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THE DYNAMICAL STATE OF THE INTERSTELLAR GAS AND FIELD

II. NON-LINEAR GROWTH OF CLOUDS AND FORCES IN THREE DIMENSIONS*

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ABSTRACT

The growth of the two-dimensional instability of a gas in a gravitational field $\boldsymbol{g}$ confining a horizontal magnetic field $\boldsymbol{B}$ is followed into the non-linear regime to provide an illustration of the formation of discrete gas clouds from a more uniform medium. A shock forms as the gas falls together, dissipating the energy and making the process irreversible.

The effective force field of an element of gas (suspended on the lines of force of a horizontal magnetic field) is calculated in three dimensions. The force field is made up of an attractive force of the same form as gravity but stronger by the factor g^2/GB^2, and a force which extends undiminished along the lines of force passing near the element of gas. The latter dominates the former at large distances and is attractive on the lines of force through, and above and below, the element of gas, and repulsive along the lines of force passing beside the element of gas. The over-all effect is a strong attraction, accompanied by vigorous streaming of gas both toward and away from an established cloud. Clouds are expected to accumulate gas in such a way as to increase their vertical dimension relative to the transverse dimension $\boldsymbol{g} \times \boldsymbol{B}$.

I. INTRODUCTION

It was demonstrated earlier (Parker 1966, hereafter referred to as "Paper I"; Lerche 1967*a*) that the pressure of the galactic magnetic field and cosmic rays both produce an instability in the interstellar gas. The process is basically a Rayleigh-Taylor instability, with the interstellar gas draining into the low places along the magnetic lines of force. It was suggested that this effect is largely responsible for the observed clumping of interstellar gas into clouds. The effect is stronger than self-gravitation by the factor $g^2/GB^2 \cong$ 5–10, where g is the acceleration of gravity perpendicular to the disk of the galaxy and B is the galactic field density. It follows that the dynamical behavior of the general interstellar gas is determined largely by g, B, and the cosmic-ray pressure P, rather than by self-gravitation. Hence, in order to understand the dynamical behavior of the interstellar gas and its evolution into stars, etc., it is necessary to understand the nature of the g, B, P forces.

In Paper I a linear perturbation analysis in two dimensions ($\boldsymbol{g}$, $\boldsymbol{B}$) was employed to demonstrate the instability of a number of equilibrium configurations, illustrating the tendency for the gas to depress the lines of force and accumulate in the newly created low places. But, of course, a linear analysis can show only the beginning of the departure from equilibrium. The next step is to examine the dynamics of the system when the departure is large. In this connection it was shown in Paper I that in two dimensions any two small condensed elements of gas supported in a large-scale field B slide along the lines of force under the influence of a force which has the same form as the two-dimensional gravitational attraction between the elements, but is larger by the factor g^2/GB^2. It follows that in two dimensions the non-linear dynamical behavior of a tenuous gas threaded by a large-scale field B is the same as one would expect from self-gravitation but more vigorous. Hence, in two dimensions, the formation and contraction of gas

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clouds is expected to proceed rapidly in the manner generally understood for self-gravitation.[1]

The present paper carries the investigation further through formal mathematical solution of a number of idealized examples, each illustrating some aspect of the general dynamical properties of the real interstellar gas-field system. The real system involves the simultaneous interplay of several effects at once, so that quantitative treatment is not possible here. The present goal is to understand as many of the basic principles as possible, on which more quantitative empirical models can be built later.

A question of no little importance is whether the observed interstellar gas clouds form by accretion of fragments of clouds left over from earlier generations, or whether completely new clouds are formed from more tenuous gas distributions through the onset of the instability described in Paper I. The formal examples which we work out explore both possibilities. Section II follows the onset of a particularly simple unstable mode into the region of finite amplitudes, illustrating how a cloud of gas might be formed from a more uniform atmosphere. Section IV works out the forces between small condensed elements of gas, applying them in § V to a cloud of finite size to show how such a cloud might accrete gas from the surrounding interstellar space.

II. DYNAMICS IN TWO DIMENSIONS

The general solution of the dynamical equations in two dimensions is beyond the scope of the present paper, but there is a class of solution which, though idealized, illustrates the clumping of the gas following the onset of instability. Consider the simple case of a cold conducting gas of density $\rho(z)$ and velocity $\boldsymbol{v}$ in a gravitational field $-\boldsymbol{e}_z g$ and an initial magnetic field $\boldsymbol{e}_y B(z)$, where $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are unit vectors in the y- and z-directions. The gas velocity has y- and z-components, with the result that additional y- and z-components of the magnetic field appear, so that in terms of the vector potential $\boldsymbol{e}_x A(y,z,t)$ the total y-component of the field is represented by $B(z) + \partial A/\partial z$ and the z-component is $-\partial A/\partial y$. The dynamical equations are

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial y}\,\rho\, v_y+\frac{\partial}{\partial z}\,\rho\, v_z=0\,, \tag{1}$$

$$\frac{\partial A}{\partial t}+v_y\frac{\partial A}{\partial y}+v_z\frac{\partial A}{\partial z}=-\,v_z B(z)\,, \tag{2}$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{1}{4\pi}\frac{\partial A}{\partial y}\left(\frac{dB}{dz}+\nabla^2 A\right), \tag{3}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{1}{4\pi}\left(B+\frac{\partial A}{\partial z}\right)\left(\frac{dB}{dz}+\nabla^2 A\right)-\rho g\,. \tag{4}$$

Now we are interested primarily in the dynamical behavior of the simple system which has the equilibrium state $\boldsymbol{v} = A = 0$,

$$\frac{B^2(z)}{B^2(0)}=\frac{\rho(z)}{\rho(0)}=\exp\left(-\frac{2z}{\Lambda}\right), \tag{5}$$

where, if the gas is cold, Λ is the scale height $V_A{}^2/g$ and V_A is the Alfvén speed $B(0)/[4\pi\rho(0)]^{1/2}$. For this system consider the special class of solutions in which the initial z-dependence is preserved:

$$\rho(y,z,t)=\rho(0)F(y,t)\exp\left(-\frac{2z}{\Lambda}\right), \tag{6}$$

[1] Self-gravitation becomes important, of course, after the clouds become very dense, or for clouds on the central plane of the galaxy where $g \cong 0$.

$$A(y,z,t) = B(0)\Lambda G(y,t)\exp\left(-\frac{z}{\Lambda}\right), \tag{7}$$

$$B(z) = B(0)\exp\left(-\frac{z}{\Lambda}\right). \tag{8}$$

Then v_y and v_z are independent of z and the equations reduce to

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial y} F v_y = \frac{2}{\Lambda} v_z F, \tag{9}$$

$$\frac{\partial G}{\partial t} + v_y \frac{\partial G}{\partial y} = v_z (G - 1) \tag{10}$$

$$F\left(\frac{\partial v_y}{\partial t} + v_y \frac{\partial v_y}{\partial y}\right) = g\Lambda \frac{\partial G}{\partial y}\left(1 - G - \Lambda^2 \frac{\partial^2 G}{\partial y^2}\right), \tag{11}$$

$$F\left(\frac{\partial v_z}{\partial t} + v_y \frac{\partial v_z}{\partial y}\right) = g(1 - G)\left(1 - G - \Lambda^2 \frac{\partial^2 G}{\partial y^2}\right) - gF. \tag{12}$$

These equations may be integrated easily if the scale of the variation in the y-direction is small compared to the scale height, i.e., they may be integrated if $k\Lambda \gg 1$. For then the weight of the gas is distributed uniformly along the magnetic lines of force except on the very smallest scale, $1/k$. The lines of force are deflected but little from their horizontal equilibrium, even in the extreme case that the gas is compressed into thin vertical sheets. The sheets are so closely spaced. Hence $B_z \ll B$. It follows that G and $\Lambda \partial G/\partial y$ are both small compared to 1. The gas motions are along the lines of force, which being nowhere far from horizontal, implies that $|v_z|/|v_y| \cong \Lambda \partial G/\partial y \ll 1$. Then the inertia of v_z can be neglected and the vertical forces, gravity and magnetic pressure, remain near equilibrium. The self-consistency of this approximation will be demonstrated further on. We will find that, while B_z, represented by the first derivative $\partial G/\partial y$, is small, as pointed out above, the gradient of the field $\partial B_z/\partial y$, represented by $\partial^2 G/\partial y^2$, may become very large where the gas is compressed into sheets.

The mathematical procedure is, then, to drop v_z from equations (9), (11), and (12) so that G is determined by the requirement for vertical equilibrium, obtained from equation (12) by ignoring the left-hand side. It follows from equation (12) that G is determined by

$$\Lambda^2 \frac{\partial^2 G}{\partial y^2} = 1 - F. \tag{13}$$

Equation (10) then gives the small values of v_z associated with the slight vertical readjustment as the gas condenses into thin vertical sheets.

Let

$$L(y,t) \equiv \int_0^y dy F(y,t), \tag{14}$$

where it is assumed that $y = 0$ is a point of symmetry, where $v_y = 0$. Equation (9) becomes

$$\frac{\partial L}{\partial t} + v_y \frac{\partial L}{\partial y} = 0 \tag{15}$$

after integrating over y. The integration constant is zero because of the choice of limits on the integral for L. Equation (13) can be integrated to give

$$\frac{\partial G}{\partial y} = \frac{1}{\Lambda^2}(y - L). \tag{16}$$

The integration constant is again zero because we have chosen $y = 0$ to be a point of symmetry where $v_y = 0$. Equation (11) can be written as

$$\frac{\partial v_y}{\partial t} + v_y \frac{\partial v_y}{\partial t} = \frac{V_A{}^2}{\Lambda^2}(y - L) \tag{17}$$

with the aid of equations (13) and (16), and the fact that F is non-vanishing.

Equations (15) and (17) are sufficient[2] to determine L and v_y. From the Lagrangian point of view equation (15) becomes

$$\frac{dL}{dt} = 0 , \tag{18}$$

so that L is constant for an observer moving with the fluid. If y_0 denotes the initial ($t = 0$) position of the fluid at y at time t, write $y = y(y_0,t)$ and $L = L(y_0)$. Then equation (17) becomes

$$\frac{d^2 y}{dt^2} = \frac{V_A{}^2}{\Lambda^2}[y - L(y_0)] . \tag{19}$$

If the initial value of dy/dt is denoted by $v(y_0)$, the solution is

$$y(y_0,t) = y_0 + [L(y_0) - y_0]\left(1 - \cosh \frac{V_A t}{\Lambda}\right) + \frac{v(y_0)\Lambda}{V_A} \sinh \frac{V_A t}{\Lambda}, \tag{20}$$

at time t. The velocity is

$$\frac{dy}{dt} = v(y_0) \cosh \frac{V_A t}{\Lambda} - \frac{V_A}{\Lambda}[L(y_0) - y_0] \sinh \frac{V_A t}{\Lambda} . \tag{21}$$

The density $F(y,t)$ follows as

$$F = \frac{\partial y_0}{\partial y} \frac{dL}{dy_0} . \tag{22}$$

The static equilibrium solution occurs for $v(y_0) = 0$ and $L(y_0) = y_0$. It is evident from the time dependence of the solution that the equilibrium is unstable, as demonstrated in Paper I from the linear analysis. An initial perturbation is represented by $v(y_0)$ and $L(y_0) - y_0$, which can have any arbitrary single-valued form. A particularly simple case is the initial density distribution $L(y_0) = y_0$ perturbed by the small velocity $v(y_0) = v_0 \sin ky_0$, with $k\Lambda \gg 1$. Then

$$\frac{\partial y_0}{\partial y} = \frac{1}{1 + (v_0/V_A)k\Lambda \cos ky_0 \sinh V_A t/\Lambda} \tag{23}$$

and

$$\rho(y,t) = \rho(y_0) \frac{\partial y_0}{\partial y} . \tag{24}$$

It is evident that $y(y_0,t)$ is a single-valued function of y_0 for all values of t up to

$$\alpha(t) \equiv \frac{v_0 k\Lambda}{V_A} \sinh \frac{V_A t}{\Lambda} = 1 ,$$

where $\partial y_0/\partial y$ becomes infinite at $ky_0 = (2n + 1)\pi$. The density $\rho(y,t)$ is plotted in Figure 1 for $\alpha(t) = 0, 0.5, 0.9$, and 1.0. The velocity profile is shown in Figure 2 in units of $v_0 \cosh (V_A t/\Lambda)$. The steepening of the wave is evident in Figure 2, together with the divergence of the density in Figure 1 as $\alpha(t)$ approaches 1. It is evident that a backward

[2] Eq. (10), $v_z \cong -dG/dt$, serves to determine v_z once L and v_y are known.

propagating shock starts from the vicinity of $ky = \pi$ just prior to $\alpha(t) = 1$, giving a density profile sketched by the broken line in Figure 1 for times a little later than $\alpha(t) = 1$. The simple solution (20) continues to be valid after $\alpha(t) = 1$ in the region between $y = 0$ and the shock front, but is invalid to the right of the shock.

The formation of the shock guarantees entropy production so that the system cannot return to its initial state.

At large values of t we suppose that the kinetic energy of the motions has been largely

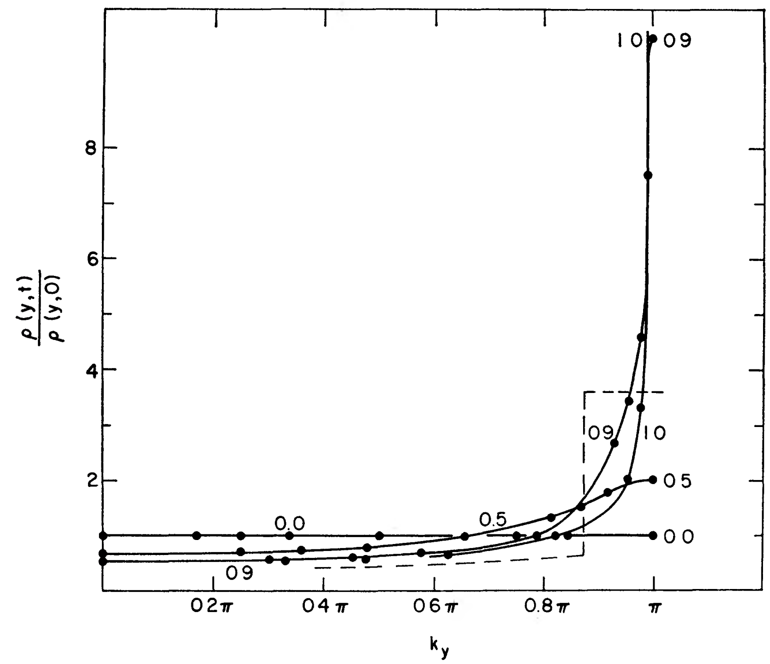

FIG. 1.—Plot of the density $\rho(y,z,t)/\rho(y,z,0)$ as a function of ky The time for each curve is given by the value of $\alpha(t) = (v_0 k\Lambda/V_A) \sinh V_A t/\Lambda = 0, 0.5, 0\,9$, and 1.0 in the diagram The dots on the curves are markers which move with the fluid, having started at $ky/\pi = 0, \frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{8}, 1$.

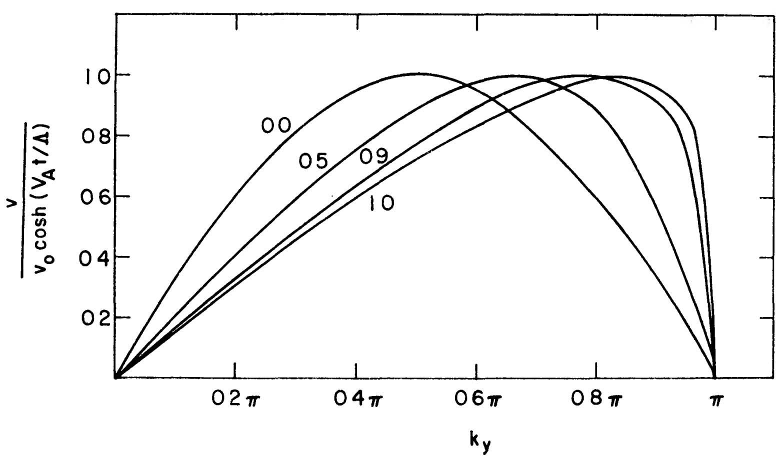

FIG. 2.—Plot of the velocity profile in units of $v_0 \cosh (V_A t/\Lambda)$. The time for each curve is given by the value of $\alpha(t) = (v_0 k\Lambda/V_A) \sinh (V_A t/\Lambda)$ marked in the diagram The velocity unit grows approximately linearly with $\alpha(t)$ for large values of t.

dissipated by the shocks into thermal energy, which has in turn been radiated away, leaving the gas cold, as it was initially. The final equilibrium solution is then a periodic array of vertical sheets:[3]

$$F(y)=\frac{2\pi}{k}\sum_{n=-\infty}^{+\infty}\delta[y-(2n+1)\pi] \tag{25}$$

with

$$G(y)=\frac{(y-2n\pi/k)^2}{2\Lambda^2} \tag{26}$$

in $(2n-1)\pi < ky < (2n+1)\pi$. Thus $B_y \simeq B_0 \exp(-z/\Lambda)$ and

$$B_z=B_0\left(\frac{ky-2n\pi}{k\Lambda}\right)\exp\left(-\frac{z}{\Lambda}\right). \tag{27}$$

The lines of force are given by the one-parameter family

$$z-z_1=-\frac{1}{2\Lambda}\left(y-\frac{2n\pi}{k}\right)^2$$

for the line of force passing through $z=z_1$ at its highest point $y=2n\pi/k$.

It is a simple matter now to demonstrate that v_z can in fact be neglected when $k\Lambda \gg 1$, as argued above on physical grounds. For the example given, $L(y_0)=y_0$, $v(y_0)=v_0 \sin ky_0$, it is evident from equations (14) and (22) that $\partial^2G/\partial y^2$ is determined by the $\partial y_0/\partial y$ given in equation (23). By writing $\partial^2G/\partial y^2$ as $(\partial y_0/\partial y)\partial^2G/\partial y_0\partial y$ it follows that

$$\Lambda^2\frac{\partial^2 G}{\partial y_0\partial y}=a(t)\cos ky_0 .$$

Hence, integrate over y_0 and note that the gas is uniformly distributed along the lines of force when $t=0$, so that $B_z=0$ then, and note from symmetry that $B_z=0$ at $ky_0=n\pi$ for all $t\geq 0$; it follows that

$$\Lambda\frac{\partial G}{\partial y}=-\frac{a(t)}{k\Lambda}\sin ky_0 .$$

Thus, even though $\partial^2G/\partial y^2$ becomes extremely large in the vicinity of $ky_0=(2n+1)\pi$ as $a(t)\rightarrow 1$, it is evident that $\partial G/\partial y$ is small, $O(1/k\Lambda)$.

This simple solution is an illustration of the manner in which a cold gas whose weight confines a large-scale magnetic field falls into clumps. More general illustrations can be constructed, in which the gas is not entirely cold and in which the characteristic wavelength along the field is not necessarily small compared to the scale height. But their construction is not without labor, and it is our feeling that the complexities which appear with the introduction of the third coordinate, perpendicular to the gB plane, make their construction of little value.

III. ENERGY STATES IN TWO DIMENSIONS

The energy of the magnetic field can be computed in the initial homogeneous atmosphere, described by equation (5), and again in the final state, such as equations (25)–(27). The reduction of the energy is responsible for the forces which drive the gas into vertical sheets or slabs.

[3] The general non-linear transformation of an initial atmosphere into vertical sheets has been treated by Lerche (1967*b*). Lerche (1967*c*) has shown that the equilibrium is unstable to horizontal displacements of the vertical sheets. Hence, if the sheets are perturbed, they will collect together into more massive sheets, the energy again being dissipated by shocks, etc.

Suppose that the gas is isothermal so that in the initial state the pressure and density are related by the constant thermal velocity u as $p(z) = u^2\rho(z)$. The scale height in equation (5) is then $\Lambda = (2u^2 + V_A{}^2)/g$. Let $z = 0$ represent the base of the system, the central plane of the galaxy, so to speak. It is readily shown that the initial energy per unit horizontal area is $B^2(0)\Lambda/8\pi$ above $z = 0$.

The reduction in magnetic and cosmic-ray energy as a result of clumping of the gas into slabs is readily demonstrated. Imagine that the magnetic field is held in its initial horizontal configuration by a suitable array of demons while the gas is slowly compressed into vertical slabs of thickness $2a$, separated by intervals of $2b$, by sliding the gas along the lines of force. The configuration, including the straight lines of force, is illustrated in Figure 3. Suppose that the slabs of gas occupy the regions $(2n + 1)b + 2na < y < (2n + 1)b + (2n + 2)a$ where n is an integer running from $-\infty$ to $+\infty$. Some small amount of work is done in compressing the interstellar gas,[4] though this is rather small in most cases because of the low temperature.

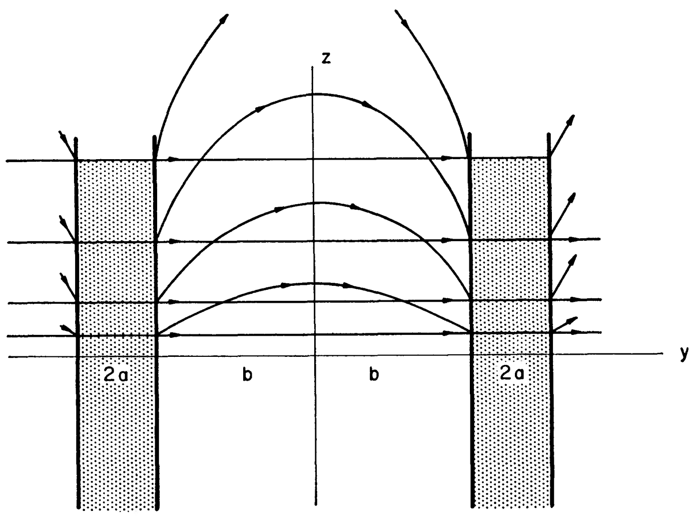

FIG. 3.—Sketch of the slabs of gas (*crosshatched*) formed by compressing the gas along the magnetic field. The magnetic lines of force are indicated before their release by the horizontal lines, and afterward by the lines which bow upward between the slabs. The x-direction is perpendicular to the yz-plane of the paper.

Following compression of the gas, hold the gas fixed and permit the magnetic field in the spaces between the slabs of gas to relax into the equilibrium form, $\boldsymbol{B} = -\nabla\psi$, $\nabla^2\psi = 0$. The boundary conditions are that $\boldsymbol{B}$ vanishes at $z = +\infty$, that no lines of force cross $z = 0$, and that the field remains frozen into the gas. This boundary value problem is easily solved (see Appendix). The field takes up the expanded form sketched in Figure 3. The total energy of the field, per unit length in the x-direction, contained in $-b < y < +b$ decreases below the initial field energy $\mathcal{E}_0 \equiv B^2(0)\Lambda/8\pi$ by the amount $\Delta\mathcal{E}$, plotted in Figure 4. It is readily shown (see Appendix) that $\Delta\mathcal{E}/\mathcal{E}_0$ increases monotonically with increasing gap width, from zero to one as b/Λ increases from zero to infinity.

We see, then, that the tendency to clump is driven at least in part by the magnetic energy decrease $\Delta\mathcal{E}$ which the clumping permits. If cosmic rays are present, they expand along with the field and make an additional contribution to $\Delta\mathcal{E}$ (Parker 1965).

[4] The work done per unit length in the x-direction in forming each slab is readily shown to be $[2p(0)\Lambda b/(s - 1)]\,(1 + a/b)\,[(1 + b/a)^{s-1} - 1]$ for the simple case that the pressure is proportional to the s-power of the density during compression If $s = 1$, the work done is $2p(0)\Lambda b(1 + a/b)\ln\,(1 + b/a)$. It is readily shown that the work reduces to $2p(0)\Lambda b$ in the limits as $b/a \to 0$ and is a monotonically increasing function of b/a when $s > 0$.

Calculation of the magnetic field in the gap between slabs of gas indicates that the net force (magnetic plus gravitational) on the slabs of gas (constrained to the density distribution $\exp[-2z/\Lambda]$) is upward at large z and downward at small z. This indicates the direction in which the gas moves when the constraints on the gas are removed. A larger gap width means that a larger portion of the sheet descends when released.[5]

Generally speaking, when released, the sheets will redistribute themselves vertically, contributing a further reduction of the total energy. Neglect the thermal energy of the gas; it follows that $\Delta\mathcal{E}$ (given in Fig. 4) is a lower limit on the energy change caused by clumping of the gas. The observed clumped state of the interstellar gas is, then, a lower energy state than a more uniform distribution over the galactic disk.

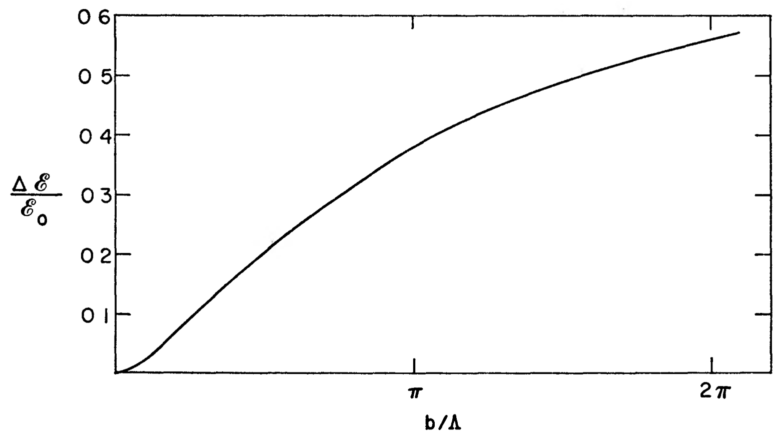

FIG. 4.—Plot of the energy decrease $\Delta\mathcal{E}$ of the magnetic field between two slabs of gas, in units of the energy $\mathcal{E}_0$ in the field before release. The initial configuration of the field is given by eq. (5) and is represented by the straight lines of force in Fig 3. The final field is described by eqs. (I.3) and (I 4), represented by the bowed lines of force in Fig. 3.

IV. FORCES IN THREE DIMENSIONS BETWEEN INDIVIDUAL ELEMENTS OF GAS

The behavior of a thermal gas confining a magnetic field and cosmic rays is rather more complicated in three dimensions than in two. Several new features are introduced by the third dimension $\boldsymbol{g} \times \boldsymbol{B}$. The three-dimensional problem will be introduced here with a calculation of the forces between elements of gas suspended on the lines of force of a large-scale field in three-dimensional space.

Imagine a uniform magnetic field B extending in the horizontal y-direction, with a uniform gravitational field g in the negative z-direction. It is sufficient for the present purposes of illustration to suppose that the large-scale field is pervaded by a tenuous plasma which is an excellent conductor of electricity but whose weight and pressure are so slight as to have no sensible dynamical effects on the magnetic field, i.e., the Alfvén speed is extremely large. Hence B may be taken as a uniform magnetic field with infinite scale height.

Into this uniform field we introduce small elements of plasma, which, as a consequence of their small size are soon threaded by the lines of force of the field B. The elements are sufficiently diffuse that their weight produces changes $\Delta\boldsymbol{B}$ which are everywhere small[6] compared to $\boldsymbol{B}$. In view of the small weight of the elements of gas, the distance which the field sags under the weight of each element is small.

[5] The general theory is given by Lerche (1967*b*).

[6] It is assumed that the lines of force are firmly anchored at some distance along the field in both directions from the region occupied by the elements of gas.

The distortion $\Delta \boldsymbol{B}$ produced by any one element of gas causes the total field $\boldsymbol{e}_y B + \Delta \boldsymbol{B}$ to slope at the positions of other elements of gas. The slope is $\Delta B_z/B$, to a first approximation. The gravitational force $-\boldsymbol{e}_z g$ then has a component $g\Delta B_z/B$ along the lines of force, causing the elements of gas to slide along the lines either toward or away from the element of gas responsible for ΔB_z. In this way it is evident that the elements of gas suspended in the field exert forces on each other. These forces are responsible for the dynamical instability and the dynamical clumping of the gas into clouds. It is important that we develop some understanding of the forces between individual elements of gas in three dimensions if we are to understand the formation and structure of the interstellar gas clouds.

Now, the problem is linear as a consequence of the light loading, $\Lambda \cong \infty$, $\Delta B \ll B$. The total distortion $\Delta \boldsymbol{B}$ is the sum of the distortions $\Delta \boldsymbol{B}$ of each of the individual elements of gas suspended alone on the field $\boldsymbol{e}_y B$. It follows that the total force exerted by a number of elements of gas is the linear sum of the force exerted by each individual element. So consider a single small element of gas, of small mass $\mathfrak{M}_1$ suspended on the

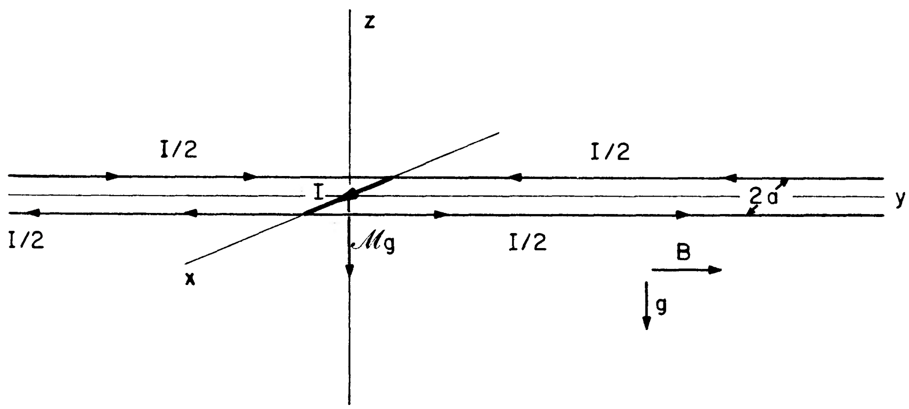

FIG. 5.—Sketch of the currents associated with a small element of gas of length $2a$ and weight $\mathfrak{M}g$ lying across a magnetic field B in the y-direction.

lines of force of $\boldsymbol{e}_y B$ at some position in space. Place the origin of the xyz-coordinate system at the position of the element of gas. It is possible to calculate the $\Delta \boldsymbol{B}$ produced by $\mathfrak{M}_1$ directly from the hydromagnetic equations. But $\Delta \boldsymbol{B}$ can be calculated correctly, and much more easily, from the currents induced by the weight of the gas.

The current system is diagrammed in Figure 5. The current I, shown by the heavy segment of length $2a$ across the middle, represents the current through the element of gas. The Lorentz force $2aIB/c$ of this current supports the weight $\mathfrak{M}_1 g$ of the gas, so that $I = \mathfrak{M}_1 cg/2aB$. The current must stream away from the ends of the element of gas but there can be no Lorentz force exerted on the tenuous plasma filling the space. Hence the current must stream away along the magnetic lines of force, as sketched in Figure 5. In the present case the current flows along the lines of force all the way to where the lines are anchored at large $|y|$.

It is evident from Maxwell's equations that $\nabla \times \boldsymbol{B} = 0$ everywhere except at the location of the currents. Hence the field has the same configuration outside the currents as in a vacuum, and the field may be computed from the currents by the same method as in a vacuum. The magnetic field of a current $\boldsymbol{I}(\boldsymbol{r}')$ at a point $\boldsymbol{r}'$ is given by the Biot-Savart law:

$$\Delta \boldsymbol{B}(\boldsymbol{r}) = \frac{1}{c} \int \frac{ds \boldsymbol{I}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3},$$

where ds is an element of length along the current $\boldsymbol{I}$. Then it is readily shown that the field $\Delta\boldsymbol{B}(\boldsymbol{r})$ produced by the current system shown in Figure 5 is

$$\Delta B_x(\boldsymbol{r}) = +\frac{Iyz}{c}\left(\frac{1}{b_1^2 s_1}-\frac{1}{b_2^2 s_2}\right) \cong +\frac{4Ia}{c}\frac{xyz}{(x^2+z^2)r}\left(\frac{1}{x^2+z^2}+\frac{1}{2r^2}\right), \tag{28}$$

$$\Delta B_y(\boldsymbol{r}) = -\frac{2Ia}{c}\frac{z}{r^3}, \tag{29}$$

$$\Delta B_z(\boldsymbol{r}) = -\frac{Iy}{c}\left(\frac{x-a}{b_1^2 s_1}-\frac{x+a}{b_2^2 s_2}\right)+\frac{2Iay}{cr^3} \cong \frac{2Iay}{cr^3}\left[1+\frac{z^2+y^2}{z^2+x^2}-\frac{2x^2r^2}{(x^2+z^2)^2}\right], \tag{30}$$

where

$$r^2 = x^2+y^2+z^2\,,$$

$$b_{1,2} = [(x\mp a)^2+z^2]^{1/2}\,,$$

$$s_{1,2} = [(x\mp a)^2+y^2+z^2]^{1/2}\,.$$

The terms $(2Ia/c)z/r^3$ and $(2Ia/c)y/r^3$ are the contribution of the current segment $2aI$. The other terms are from the currents flowing along the lines of force. The first line of each formula for the components of $\Delta\boldsymbol{B}$ is exact for the idealized current system shown in Figure 5. The approximate forms are for either or both x or z large compared to a. When $x \ll a$, we have $\Delta B_x \cong 0$. The formula for ΔB_y is unchanged, and

$$\Delta B_z \cong \frac{2Iay}{c}\left[\frac{1}{(a^2+z^2)(a^2+y^2+z^2)^{1/2}}+\frac{1}{r^3}\right]. \tag{31}$$

Now consider what effect the perturbation $\Delta\boldsymbol{B}$ has on the equilibrium of another small element of gas of mass $\mathfrak{M}_2$ suspended elsewhere on the field. The weight of $\mathfrak{M}_2$ is supported in equilibrium by the large-scale field $B\boldsymbol{e}_y$. To calculate the force on $\mathfrak{M}_2$ caused by $\Delta\boldsymbol{B}$ note that the currents flowing to and from the ends of $\mathfrak{M}_2$ will flow along the *perturbed* lines of force, so there is no Lorentz force exerted on them by $\Delta\boldsymbol{B}$. The only force is $F = 2a\boldsymbol{I}_2 \times \Delta\boldsymbol{B}/c$, where $\boldsymbol{I}_2 = \boldsymbol{e}_x\mathfrak{M}_2 cg/2aB$ is the supporting current through $\mathfrak{M}_2$. Obviously $F_x = 0$ and

$$F_y = -\mathfrak{M}_2 g\frac{\Delta B_z}{B},$$

$$F_z = +\mathfrak{M}_2 g\frac{\Delta B_y}{B}. \tag{32}$$

The vertical component of the force, F_z, is of little present interest. The elements of gas are threaded by the field and hence constrained to slide along the lines of force.

The force F_y causes the element of gas $\mathfrak{M}_2$ to slide along the magnetic lines of force, either toward or away from $\mathfrak{M}_1$. To illustrate the general form of F_y consider first the case that $\mathfrak{M}_2$, at (x,y,z), is on a line of force passing near $\mathfrak{M}_1$, as compared to the distance along the lines of force between $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Then $y^2 \gg x^2,z^2$, and

$$\Delta B_z \cong \frac{2Ia}{c}\frac{(z^2-x^2)}{(z^2+x^2)^2},$$

so that

$$F_y = -\mathfrak{M}_1\mathfrak{M}_2\frac{g^2}{B^2}\frac{(z^2-x^2)}{(z^2+x^2)^2}. \tag{33}$$

The force is thus attractive if $z^2 > x^2$ and repulsive if $z^2 < x^2$. The force falls off inversely as the square of the separation $(z^2 + x^2)^{1/2}$ of the magnetic lines of force on which the elements slide. The force is undiminished by the distance y between $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

On the other hand, if $a^2, y^2 \ll x^2, z^2$ we have

$$\Delta B_z = \frac{2Iay}{c} \frac{(2z^2 - x^2)}{(z^2 + x^2)^{5/2}},$$

so that

$$F_y = -\mathfrak{M}_1\mathfrak{M}_2 \frac{g^2}{B^2} \frac{y(2z^2 - x^2)}{(z^2 + x^2)^{5/2}}. \tag{34}$$

The force is attractive if $z^2 > x^2/2$ and repulsive if $z^2 < x^2/2$. The force is directly proportional to y. If $\mathfrak{M}_2$ were threaded on a line of force directly above the line through $\mathfrak{M}_1$, then $x = 0$ and the force is

$$F_y = -\mathfrak{M}_1\mathfrak{M}_2 \frac{2g^2}{B^2} \frac{y}{z^3}. \tag{35}$$

The gravitational force between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is $G\mathfrak{M}_1\mathfrak{M}_2 y/z^3$ along the lines of force, indicating that F_y is larger by the factor $2g^2/GB^2$ than self-gravitation.

It is evident from equation (34) that the force in $x^2 > z^2 > x^2/2$ is attractive when y is small, and it is evident from equation (33) that the force is repulsive when y is large. We would expect, then, that in passing from small to large y along a line in $x^2 > z^2 > x^2/2$ the force F_y must pass through zero. It is readily shown, by equating the right-hand side of equation (30) to zero, that the force F_y vanishes at

$$y^2 = (x^2 + z^2)\left(\frac{2z^2 - x^2}{x^2 - z^2}\right). \tag{36}$$

The equilibrium position at this value of y is unstable, of course.

Finally, if the two elements of gas are exactly on the same lines of force, then $x = z = 0$ and it follows from equation (31) that

$$F_y = -\mathfrak{M}_1\mathfrak{M}_2 \frac{g^2}{B^2 a^2}\left(1 + \frac{a^2}{y^2}\right). \tag{37}$$

The force is attractive for all values of y, approaching the limit $-\mathfrak{M}_1\mathfrak{M}_2 g^2/B^2 a^2$ as y^2 becomes large compared to a^2.

Finally it should be noted that the discussion is easily generalized to the case that the gravitational acceleration is a function of position. The force depends only upon the weights of the two elements of gas, so if $g(z_1)$ represents gravity at the position of $\mathfrak{M}_1$, and $g(z_z)$ at $\mathfrak{M}_2$, it is necessary only to replace $\mathfrak{M}_1 g$ by $\mathfrak{M}_1 g(z_1)$ and $\mathfrak{M}_2 g$ by $\mathfrak{M}_2 g(z_2)$ in the formulae. Hence the form F_y is unchanged. The magnitude is proportional to $\mathfrak{M}_1\mathfrak{M}_2 g(z_1) g(z_2)$.

Altogether it is evident that elements of gas will tend to group in vertical columns, with columns on neighboring lines of force moving apart.

V. FORCES WITHIN CLOUDS OF GAS

The gas suspended on the lines of force of the uniform horizontal field $\boldsymbol{e}_y B$ of the previous section is constrained to motions along the magnetic lines of force. The calculations show that the y-component of the force between any two elements of gas, given by equation (32), is attractive or repulsive depending upon the relative location of the lines through the two elements of gas. It is evident therefore that the formation of an interstellar gas cloud cannot be as simple as with self-gravitation alone, where all forces are attractive. It is evident that the gas in some regions is strongly repelled from the cloud, so that a complex flow pattern must result. A complete investigation of the flow associated with the formation of a cloud is beyond the scope of this paper, but the question is so basic that a precise illustration of the force field around a simple finite distribution of gas is important for understanding the problem which still lies ahead.

First of all, it is a simple matter to show that the net or average force produced by each element of gas is attractive. On any plane y = const. perpendicular to the magnetic field, the total force per unit mass is readily shown from equations (32) to be

$$\mathfrak{F} = -\frac{g}{B}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dz \Delta B_z \text{ dynes cm}^2/\text{gm}.$$

It is obvious from symmetry (see Fig. 5) that the fields of the currents flowing to and from the element of gas produce no net ΔB_z. So only the term $2Iay/cr^3$ for the current across the element of gas contributes. This has exactly the same form, y/r^3, as the gravitational attraction of $\mathfrak{M}$ along the lines of force. The total force is

$$\mathfrak{F} = -\frac{4\pi a g I}{cB} = -\frac{2\pi g^2 \mathfrak{M}}{B^2}. \tag{38}$$

Another way to derive this same result is to note that the total magnetic stress in the upward direction across any plane y = const. on either side of the suspended mass $\mathfrak{M}$ must support half of the weight of $\mathfrak{M}$. Hence $\int dx \int dz\, BB_z/4\pi = \frac{1}{2}\mathfrak{M}g$, which yields $\int dx \int dz\, B_z = 2\pi \mathfrak{M} g/B$ and gives the above result for $\mathfrak{F}$.

For comparison the total gravitational force $\mathfrak{F}_G$ parallel to the lines of force is

$$\mathfrak{F}_G = -\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dz \frac{G\mathfrak{M}y}{r^3} = -2\pi G \mathfrak{M},$$

so that the force produced by the current segment is larger by the usual factor g^2/GB^2.

Consider the forces produced by the currents flowing away from along the lines of force. As already pointed out, these forces are both attractive and repulsive, and contribute no net force. But they do not diminish with increasing distance $|y|$ along the lines from $\mathfrak{M}$. The forces are concentrated along the lines of force which pass through, and near, $\mathfrak{M}$. The total force is thus a composite of a general gravitational-like force, giving the net over-all attraction (38), and a force concentrated along the lines of force through $\mathfrak{M}$ which does not diminish with distance from $\mathfrak{M}$. It is this latter force which is the dominant term (33) when $y^2 \gg x^2, z^2$.

To investigate the nature of the forces concentrated along the lines of force through a cloud of mass $\mathfrak{M}$, consider the simple idealized case of a homogeneous rectangular cloud,[7] of length $2l$ along the lines of force, of width $2a$ and height $2b$. Place the center of the cloud at the origin. Gas can enter or leave the actual rectangular volume of the cloud along the lines of force which pass through the volume. The variation of the dominant force across the lines of force passing through or near the cloud is readily calculated. The cloud is homogeneous, so the only currents flowing away from the cloud along the lines of force are the sheets $x = \pm a$, $z^2 < b^2$. The current density is $\mathfrak{M}gc/8abB$ in the sheets. The current has the sign of y in the sheet $x = +a$ and the opposite sign in $x = -a$. The z-component of the field due to the current sheets is (for $y^2 \gg a^2, b^2$)

$$\Delta B_z(x,z) = \frac{\mathfrak{M}g}{4abB}\left[-\int_{-b}^{+b} \frac{dz'(x-a)}{(x-a)^2+(z'-z)^2} + \int_{-b}^{+b} \frac{dz'(x+a)}{(x+a)^2+(z'-z)^2}\right].$$

[7] The interested reader may wish to make the calculation for a circular cylindrical cloud lying along the lines of force. The force field is expressible in terms of elementary functions, as with the rectangle. Elliptical cross-sections give forces expressible in terms of elliptic functions.

The force per unit mass $f(x,z)$ as a consequence of the slope is, then,

$$f(x,z) = -g\frac{\Delta B_z}{B} = -\frac{\mathfrak{M}g^2}{4abB^2}\left(\arctan\frac{b-z}{a-x} + \arctan\frac{b+z}{a-x} + \arctan\frac{b-z}{a+x} + \arctan\frac{b+z}{a+x}\right) \tag{39}$$

along the lines of force. A negative value for f implies attraction. The principal values of the inverse tangents are to be used for all values of x and z.[8]

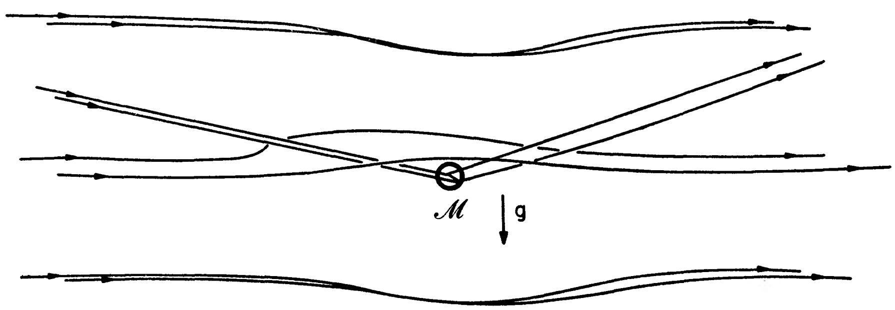

FIG 6.—Sketch of the displaced lines of force in the neighborhood of a small element of gas $\mathfrak{M}$ suspended in the field. The lines of force above and below $\mathfrak{M}$ are displaced downward along with the lines threading $\mathfrak{M}$. Consequently the lines to the side are displaced upward.

Note that if the field point (x,z) is made to approach from outside ($x^2 > a^2$) toward either of the sides $|x| = a$, $|z| < b$ the limiting value is

$$f(a,z) = \frac{\mathfrak{M}g^2}{4abB^2}\left(\pi - \arctan\frac{b-z}{2a} - \arctan\frac{b+z}{2a}\right), \tag{40}$$

representing repulsion, whereas approaching from the inside ($x^2 < a^2$)

$$f(a,z) = -\frac{\mathfrak{M}g^2}{4abB}\left(\pi + \arctan\frac{b-z}{2a} + \arctan\frac{b+z}{2a}\right), \tag{41}$$

representing attraction. The discontinuity reflects the abrupt change in slope of the lines of force in passing across the surface (current sheet) $|x| = a$ of the rectangle. In hydromagnetic terms, the lines of force which pass through the cloud are deflected downward by the weight of the cloud. The crowding of the lines of force which occurs underneath the cloud displaces the external lines of force upward to make room. The effect is illustrated in Figure 6, which is a sketch of the lines of force passing through, and near, a cloud of gas supported on the field.

The important point which follows from this simple model is that the force is attractive everywhere along the lines of force threading the cloud, and on the lines passing above or below the cloud. The force is repulsive along the lines of force passing by beside the cloud. To see this note that within the rectangle $|x| < a$, $|z| < b$ the arguments of all the inverse tangents on the right-hand side of equation (39) are positive. Hence f is negative and attractive. Outside the rectangle, the forces immediately above and below

[8] It is possible to calculate f from eq (33), but it is then not so easy to see which branch of the inverse tangents to choose. For instance, outside the rectangle $x^2 = a^2$, $z^2 = b^2$ the arguments of the inverse tangents may be inverted if the minus sign in front is changed to plus But inside the rectangle only the form given by eq. (39) is correct. The symmetry of eq (33) is misleading in that it suggests that f should vanish at the origin if $a = b$. The more complete formula for ΔB_z must then be consulted.

the cloud are attractive. Along the sides $|x| = a$, the forces are repulsive, as may be seen from equation (40). At larger (x,z) distances from the cloud the forces go over into the symmetric pattern of equation (33) with attraction for $|z| > |x|$ and repulsion for $|z| < |x|$.

Altogether it is evident that any gas cloud tends to attract gas into its general region with a force g^2/GB^2 times stronger than gravity. The cloud attracts gas along the lines of force passing through, and above and below, the cloud with a force which does not diminish with distance and which is at large distances stronger even than g^2/GB^2, compared to gravity.

The example of a homogeneous gas cloud with sharply defined plane rectangular boundaries is highly idealized, of course, so we must exercise caution in extrapolating some of the results to the real gas clouds in interstellar space. For instance, in the present example we found that the force was attractive everywhere along *every* line of force threading through the cloud. But we should note that the attractive force changed discontinuously to repulsion upon crossing onto lines which passed beside the cloud without actually passing through it. It is obvious that dissipative effects would soften this artificial discontinuity in the real case. More important, consider what happens if the cloud boundary is relatively diffuse and gradual. Suppose that we had made the cloud boundary less sharp by introducing a small amount of gas immediately outside the sharp cloud boundary $|x| = a$ in such a way that the gas density declined smoothly, instead of abruptly, to zero. It is evident that some of the additional gas lies out in the repulsion zone, so that it would be expelled from the cloud, thereby tending to restore the sharp boundary. Or to put the situation the other way around, a cloud with sharp side boundaries will accumulate little or no gas in the region immediately outside the sharp boundaries. We suggest, then, that there should be a tendency for the sides of the gas clouds to be less diffuse than, say, the top and bottom.

VI. SUMMARY AND CONCLUSION

We have, through an idealized example, explored the way in which an initial horizontal atmosphere confining a magnetic field becomes unstable and crashes into vertical sheets. A simple illustration of the energy decrease of the system was given too.

Probably the most serious idealization of this illustration is that it is restricted to the two dimensions defined by the gravitational acceleration $\boldsymbol{g}$ and the large scale horizontal magnetic field $\boldsymbol{B}$. In two dimensions the forces exerted within the gas are of exactly the same form, though stronger, than self-gravitation (in the linear approximation $\Delta B \ll B$). Hence, we are forced to believe that the system will behave in essentially the same way as we presently believe it would behave under self-gravity alone, viz., forming gas clouds and finally stars, etc.

In three dimensions $(\boldsymbol{g}, \boldsymbol{B}, \boldsymbol{g} \times \boldsymbol{B})$, however, the situation becomes more complicated. In addition to the force which has exactly the same form as the gravitational attraction, there is an additional force transmitted undiminished[9] along the lines of force passing near the gas. This force is attractive along the lines of force passing through, and above and below, a cloud. The force is repulsive along the lines of force passing beside the cloud. The average of this additional force over all lines is identically zero. But the force, whether in the region of attraction or repulsion, dominates the gravitational and the pseudo-gravitational force at large distances along the lines of force passing through, or near, the cloud.

In a very general way we conclude that the growth of gas clouds is $(1 + g^2/GB^2)$ times stronger than under self-gravitation alone. But if gas is available to the cloud principally at some distance along the lines of force through the cloud, then the accumulation

[9] The force may, in fact, be diminished if there is a general twisting of the lines of force into a rope, because the loading of the lines is then redistributed as the lines circle each other, smearing out the effect and reducing it to the over-all mean value of zero after about one full twist.

to the ends (in the $\boldsymbol{B}$ dimension) and top and bottom (in the $\boldsymbol{g}$ dimension) may be more vigorous still. At the same time the sides of the cloud (in the $\boldsymbol{g} \times \boldsymbol{B}$ direction) may be sheared away by the repulsive force there, perhaps sharpening the boundary at the sides. So that altogether we would expect the gas clouds to grow principally in the $\boldsymbol{g}$ and $\boldsymbol{B}$ dimensions, with relatively less growth in the $\boldsymbol{g} \times \boldsymbol{B}$ direction. The growth of a cloud is presumably accompanied by considerable streaming of gas along the field both toward and away from the cloud.

We must not, however, conclude that there is no more to the picture than this. The calculations have been based on the existence of a well-defined equilibrium cloud. Yet if such a cloud were thought of as two clouds placed side by side, it is obvious that the equilibrium is unstable, with a tendency for the two halves to separate by sliding along the lines of force. This tendency may be seen directly in equation (33), showing that any two elements of gas placed side by side tend to repel each other. This complication is pursued in the next paper (Parker 1967). All we can say here is that the formation of gas clouds in interstellar space must be both a vigorous and a complicated dynamical process.

The author wishes to thank Dr. I. Lerche for stimulating discussion and several helpful suggestions in preparing this paper. The author is indebted to Dr. S. Chandrasekhar for discussion of the over-all problem of presenting the dynamical behavior of the interstellar gas.

APPENDIX

THE MAGNETIC FIELD BETWEEN SLABS OF GAS

Consider the magnetic field $\boldsymbol{B} = -\nabla\psi$, $\nabla^2\psi = 0$ in the region $-b < y < +b$, $z \geq 0$ subject to the boundary conditions that $B_z = 0$ on $z = 0$, $\boldsymbol{B} = 0$ at $z = +\infty$, and $B_y = B(z)$ (given by eq. [5]) at $y = \pm b$. Then, in terms of solutions of $\nabla^2\psi = 0$, write

$$\psi(y,z) = \int_{-\infty}^{+\infty} dk\, f(k) \sinh ky \exp ikz\,, \tag{I.1}$$

with the understanding that $f(k)$ must be an odd function of k so that B_z will vanish at $z = 0$. The transform $f(k)$ is evaluated from the requirement that $B_y = B(z)$ at $y = \pm b$, yielding

$$f(k) = -\frac{B(0)}{\pi k \cosh kb} \frac{\Lambda}{1 + k^2\Lambda^2}\,. \tag{I.2}$$

The resulting integrals for B_y and B_z are readily evaluated by closing the contour around the upper half plane and applying Cauchy's theorem,

$$\begin{aligned} B_y(y,z) = B(0) \Bigg\{ & \frac{\cos y/\Lambda}{\cos b/\Lambda} \exp\left(-\frac{z}{\Lambda}\right) \\ & + \frac{2\Lambda}{b} \sum_{n=0}^{\infty} \frac{(-1)^n \cos(n+\frac{1}{2})\pi y/b \exp[-(n+\frac{1}{2})\pi z/b]}{1-(n+\frac{1}{2})^2\pi^2\Lambda^2/b^2} \Bigg\}, \end{aligned} \tag{I.3}$$

$$\begin{aligned} B_z(y,z) = -B(0) \Bigg\{ & \frac{\sin y/\Lambda}{\cos b/\Lambda} \exp\left(-\frac{z}{\Lambda}\right) \\ & + \frac{2\Lambda}{b} \sum_{n=0}^{\infty} \frac{(-1)^n \sin(n+\frac{1}{2})\pi y/b \exp[-(n+\frac{1}{2})\pi z/b]}{1-(n+\frac{1}{2})^2\pi^2\Lambda^2/b^2} \Bigg\}. \end{aligned} \tag{I.4}$$

The change in field energy is

$$\Delta\mathcal{E} = \mathcal{E}_0 - \int_{-b}^{+b} dy \int_0^{\infty} dz \frac{B_y^2 + B_z^2}{8\pi} \tag{I.5}$$

per unit length in the x-direction, where

$$\mathcal{E}_0 = \int_{-b}^{+b} dy \int_0^{\infty} dz \frac{B^2(z)}{8\pi} = \frac{B^2(0)}{8\pi} \Lambda b \tag{I.6}$$

is the energy of the initial distribution in $-b < y < +b$, given by equation (5). The easiest way to perform the integration is to go back to equation (I.1). Then

$$\int_{-b}^{+b} dy \int_0^{\infty} dz B_y^2 = \tfrac{1}{2} \int_{-b}^{+b} dy \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dk' f(k') k' \cosh k'y \exp ik'z$$

$$\times \int_{-\infty}^{+\infty} dk'' f(k'') k'' \cosh k''y \exp ik''z = \pi \int_{-b}^{+b} dy \int_{-\infty}^{+\infty} dk k^2 f^2(k) \cosh^2 ky ,$$

after noting that the integration over z gives $2\pi\delta(k' + k'')$ and that $f(-k) = -f(k)$. The integral of B_z^2 gives

$$\int_{-b}^{+b} dy \int_{-\infty}^{+\infty} dz B_z^2 = \pi \int_{-b}^{+b} dy \int_{-\infty}^{+\infty} dk k^2 f^2(k) \sinh^2 ky .$$

The integrands have poles at $k\Lambda = \pm i$ and at $kb = i\,(n + \tfrac{1}{2})\pi$ as a consequence of the denominator of $f^2(k)$. The residues of $\int dk k^2 f^2(k) \sinh^2 ky$ are

$$\frac{-i \cos 2y/\Lambda}{4\Lambda \cos^2 b/\Lambda} \left(1 + \frac{2y}{\Lambda} \tan \frac{2y}{\Lambda} - \frac{2b}{\Lambda} \tan \frac{b}{\Lambda}\right)$$

and

$$-\frac{2i}{b^2} \left\{ \frac{y \sin(2n+1)\pi y/b}{[1 - (n+\frac{1}{2})^2 \pi^2 \Lambda^2/b^2]^2} - \frac{(2n+1)\pi(\Lambda^2/b) \cos(2n+1)\pi y/b}{[1 - (n+\frac{1}{2})^2 \pi^2 \Lambda^2/b^2]^3} \right\},$$

respectively. Then, closing the contour around the upper half of the complex plane and using Cauchy's theorem, we can evaluate the integral over k. The remaining integration over y is then elementary, the final result being

$$\Delta\mathcal{E} = \mathcal{E}_0 \left\{ 1 + \sec^2 \frac{b}{\Lambda} - 2 \frac{\Lambda}{b} \tan \frac{b}{\Lambda} - \frac{4\Lambda}{\pi b} \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})[1 - (n+\frac{1}{2})^2 \pi^2 \Lambda^2/b^2]^2} \right\} \tag{I.7}$$

for the decrease of the field energy below the initial value $\mathcal{E}_0$.

The general behavior of $\Delta\mathcal{E}$ is readily established. Beginning with a narrow gap between the slabs of gas, we have $b/\Lambda \ll 1$. Expanding in ascending powers of b/Λ yields

$$\Delta\mathcal{E} = \tfrac{1}{3} \mathcal{E}_0 \left(\frac{b}{\Lambda}\right)^2 \left[1 + O\left(\frac{b}{\Lambda}\right)\right], \tag{I.8}$$

so that $\Delta\mathcal{E}/\mathcal{E}_0$ increases as the square of the gap width over which the field is freed.

It is readily apparent that the individual terms, including the summations, have singularities at $b/\Lambda = (n + \frac{1}{2})\pi$. The complete expression remains finite however, as can be demonstrated by writing $b/\Lambda = (n + \frac{1}{2})\pi + \epsilon$ and carrying out the expansion in ascending powers of ϵ. The terms of order $1/\epsilon$ and $1/\epsilon^2$ vanish identically.

Finally, in the limit of very large b/Λ it is possible to do the sum by noting that the major contribution is from n in the vicinity of $b/\pi\Lambda$. We know that $\Delta\mathcal{E}$ is a smoothly varying function of b/Λ, so for simplicity suppose that $b/\Lambda = m\pi$ where m is a large integer. Then let $n = m + \mu$. We have

$$\Delta\mathcal{E} = \mathcal{E}_0 \left\{2 - \frac{4}{m\pi^2}\sum_{-\infty}^{+\infty}\frac{m^2}{(m+\frac{1}{2}+\mu)(2\mu+1)^2[1+(2\mu+1)/4m^2]^2}\right\}, \quad \text{(I.9)}$$

where the sum is now over μ. Examination of the terms in the vicinity of $\mu = 0$ shows that the terms in the sum which are $O(m)$ are $(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \ldots)$, yielding $m\pi^2/4$. The terms $O(1)$ vanish identically, and the terms $O(1/m)$ yield the divergent series $\ldots + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\ldots$. It follows, then, that $\Delta\mathcal{E} = \mathcal{E}_0$ in the limit of large b/Λ. The final field energy is small compared to the initial energy because of the unlimited expansion which has taken place. The asymptotic approach of $\Delta\mathcal{E}$ to $\mathcal{E}_0$ is rather slow with increasing b/Λ, as may be seen from Figure 4 and from the fact that even with $b/\Lambda = 4\pi$ we have $\Delta\mathcal{E}$ equal only to 0.61 $\mathcal{E}_0$.

Finally, it is of interest to consider what stresses are exerted on the gas, so that we may have some idea whether the gas would be raised or lowered if it were released. The magnetic force in the upward direction is

$$F_B(z) = \frac{1}{4\pi}B_y(b,z)B_z(b,z) = \frac{B^2(0)}{4\pi}\left\{\tan\frac{b}{\Lambda}\exp\left(-\frac{2z}{\Lambda}\right) + \frac{2\Lambda}{b}\sum_{n=0}^{\infty}\frac{\exp\{-[(n+\frac{1}{2})\pi/b+1/\Lambda]z\}}{1-(n+\frac{1}{2})^2\pi^2\Lambda^2/b^2}\right\}\ \text{dynes/cm}^2$$

on each face of the gas slabs. The total downward force on the gas per square centimeter of face is the difference between the gravitational force and the total pressure gradient in the gas,

$$F_G(z) = (a+b)\,g\rho(0)\exp\left(-\frac{2z}{\Lambda}\right) \times \left\{1 - \left(\frac{a}{a+b}\right)\left(\frac{\frac{1}{2}V_A{}^2}{u^2+\frac{1}{2}V_A{}^2}\right) - \left(\frac{a+b}{a}\right)^{s-1}\left(\frac{u^2}{u^2+\frac{1}{2}V_A{}^2}\right)\right\},$$

if we suppose that the pressure increased in proportion to the s-power of the density during the compression. If the upward magnetic force $F_B(z)$ exceeds the net downward force, $F_G(z)$, the gas will tend to move upward when released. If $F_B(z) < F_G(z)$, the gas will move downward. In either case the energy of the system will decrease further. In the simplest case suppose that $a \ll b$ as a consequence of the gas being very cold $u^2 \ll \frac{1}{2}V_A{}^2$. Then the only pressure is magnetic, and $\Lambda = V_A{}^2/g$. It follows that

$$F_G = \frac{B^2(0)}{4\pi}\frac{b}{\Lambda}\exp\left(-\frac{2z}{\Lambda}\right)$$

and

$$F_B(z) - F_G(z) = \frac{B^2(0)}{4\pi}\left\{\left(\tan\frac{b}{\Lambda} - \frac{b}{\Lambda}\right)\exp\left(-\frac{2z}{\Lambda}\right) + \frac{2\Lambda}{b}\sum_{n=0}^{\infty}\frac{\exp\{-[(n+\frac{1}{2})\pi/b+1/\Lambda]z\}}{1-(n+\frac{1}{2})^2\pi^2\Lambda^2/b^2}\right\}.$$

Now consider whether the magnetic or gravitational forces dominate. For $z \ll b$, Λ the exponential can be put equal to 1 in the sum. Then when $b \ll \Lambda$ each term in the sum can be expanded in descending powers of $n\Lambda/b$, yielding

$$F_B(z) - F_G(z) \cong -\frac{B^2(0)}{4\pi}\frac{b}{\Lambda}\left[1 + O\left(\frac{b^4}{\Lambda^4}\right)\right].$$

When $b \gg \Lambda$ suppose that $b = m\pi\Lambda$ where m is an integer. Then write $n = m + \mu$. The sum can be written

$$-\left\{\left[\sum_{\mu=-m}^{m-1} + \sum_{\mu=m}^{\infty}\right] \frac{n}{(2\mu+1)[1+(2\mu+1)/m]}\right\}$$

$$= \left\{\frac{1}{2}\sum_{\mu=0}^{m-1} \frac{1}{[1-(2\mu+1)^2/16m^2]} + m\sum_{\mu=m}^{\infty} \frac{1}{(2\mu+1)[1+(2\mu+1)/4m]}\right\}.$$

The individual terms in the first sum are all $O(1)$, so the sum is $O(m)$. The first $2m$ terms in the second sum are $O(1/m)$. Thereafter the individual terms decline as $1/\mu^2$, so that the whole quantity in curly braces is $O(m)$. It follows that

$$F_B(z) - F_G(z) \cong -\frac{B^2(0)}{4\pi}\frac{b}{\Lambda}.$$

Intermediate values such as $b = \pi\Lambda$ also give $F_B < F_G$. Thus for small z the gas moves downward when released for all values of b/Λ.

The situation is rather different when $z \gg b, \Lambda$. If $b \ll \Lambda$, then the sum is negligible because of the exponential factor in each term,

$$F_B(z) - F_G(z) \cong +\frac{B^2(0)}{4\pi}\frac{b^2}{3\Lambda^2}\exp\left(-\frac{2z}{\Lambda}\right).$$

If $b \gg \Lambda$, put $b = m\pi\Lambda$ where m is a large positive integer. The exponential factors in the series are all larger than the factor $\exp(-2z/\Lambda)$ for $n + \frac{1}{2} < m$. These terms are also positive. The first term in the series dominates all the others in the limit of large z, so that $F_G > F_B$. For intermediate values, such as $b = \pi\Lambda$, the same result is obtained. Hence at large z the material moves upward when released.

At intermediate values of z ($z \cong \Lambda$) it is readily shown that $F_B > F_G$ if $b \ll \Lambda$, and $F_B < F_G$ if $b \gg \Lambda$, as one would expect: When b is large, there is so much mass packed into each sheet that the local field cannot support it without sagging.

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