, if the lunar brightness distribution is symmetrical about the equator, so that a scan at angle θ is the same as a scan at angle $-\theta$, then half as many scans should suffice. Scans in effectively fourteen position angles were available.

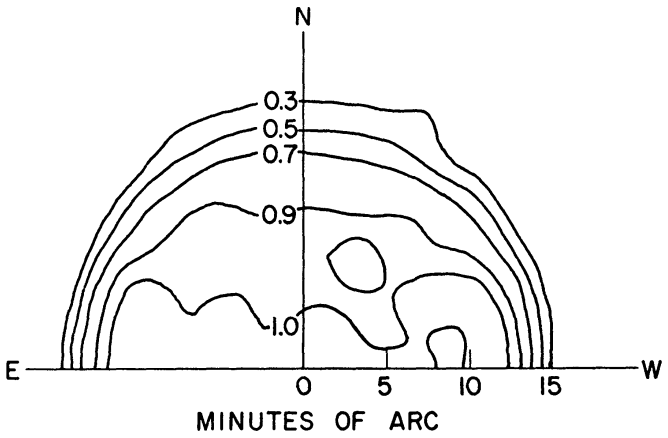


FIG. 2.—Moon brightness at 9.1 cm reconstructed from fan-beam scans to a peculiar interval of 1′.28

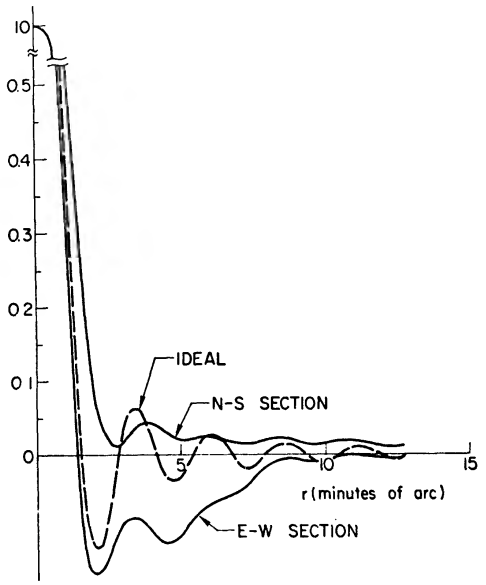


FIG. 3.—Principal sections of equivalent reception pattern compared with the ideal

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The brightness distribution shown in Figure 2 was reconstructed as a test of the method. The result cannot be compared with direct pencil beam observations of the Moon, because the resolution exceeds that available at 9 cm from the largest parabolic reflectors, but the general character is in agreement with expectation and the significance of the irregular features present can be studied by recourse to the equivalent reception pattern, shown in two principal sections in Figure 3.

VI. FEW UNEQUALLY SPACED SCANS

It may happen that the number of available scans is too small to permit reconstruction of a given source to the spatial frequency achieved on the strip scans. This is often the case with lunar occultations. One approach would be to degrade the strip scans by convolution with the appropriate sinc function (convolution with a Gaussian function will result in loss of information which, depending on the signal-to-noise ratio, may or may not be significant). Alternatively, we propose that the data be used directly by substitution in equation (3) with coefficients a_i determined from equation (4), and that the resulting map and the associated equivalent reception pattern be made the basis of a choice of model with parameters to be fixed by recourse to the data. This procedure recognizes the success of models in interpreting lunar occultations and the need for selecting a model whose parameters are strongly delimited by the data.

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