1967ApJ...148..803R

THE ASTROPHYSICAL JOURNAL, Vol. 148, June 1967

INSTABILITIES IN HIGHLY EVOLVED STELLAR MODELS*

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Received June 1, 1966; revised November 18, 1966

ABSTRACT

The structure of heavy stars at highly evolved stages of evolution is considered. It was found that radiation relativistic effects on the electrons and pair creation induce a dynamical instability in stars heavier than about 30 $M_{\odot}$. The evolution of a 30 $M_{\odot}$ oxygen star is described in some detail up to the point where it becomes dynamically unstable.

I. INTRODUCTION

Late stages of evolution have been discussed quite thoroughly in recent years. In particular the effect of intense neutrino emission on stellar evolution and the mechanism of supernovae has been considered (Hayashi and Cameron 1962; Reeves 1963; Fowler and Hoyle 1964; Deinzer and Salpeter 1966; Chiu 1966).

In order to check the various ideas about supernovae in more detail, the evolutionary calculations of stellar models must be continued up to the very evolved models in which the instabilities supposed to result in supernova explosions can show up. A computer program capable of performing such evolutionary calculations efficiently has been developed (Rakavy, Shaviv, and Zinamon 1966). As first sample calculation, which may be of some interest, we chose a model of a pure oxygen star of 30 $M_{\odot}$. We have studied the behavior of this model under various assumptions from the stage of contraction due to photon emission, continuing through a stage in which energy losses due to neutrino emission predominate, up to the ignition of oxygen burning. During these calculations it was learned that stars heavier than about 30 $M_{\odot}$ arrive at an instability due to pair creation whenever their central temperature rises to 1.8—2.3 $\times$ 10^9 ° K. This instability seems to be quite independent of the nature of the nuclear reactions, the mode of energy losses, convection, and probably also composition.

In the present paper the general question of structure and stability of a star losing energy predominantly through neutrino emission is discussed. The discussion is based on results from the above-mentioned calculation of a 30 $M_{\odot}$ model and from exploratory calculations of a few lighter models. We hope to publish detailed results on configurations of stars at the limit of stability after performing more accurate calculations, starting from earlier evolutionary models.

In § II the structure of a "neutrino star" (i.e., a star losing energy predominantly through neutrino losses) is discussed. The equations of evolution are formulated in § III. As the conditions for dynamical stability have been thoroughly discussed in the literature, we make only a few comments on that problem. A short résumé of methods for treating dynamical instabilities which we use in the present work is given in Appendix A. (For a full review of the subject cf. Ledoux 1958, 1965.) In § IV, the particular case of dynamical instabilities due to pair formation is discussed. In § V a brief account is given of the results of the calculation of the evolution of the 30 $M_{\odot}$ model. Details of the nuclear reaction rates and the equation of state used in these calculations can be found in Appendices B and C.

* Supported in part by the National Science Foundation (GP-5391) and the Office of Naval Research (Nonr-220(47)).

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II. REMARKS ON THE STRUCTURE OF NEUTRINO STARS

When a star contracts and the central temperature reaches about 0.6×10^9 ° K the rate of energy losses by means of various neutrino processes equals the rate of energy lost by photon emission. At higher temperatures the neutrino losses dominate over photon losses by several orders of magnitude. It has become customary to refer to such a star, in which heat conduction by photons (or electrons) can be completely neglected (although convection may still be important) as a "neutrino star." Stars with densities less than 10^{11} gm cm^{-3} are transparent to neutrinos (Bahcall 1964; Bahcall and Frautschi 1964). Thus the mechanism of neutrino losses differs radically from the mechanism of energy losses by photons. While the latter is a conduction mechanism and is influenced by the temperature distribution and temperature gradients all over the star, the rate of neutrino losses from a given mass element depends only on the local temperature and density.

The neutrino losses accelerate the evolution. Both contraction and nuclear-burning periods become shorter by orders of magnitude. As a result of this acceleration the temperature at which a specific nuclear fuel burns is somewhat increased (for example, the temperature of oxygen burning is increased from around 1.6×10^9 ° K to approximately 2.1×10^9 ° K). In addition to these quantitative differences there exists a rather more essential difference between "neutrino stars" and "photon stars" (stars losing energy predominantly by photon emission). Sometime after a nuclear fuel is ignited, a "photon star" tends to settle in a quasi-stationary state.

In this state the rate of heat removed by nuclear reactions equals the divergence of the heat current conducted and convected away by the photons at each point in the star, namely,

$$T \frac{\partial s}{\partial t} = q_{\text{nuc}} - \frac{\partial F_{\text{rad}}}{\partial m} - \frac{\partial F_{\text{conv}}}{\partial m} = 0 , \tag{1}$$

where s is the entropy density, q_{nuc} the energy produced by nuclear reactions, and F_{rad} and F_{conv} are the radiative and convective heat fluxes, respectively. F_{conv} is different from zero only if the density gradient is smaller than the adiabatic one.

The situation is different in a "neutrino star." A term q_ν—the energy losses by neutrinos—should be added, and the term $\partial F_{\text{rad}}/\partial m$ can be neglected. We have now

$$T \frac{\partial s}{\partial t} = q_{\text{nuc}} - q_\nu - \frac{\partial F_{\text{conv}}}{\partial m} . \tag{2}$$

It is evident that if $F_{\text{conv}} = 0$ a neutrino star cannot settle on a quasi-stationary state in which $(\partial s/\partial t) = 0$ locally. Even if there is a solution of the hydrostatic equilibrium equation consistent with the condition that at each point of the star the energy generated by nuclear reactions is exactly balanced by the energy removed through neutrino emission, such a solution is unstable in the sense that a small (localized) perturbation to the temperature will grow rather than decay. Assume that the entropy rises by an infinitesimal amount over a small region of the star. If the region is small enough, the pressure will not change. Adding entropy under isobaric conditions elevates the temperature. As the nuclear reactions increase faster with temperature than the neutrino losses, more entropy is added and the balance is destroyed.

As detailed calculations have shown, the nuclear reaction rates in a small region become several times greater than the neutrino losses. In the rest of the star neutrino losses dominate. A growing convective core develops and tends to spread the energy supply. If the convective core can extend so far that $\int(q_{\text{nuc}} - q_\nu)\,\text{dm} = 0$ where the integration is carried over the whole convective zone and $q_{\text{nuc}} = q_\nu = 0$ outside this region, one would have $(\partial s/\partial t) = 0$ locally. However, it is found that generally this is

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not the case. The extent of the convective core depends on the relation between the oxygen Q value (erg per gram of fuel) and the duration of neutrino losses before oxygen ignition. Immediately after the ignition of the nuclear burning the reaction rates increase very rapidly at the center becoming several times faster than the neutrino losses. A convective core starts to grow around the burning region. When the heat generated by the nuclear reactions is spread over a wide enough region (either directly or through convective transfer) the burning is stabilized and further changes of reaction rates are comparatively slow. The burning is stabilized so as to maintain an approximate balance between the *integrated* rates of energy release by nuclear reactions and energy loss by neutrino emission. The excess energy in the region in which $(\partial s/\partial t) > 0$ is transformed into internal energy and work. The center with the convective core expands slowly while the outer layers contract. Because the time scale dictated by the neutrinos (a few days at least) is longer than the freefall time (a few hundredths of a second for a core collapse) the motions that result from the work done are so slow that kinetic energy can be neglected compared to internal energy and the problem does not become dynamical, namely, we can assume that hydrostatic equilibrium is maintained at all times. (It should be noted, however, that under special circumstances one has also in a "photon star" $(\partial s/\partial t) \neq 0$; Hofmeister, Kippenhahn and Weigert 1964.)

Another important difference between "photon stars" and "neutrino stars" exists during periods of contraction without fuel burning. A contracting "photon star" with a conventional opacity law tends to become fully convective, while a "neutrino star" can never become convective when there is no nuclear burning.

III. THE EQUATIONS OF EVOLUTION

The variation of the static energy

$$W = W_{(\text{gravitational})} + W_{(\text{thermal})} = -G\int \frac{m\,dm}{r} + \int u\,dm \tag{3}$$

due to an adiabatic displacement $\delta r(m)$ can be written (Dyson 1961)

$$\delta W = \int dV \chi \left(\frac{dp}{dV} + \frac{Gm\rho}{4\pi r^4}\right) + \tfrac{1}{2}\int dV(\chi H \chi) + \ldots ,$$

where

$$V = \frac{4\pi}{3} r^3 , \qquad \chi = \delta V = 4\pi r^2 \delta r(m) ,$$

$$H = -\frac{d}{dV}\gamma p \frac{d}{dV} - \tfrac{4}{9}\frac{Gm\rho}{rV^2} , \qquad \gamma = \frac{\rho}{p}\left(\frac{\partial p}{\partial \rho}\right)_{s=\text{const}} . \tag{4}$$

Here m, the mass contained in a sphere of radius r, is used as an independent variable. Sometimes we write integrations on the volume element $dV = dm/\rho$. This should be distinguished clearly from the variation $\delta V = \chi(m)$ which is an arbitrary finite function of position (with small absolute value). The only condition on $\chi(m)$ is that it should vanish at the origin: $\chi(0) = 0$. In equation (4), only terms up to the second order in δV have been retained.

A configuration is in hydrostatic equilibrium (H.E.) when its static energy W is stationary with respect to an arbitrary adiabatic displacement; i.e., the first term in equation (4), which is linear in χ, must vanish identically in χ. Equating the coefficient of χ in that term to zero, one obtains the equation of H.E.

$$\frac{dp}{dm} + \frac{Gm}{4\pi r^4} = 0 . \tag{5}$$

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If the entropy density distribution in the system $s(m)$ is changed slowly, H.E. is maintained. Let the heat supplied to a unit mass per unit time be q. For the rate of change of entropy density, we have

$$\frac{\partial s(m,t)}{\partial t}=\frac{q}{T}. \tag{6}$$

The heat equation (6), the equation of H.E. (5) and the equation of change in chemical composition, augmented by the equations of state connecting p, ρ, T and s, ρ, T are the basic equations for the calculation of stellar evolution.

It is quite evident that H.E. is stable if the static energy W has a minimum with respect to all adiabatic variations. If this is not the case, fast motions may develop independent of the rate of heat supply q. The condition for a stationary point of W to be a minimum is that the operator H defined in equation (4) should be positive definite. This condition has been formulated in the literature in various forms. A short résumé is given in Appendix A.

Thermal instabilities, if they occur, must show up in a natural way while integrating the time-dependent equations (5)–(6). During integration one has to check continuously if the rates of change of entropy density, etc., are slow enough so that the assumptions of complete mixing by convection and the existence of instantaneous H.E. are not violated. It was found that in heavy stars no fast changes occur before some kind of dynamical instability is encountered.

IV. THE DYNAMICAL INSTABILITY DUE TO PAIR FORMATION

As is well known, the dynamical stability is governed by the average value of γ over the star. In Figure 1, lines of constant $\gamma - \frac{4}{3}$ are drawn in the ρ,T plane. The equation of state used to obtain these values of γ is described in Appendix B. The depression below $\gamma = \frac{4}{3}$ around $T_9 = 2.5$ is due to pair formation (Fowler and Hoyle 1964; Souffrin 1960). At high densities, γ never goes below $\frac{4}{3}$, both due to a larger contribution from the ion pressure and to suppression of pair formation. At higher temperatures, the peculiar behavior of the equation of state due to the energy gap between the negative and positive energy states of the electrons is less pronounced and γ stays above $\frac{4}{3}$ even for zero density (cf. Fig. 2). Both for high temperatures and high densities, the relativistic effects predominate and γ tends toward $\frac{4}{3}$. At high temperatures, the pressure is determined primarily by radiation and pairs; at high densities the degenerate electron pressure predominates. In both cases, ion pressure is unimportant.

As we see from Figure 1, γ is not very far from $\frac{4}{3}$ in the whole region where oxygen or heavier isotopes can react. The stars being very soft, high accuracy must be maintained in the calculations in order to obtain reliable results. In particular, one can never assume the electron gas to be really non-degenerate (cf. Appendix B), and accurate expressions for the equation of state must be used.

A heavy star, while losing entropy by radiation or neutrino losses, contracts and raises its temperature and density; in Figure 1, the point representing these quantities for any given mass point, moves up and to the right. With evolution proceeding, γ in the central region approaches $\frac{4}{3}$ and the star becomes softer and softer. At last, a minute decrease in entropy raises the central temperature and density to infinity. For completely convective models, we can assign for each mass (larger than the Chandrasekhar limit) a minimum value for the sum of leptonic and electromagnetic entropies (assuming Z/A given, say $\frac{1}{2}$) or inversely: for any entropy a maximum mass. In general, no such unique limits exist, but actually even stars of only a few solar masses move along lines quite near to lines of constant entropy when the central temperature is around 10^9 ° K. A number of values of limiting masses are indicated in Figure 1.[1]

[1] The ionic contribution to the entropy in the region of interest varies only slightly and is normalized to vanish at $T_9 = 1$, $\rho = 1\ 66 \times 10^3$. The calculation of limiting masses will be discussed in a forthcoming publication.

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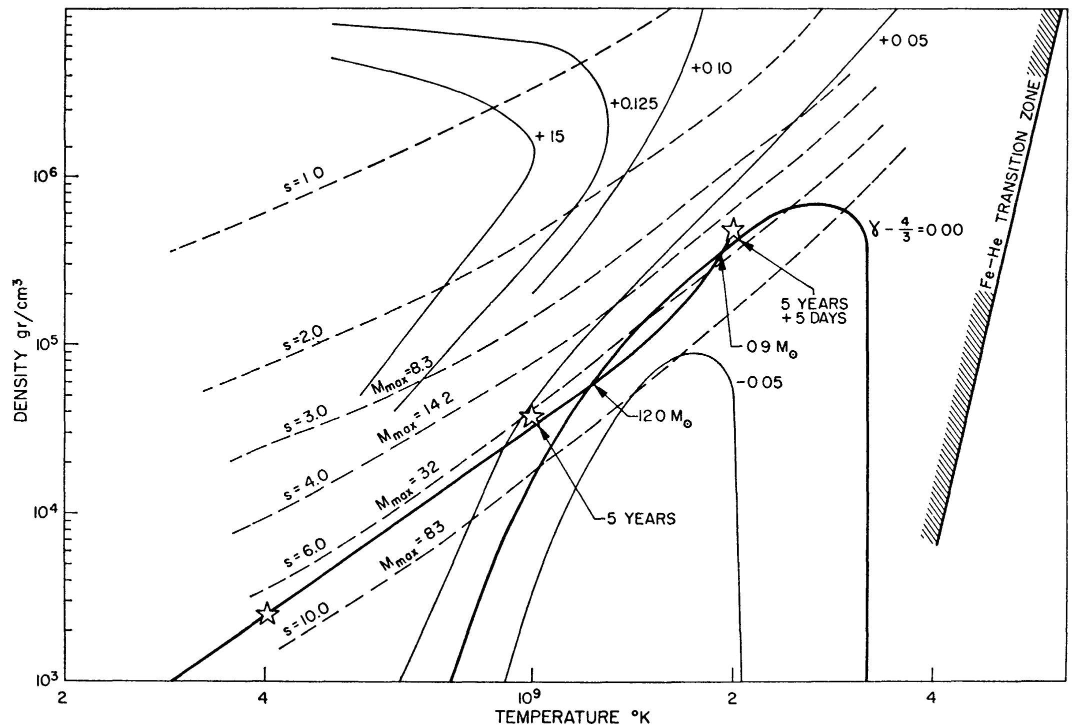

FIG. 1 —Contours of constant $\gamma - \frac{4}{3}$ are represented by thin full lines The contour $\gamma = \frac{4}{3}$ is emphasized by a heavy line Contours of constant s are drawn by broken lines On some of the lines of constant entropy limiting masses for isentropic models are indicated. The Fe–He transition region is indicated by a hatched stripe The stars represent the central condition at three instances during the evolution. The lowest star represents the moment at which neutrino losses start to dominate. The middle star represents the moment of ignition of nuclear reactions, and the highest the moment the model becomes unstable The heavy line descending to the left from the upper star presents the run of ρ versus T along the radius of the model at the moment of instability. It is seen that the part of the star between 0 9 and 12 0 $M\odot$ has γ below $\frac{4}{3}$ at that moment.

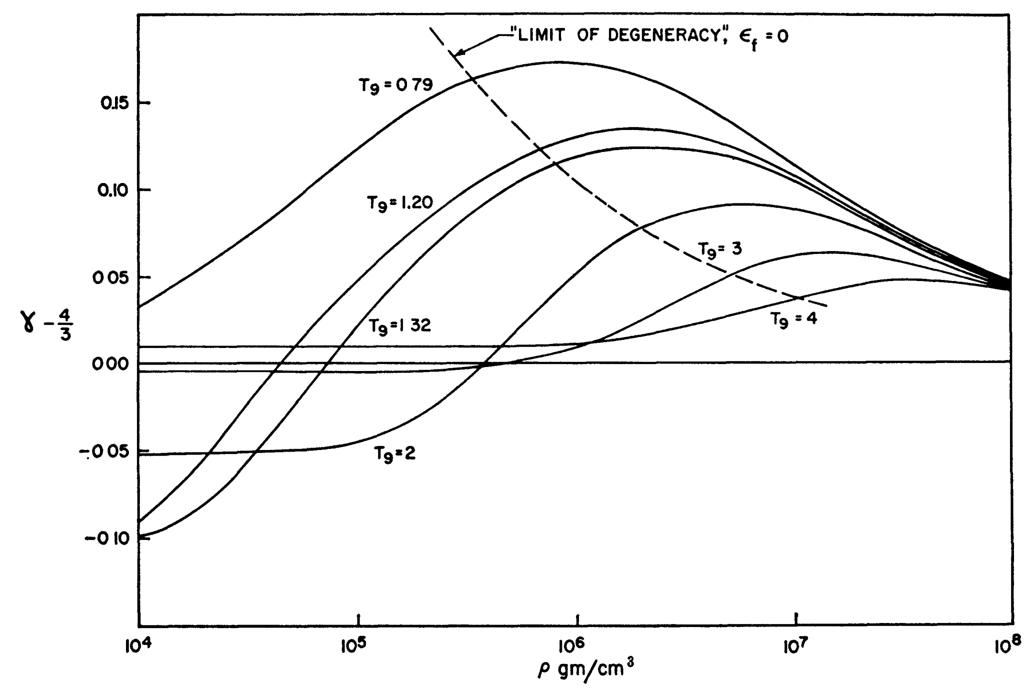

FIG. 2.—($\gamma - \frac{4}{3}$) as function of density for various temperatures

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The central conditions at various stages of evolution for a 30 $M_\odot$ model are indicated in Figure 1 by "stars." These values are taken from a calculation including neutrino losses. If we assume a purely convective model (as actually is the case without neutrino losses) the line of evolution of the central conditions lies just below the "stars" on Figure 1 and approaches asymptotically an entropy $s \sim 6.0$. The line of evolution of the central conditions for a heavier model would lie even lower and thus inevitably enter the region of $\gamma < \frac{4}{3}$ and become unstable.

In general, it is very difficult to state anything about the behavior of a star after a sufficient large fraction of its mass entered the "valley" of instability and a dynamical instability sets in. The increased rate of nuclear reactions at its center would not necessarily result in a vigorous explosion. It may just "push back" the star and hold it. If not much matter is lost until all the oxygen is consumed, a fast collapse may follow with an explosion due to the ignition of nuclear reaction between elements in the S-Si region (produced by the oxygen burning).

V. EVOLUTION OF A 30 $M_\odot$ OXYGEN STAR

As a first exploratory calculation, we calculated the evolution of a 30 $M_\odot$ star of pure oxygen. We started with a density distribution similar to a polytropic model of index $n = 3$ and with a central temperature of about 10^8 ° K. The star started contracting due to photon losses, soon becoming completely convective. Thus it seems that the later history is quite independent of the exact initial entropy distribution. The rate of energy loss of a completely convective star depends crucially on the boundary layer, but the structure is completely determined by the value of the entropy density (which is constant throughout the star).

The choice of pure oxygen for the composition may be an oversimplification. In lieu of detailed evolutionary calculations, we can only give the following general arguments in favor of this model: (1) In massive stars ($M \geq 10\ M_\odot$) the main product of He burning is oxygen (Hoyle 1954; Faulkner 1966). (2) When the star is contracting under photon losses, the whole star is convective, and later when contracting under neutrino losses, the temperature gradient is rather low. Any of these phenomena could result in the formation of a homogeneous oxygen star with almost no envelope of lighter elements.

When the central temperature rises to about $T_9 = 0.5$, neutrino losses quickly take over and convection is rapidly stopped. The further evolution, although predominantly due to neutrino losses, does not seem to be very different from evolution due to photon losses. The loci of temperatures versus density along the radius still lie on a line not too far from a constant entropy line (cf. the line describing the final structure in Fig. 2). The temperature gradient in the central region is somewhat reduced but does not become negative.

When the temperature rises to about $T_9 \cong 1.6$ oxygen starts to burn to sulfur. The temperature continues to rise up to $T_9 \cong 2.1$. In this period, a growing zone of γ less than $\frac{4}{3}$ forms; the lowest value of γ lies a few solar masses from the center and moves slowly outward. The value of γ in the center never does fall below $\frac{4}{3}$ in this particular case. At the same time, nuclear reaction rates in a small region containing a few tenths of a solar mass around the center rise faster than neutrino losses, becoming 5–10 times greater than the latter.

Outside this small central region, neutrino losses predominate. The total rate of energy loss from the star is equal within a factor of 2 to the energy produced by nuclear reactions. Some time after the nuclear reactions ignite, a slowly growing convective zone forms in the center around the nuclear reaction zone.

While nuclear burning goes on at the center at a rather constant rate, the rest of the star goes on contracting due to neutrino losses. The energy released by nuclear reactions

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is transferred only within the convective zone containing less than 2.5 M$\odot$. Before much nuclear fuel was consumed, the stellar model became dynamically unstable and the evolution could not be followed further by our methods. In Table 1 the structure of the model is described at the moment it becomes unstable.

Although this dynamical instability is expected to be a rather general phenomenon in stars heavier than approximately 25 $M\odot$, it was of some interest to check whether the occurrence of the instability in a 30 $M\odot$ model is independent of the detailed physical processes assumed in our calculation. We have repeated the calculation with the following assumptions: (1) no neutrino losses; (2) no nuclear reactions; (3) no convection. Actually, none of these modifications seriously affected the structure of the star. The dynamical instability always occurred at approximately the same central conditions. Only the time scale of evolution is affected by the above-mentioned modifications. In these calculations a radiative zero boundary condition was assumed. As these stars are

TABLE 1

RUN OF IMPORTANT PARAMETERS THROUGH A 30 $M\odot$ STAR JUST BEFORE DYNAMICAL INSTABILITY IS CAUSED BY PAIR FORMATION*

m	r	ρ	T	β	γ	ν	ϵ_f	$q_{\rm nuc}$	q_ν	q
0 4	8 2 $\times 10^{-3}$	1 04$\times 10^{6}$	2 15$\times 10^{9}$	0 678	1 372	1 054	−0 89	1 61$\times 10^{13}$	2 00$\times 10^{12}$	1 41$\times 10^{13}$
0 9	1 1 $\times 10^{-2}$	8 02$\times 10^{5}$	2 07$\times 10^{9}$	0 647	1 362	1 063	−1 08	4 09$\times 10^{12}$	1 68$\times 10^{12}$	2 41$\times 10^{12}$
1 8	1 6 $\times 10^{-2}$	5 46$\times 10^{5}$	1 99$\times 10^{9}$	0 589	1 353	1 093	−1 40	8 09$\times 10^{11}$	1 56$\times 10^{12}$	−7 49$\times 10^{11}$
3 1	2 1 $\times 10^{-2}$	3 40$\times 10^{5}$	1 88$\times 10^{9}$	0 520	1 337	1 135	−1 76	7 95$\times 10^{10}$	1 27$\times 10^{12}$	−1 19$\times 10^{12}$
4 7	2 69$\times 10^{-2}$	2 19$\times 10^{5}$	1 75$\times 10^{9}$	0 467	1 325	1 157	−2 06	4 92$\times 10^{9}$	8 22$\times 10^{11}$	−8 17$\times 10^{11}$
6 6	3 34$\times 10^{-2}$	1 50$\times 10^{5}$	1 62$\times 10^{9}$	0 428	1 319	1 144	−2 30	2 32$\times 10^{8}$	4 37$\times 10^{11}$	−4 37$\times 10^{11}$
8 6	3 98$\times 10^{-2}$	1 10$\times 10^{5}$	1 49$\times 10^{9}$	0 404	1 320	1 110	−2 48	9 71$\times 10^{6}$	2 02$\times 10^{11}$	−2 02$\times 10^{11}$
10 8	4 66$\times 10^{-2}$	8 18$\times 10^{4}$	1 37$\times 10^{9}$	0 386	1 327	1 074	−2 64	3 27$\times 10^{5}$	8 25$\times 10^{10}$	−8 25$\times 10^{10}$
13 2	5 39$\times 10^{-2}$	6 05$\times 10^{4}$	1 25$\times 10^{9}$	0 375	1 338	1 042	−2 79	6 99$\times 10^{3}$	2 83$\times 10^{10}$	−2 83$\times 10^{10}$
15 6	6 16$\times 10^{-2}$	4 43$\times 10^{4}$	1 13$\times 10^{9}$	0 368	1 352	1 020	−2 93		8 13$\times 10^{9}$	−8 13$\times 10^{9}$
20 4	7 90$\times 10^{-2}$	2 22$\times 10^{4}$	8 95$\times 10^{8}$	0 366	1 375	1 002	−3 22		3 42$\times 10^{8}$	−3 42$\times 10^{8}$
25 2 .	1 05$\times 10^{-1}$	7 67$\times 10^{3}$	6 49$\times 10^{8}$	0 343	1 381	1 000	−3 74		2 77$\times 10^{6}$	−2 92$\times 10^{6}$

* No convection has developed yet Nuclear burning is already stabilized The mass m within a radius r is given in solar mass units and the radius in solar radii Density ρ and temperature T are given in gm cm^{-3} and ° K, respectively The quantity β is the ratio of gas pressure to total pressure, γ is defined in eq. (4), and ν and ϵ_f are defined as the number of leptons divided by the number of protons and the Fermi energy divided by kT The quantities $q_{\rm nuc}$, q_ν, and q are given in ergs gm^{-1} sec^{-1}.

red supergiants, they have very extended and massive convective envelopes. The exact mass at which the dynamical instability sets in may therefore depend on the boundary condition.

VI. CONCLUDING REMARKS

We have seen that stars heavier than about 30 $M\odot$ become dynamically unstable due to pair formation before forming any iron core, and they may lose their dynamical stability without burning much of their oxygen. Somewhat lighter stars are supposed to form an appreciable iron core and get into another dynamical instability due to a Fe $\rightleftarrows$ 13 He + 4n "phase transition" (Fowler and Hoyle 1964). The lightest stars, just above the Chandrasekhar limit, are supposed to be thermally unstable (Hoyle and Fowler 1960) when their fuel (probably C, O, or Na-Mg) ignites. Intermediate mass stars may be very near to their limiting entropy when they arrive at the point of oxygen or Mg burning; thus they may contract very rapidly, get far out of hydrostatic equilibrium, greatly "overshoot" the ignition temperature and explode. Still other possibilities have been suggested (Hamada and Salpeter 1961): at high densities electron-capture processes proceeding at a high rate shift the equilibrium Z to lower values. Effectively γ decreases below $\frac{4}{3}$ and a dynamical instability of still another origin develops. All these possibilities should be checked by at least rough evolutionary calculations.

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Primarily, we want to thank Professor William A. Fowler for many illuminating discussions and for showing great interest in the work. We want to thank Professor R. F. Christy for very helpful discussions and for his comments on the work. Thanks are also due to Dr. N. Lebovitz and Professor F. Hoyle for several conversations. It is a pleasure to acknowledge the important contribution of Mr. Z. Zinamon to the development of the mathematical and numerical methods used in this work. We are grateful to Professor Fowler for extending to us the hospitality of the W. K. Kellogg Radiation Laboratory at the California Institute of Technology.

APPENDIX

A. FORMULATION OF THE CONDITION FOR DYNAMICAL STABILITY

The condition for dynamical stability, i.e., that the operator H defined in equation (2) be positive definite, can be formulated in various ways. (For detailed treatment and further references, see the review articles by Ledoux 1958, 1965.) Using equation (3), we can define H at a stationary point in a way that is sometimes more convenient, namely,

$$H = -\frac{d}{dV}\gamma p \frac{d}{dV} + \frac{4}{3V}\left(\frac{dp}{dV}\right). \tag{A1}$$

One obvious way of stating the condition for dynamical stability is that the sign of the lowest eigenvalue E_0 of H should be positive. The boundary conditions for the eigenfunction Ψ_0 are: $\Psi(0) = 0$ and Ψ_0 finite everywhere. One can prove that the sign of E_0 and of the lowest eigenvalues λ_0 of the more general equation

$$H\phi_0 = \lambda_0 g \phi_0 \tag{A2}$$

must be equal. Here, g is an arbitrary positive function of V,[2] and the boundary values for the eigenfunctions ϕ are $\phi(0) = 0$, $(\gamma p d\phi/dV)_{m=M} = 0$.

The relation between the signs of E_0 and λ_0 is easily established by noticing that

$$\lambda_0 = \min\left(\frac{\langle\phi\,|\,H\,|\,\phi\rangle}{\langle\phi\,|\,g\,|\,\psi\rangle}\right). \tag{A3}$$

It is obvious that the specific choice of g does not affect the sign of λ_0. In particular, for $g = 1$, we have $\lambda_0 = E_0$. At least three different choices for the normalization g have been used in the literature:

1. The simplest case $g = 1$ is sometimes referred to as the "energy method."
2. Dyson's equation (Dyson 1961) is obtained by choosing

$$\lambda = \Gamma - \tfrac{4}{3}, \qquad g = -\frac{1}{V}\frac{dp}{dV}. \tag{A4}$$

3. The equation of small oscillations gives a result equivalent to the choice

$$\lambda = \omega^2, \qquad g = \frac{1}{16\pi^2}\frac{\rho}{r^4}. \tag{A5}$$

It can easily be seen by direct substitution that for a constant γ, $\Psi = V$ is an eigenfunction of Dyson's equation with an eigenvalue $\Gamma = \gamma$.

[2] Actually, one can generalize the correspondence in signs of eigenvalues also to cases that g is a more general operator.

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In the general case, we have no analytical expressions, but the lowest eigenvalue E_0 of H can be approximated by using $\Psi = V$ as a trial function in equation (A4). We then find that

$$E_0 \cong \frac{\langle V \mid H \mid V \rangle}{\langle V \mid V \rangle} = \frac{\int_0^{V_0} \left(\gamma - \frac{4}{3} \right) p dV}{\frac{1}{3} V_0{}^3}, \tag{A6}$$

where V_0 is the volume of the star Using the same approximation, we obtain for the eigenvalue of Dyson's equation

$$\Gamma = \frac{\int \gamma p dV}{\int p dV}. \tag{A7}$$

In order to obtain the sign of the eigenvalue of H, we do not need actually to solve equations (A2) or (A3); it is enough to integrate the equation

$$H\Psi = 0,$$

starting from the origin with $\Psi = 0$. If Ψ has no nodes, the lowest eigenvalue is positive and the model is dynamically stable.

B. THE EQUATION OF STATE

We consider a plasma of ions, electrons, positrons, and radiation in thermodynamic equilibrium; the equations of state are expressed naturally as functions of the temperature T, the density ρ, and the chemical potential of the leptons μ. Actually μ is not an independent variable; it must be determined by the condition of charge neutrality,

$$\left\langle \frac{Z}{A} \right\rangle \rho = \rho_{e^-} - \rho_{e^+} = \rho_e(\mu, T). \tag{B1}$$

An explicit expression $\mu(\rho,T)$ can be obtained when the electrons are non-degenerate, i.e , when

$$\frac{\mu - m c^2}{T} = \epsilon_f < -6. \tag{B2}$$

Otherwise, equation (B1) must be solved numerically. Electrons and positrons are treated relativistically when

$$\frac{kT}{m c^2} > \frac{1}{30}. \tag{B3}$$

Otherwise, non-relativistic expressions are used and the density of pairs, i.e., ρ_{e^+} is assumed to vanish.

Each of the thermodynamic quantities is expressed as a sum of three terms originating from the ions, radiation and leptons. Using "astrophysical units" (Frank-Kamenetskii 1959) defined below, we have

$$p_{\text{ions}} = \left\langle \frac{1}{A} \right\rangle \rho T, \qquad u_{\text{ions}} = \frac{3}{2} \left\langle \frac{1}{A} \right\rangle T, \qquad s_{\text{ions}} = \left\langle \frac{1}{A} \right\rangle \rho T, \tag{B4}$$

$$p_{\text{rad}} = \frac{1}{3} a T^4, \qquad u_{\text{rad}} = a \frac{T^4}{\rho}, \qquad s_{\text{rad}} = \frac{4}{3} a \frac{T^3}{\rho}, \tag{B5}$$

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$$p_e = mc^2 C F_1(\phi,\beta), \qquad u_e = mc^2 C F_3(\phi,\beta) - \left\langle\frac{Z}{A}\right\rangle mc^2, \tag{B6}$$

$$s_e = \frac{C\beta}{\rho}\left[F_1(\phi,\beta) + F_3(\phi,\beta) - \frac{\phi}{\beta} F_2(\phi,\beta)\right], \tag{B7}$$

and for the lepton charge density we have

$$\rho_e(\mu,T) = C F_2(\phi,\beta). \tag{B8}$$

Here, we have used the following quantities:

$$\left\langle\frac{1}{A}\right\rangle = \text{average reciprocal atomic weight},$$

$$\left\langle\frac{Z}{A}\right\rangle = \text{average number of electrons per unit mass},$$

$$a = 0.1854, \quad C = \frac{1}{\pi^2}\left(\frac{mc}{\hbar}\right)^3 = 0.5075\times 10^6, \quad \beta = \frac{mc^2}{T} = \frac{259}{T}, \quad \phi = \frac{\mu}{T} = \epsilon_f + \beta.$$

The functions $F_\nu(\phi,\beta)$ are defined below. Often one needs various derivatives of the thermodynamic functions. The contributions of the ions and radiation to these derivatives are elementary. The contributions of the leptons are

$$\frac{\partial\rho_e(\phi,\beta)}{\partial\phi} = \frac{CF_4}{\beta}, \qquad \frac{\partial\rho_e(\phi,\beta)}{\partial\beta} = -\frac{C(F_2+F_5)}{\beta}, \tag{B9}$$

$$\left(\frac{\partial\phi}{\partial\beta}\right)_\rho = -\frac{\partial\rho_e/\partial\beta}{\partial\rho_e/\partial\phi} \tag{B10}$$

$$\frac{\partial p_e(\rho,T)}{\partial\rho} = \frac{\rho_e}{\rho} C mc^2 \frac{\partial F_1/\partial\phi}{\partial\rho_e/\partial\phi} = \frac{\rho_e}{\rho} CT \frac{F_2}{\partial\rho_e/\partial\phi} \tag{B11}$$

$$\frac{\partial p_e(\rho,T)}{\partial T} = C\beta\left[F_1 + F_3 - F_2\left(\frac{\partial\phi}{\partial\beta}\right)_\rho\right] \tag{B12}$$

$$\frac{\partial s_e(\phi,\beta,\rho)}{\partial\phi} = \frac{C}{\rho}\left(F_2 + F_5 - \frac{\phi}{\beta}F_4\right) \tag{B13}$$

$$\frac{\partial s_e(\phi,\beta,\rho)}{\partial\beta} = \frac{C}{\rho}\left[-2F_3 - F_6 + \frac{\phi}{\beta}(F_2+F_5)\right] \tag{B14}$$

$$\frac{\partial s_e(\rho,T)}{\partial\rho} = \frac{\rho_e}{\rho}\left\{-\frac{s_e}{\rho_e} + \frac{\partial s_e(\phi,\beta,\rho)/\partial\phi}{\partial\rho_e/\partial\phi}\right\} \tag{B15}$$

$$\frac{\partial s_e(\rho,T)}{\partial T} = -\frac{\beta}{T}\left[\frac{\partial s_e(\phi,\beta,\rho)}{\partial\beta} + \frac{\partial s_e(\phi,\beta,\rho)}{\partial\phi}\left(\frac{\partial\phi}{\partial\beta}\right)_\rho\right]. \tag{B16}$$

The ratio between the total number of leptons (electrons and positrons) to the electric charge of the leptons (measured in units of the electron charge) is

$$\nu = \frac{F_2^+(\phi,\beta)}{F_2(\phi,\beta)}. \tag{B17}$$

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The functions $F_\nu(\phi,\beta)$ are defined by the following integrals:

$$
\begin{aligned}
F_1(\phi,\beta) &= \int_{\epsilon=\beta}^{\infty} \Gamma\,\frac{\epsilon}{\beta}\,D^+(\epsilon,\beta)\frac{d\epsilon}{\beta} = F_{1n}+F_{1p}\,,\\
F_2(\phi,\beta) &= \beta\,\frac{\partial}{\partial\phi}F_1 = \int \Gamma' D^-\,\frac{d\epsilon}{\beta} = F_{2n}-F_{2p}\,,\\
F_2{}^+(\phi,\beta) &= \int \Gamma' D^+\,\frac{d\epsilon}{\beta} = F_{2n}+F_{2p}\,,\\
F_3(\phi,\beta) &= -\frac{\partial}{\partial\beta}(\beta F_1) = \int\left(\frac{\epsilon}{\beta}\right)\Gamma' D^+\,\frac{d\epsilon}{\beta} = F_{3n}+F_{3p}\,,\\
F_4(\phi,\beta) &= \beta\,\frac{\partial F_2}{\partial\phi} = \int \Gamma'' D^+\frac{d\epsilon}{\beta} = F_{4n}+F_{4p}\,,\\
F_5(\phi,\beta) &= -\frac{\partial}{\partial\beta}(\beta F_2) = \int\left(\frac{\epsilon}{\beta}\right)\Gamma'' D^-\frac{d\epsilon}{\beta} = F_{5n}-F_{5p}\,,\\
F_6(\phi,\beta) &= -\frac{1}{\beta}\,\frac{\partial}{\partial\beta}(\beta^2F_3) = \int\left(\frac{\epsilon}{\beta}\right)^2\Gamma'' D^+\frac{d\epsilon}{\beta} = F_{6n}+F_{6p}\,,
\end{aligned}
\tag{B18}
$$

where $\phi = \mu/T$;

$$
\Gamma(x) = \tfrac{1}{3}(x^2-1)^{3/2}\,; \qquad \Gamma' = \frac{d\Gamma(x)}{dx}\,; \qquad \Gamma'' = \frac{d^2\Gamma(x)}{dx^2}\,;
$$

$$
D^\pm(\epsilon,\phi) = \frac{1}{e^{\epsilon-\phi}+1} \pm \frac{1}{e^{\epsilon+\phi}+1}\,.
$$

The method of evaluation of the terms $F_{\nu n}$ originating from the first term of $D^\pm$ depends on the value of $\epsilon_f = \phi - \beta$. For $\epsilon_f < 0$ (weak degeneracy) the integrals can be evaluated by a series expansion. The contribution $F_{\nu p}$ to the integrals from the second term of $D^\pm$ can always be evaluated by the same series by replacing ϕ by $-\phi$. For $\epsilon_f < 0$, we thus have

$$
\begin{aligned}
F_1 &= \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{(n\beta)^2}\,, \qquad F_2 = \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n\beta}\,e^{n\phi}K_2(n\beta) - F_{2p}\,,\\
F_2{}^+ &= F_{2n}+F_{2p}\,, \qquad F_3 = \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{4n\beta}\,e^{n\phi}[\,3K_3(n\beta)+K_1(n\beta)\,]\,,\\
F_4 &= \sum_{n=1}^{\infty}(-)^{n+1}e^{n\phi}K_2(n\beta)+F_{4p}\,,\\
F_5 &= \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{2}e^{n\phi}[\,K_3(n\beta)+K_1(n\beta)\,]-F_{5p}\,,\\
F_6 &= \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{2n\beta}\,e^{n\phi}(3K_3-K_1+2n\beta K_2)+F_{6p}\,,
\end{aligned}
\tag{B19}
$$

where K_ν denotes the modified Bessel function of order ν.

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For $\epsilon_f \gg 1$, the integrals are evaluated by a method due to Sommerfeld (1928). One obtains

$$F_1(\phi,\beta) = L(x,\beta) = \tfrac{1}{24}\left[f(x) + \frac{4\pi^2}{\beta^2}(x^2+1)^{1/2} + \tfrac{7}{15}\frac{\pi^4}{\beta^4}\left(\frac{2}{x}-\frac{1}{x^3}\right)(x^2-1)^{1/2}\right] + F_{1p}(\phi,\beta) \tag{B20}$$

where

$$x = [(\phi/\beta)^2 - 1]^{1/2}, \qquad f(x) = x(2x^2-3)(x^2+1)^{1/2} + 3\sinh^{-1}(x).$$

The other functions are obtained from F_1 by differentiation. For intermediate values of $\epsilon_f(-0.1 < \epsilon_f < 4)$, the integrals are evaluated by direct numerical integration.

In the non-relativistic limit, $\beta \gg 1$, the functions $F_\nu(\phi,\beta)$ can be expanded in powers of $1/\beta$ in terms of the functions (11):

$$I_k(\epsilon_f) = \int_0^\infty \frac{x^k}{e^{x-\epsilon_f}+1}\,dx\,, \tag{B21}$$

for example,

$$F_1 = \frac{2^{3/2}}{3}\left(\frac{1}{\beta^{5/2}} I_{3/2} + \tfrac{3}{4}\frac{1}{\beta^{7/2}} I_{5/2} + \ldots\right),$$

$$F_2 = 2^{1/2}\left(\frac{1}{\beta^{3/2}} I_{1/2} + \tfrac{1}{4}\frac{1}{\beta^{5/2}} I_{3/2} + \ldots\right), \tag{B22}$$

$$F_3 = 2^{1/2}\left(\frac{1}{\beta^{3/2}} I_{1/2} + \tfrac{5}{4}\frac{1}{\beta^{5/2}} I_{3/2} + \ldots\right);$$

when $\beta > 30$ we obtain an accuracy of a few per cent using the first term in these expansions. The thermodynamic quantities and their derivatives we write as follows:

$$\rho_e = \frac{2^{1/2}C}{\beta^{3/2}} I_{1/2}(\epsilon_f), \qquad p_e = \rho_e T D_1\,, \tag{B23}$$

$$u_e = 1.5(\rho_e/\rho)TD_1\,, \qquad s_e = (\rho_e/\rho)(\tfrac{5}{2}D_1 - \epsilon_f), \tag{B24}$$

and for the derivatives of these quantities, we have

$$\frac{\partial\rho_e(\epsilon_f,\beta)}{\partial\epsilon_f} = \frac{\rho_e}{D_2}, \qquad \frac{\partial\rho_e(\epsilon_f,\beta)}{\partial\beta} = -\tfrac{3}{2}\frac{\rho_e}{\beta}, \tag{B25}$$

$$\left(\frac{\partial\epsilon_f}{\partial\beta}\right)_\rho = -\frac{\partial\rho_e/\partial\beta}{\partial\rho_e/\partial\epsilon_f} = \tfrac{3}{2}\frac{D_2}{\beta}, \tag{B26}$$

$$\frac{\partial p_e(\rho,T)}{\partial\rho} = \frac{\rho_e}{\rho}TD_2\,, \qquad \frac{\partial p_e(\rho,T)}{\partial T} = \rho_e D_3\,, \tag{B27}$$

$$\frac{\partial s_e(\rho,T)}{\partial\rho} = -\frac{\rho_e}{\rho^2}D_3\,, \qquad \frac{\partial s_e(\rho,T)}{\partial T} = \tfrac{3}{2}\frac{\rho_e}{\rho}\frac{D_3}{T}. \tag{B28}$$

We used

$$D_1 = \frac{I_{3/2}(\epsilon_f)}{dI_{3/2}(\epsilon_f)/d\epsilon_f}, \qquad D_2 = \frac{I_{1/2}(\epsilon_f)}{dI_{1/2}(\epsilon_f)/d\epsilon_f}, \qquad D_3 = \tfrac{5}{2}D_1 - \tfrac{3}{2}D_2\,. \tag{B29}$$

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The non-degenerate approximation is valid only if ϵ_f is negative and large in absolute value. To preserve an accuracy of 1 part in 10^3, we need $\epsilon_f < -7$. In the non-relativistic region, the non-degenerate case is trivial; it is obtained by putting $D_1 = D_2 = D_3 = 1$. In the relativistic region, it is difficult to speak of a really non-degenerate case. As an excess of negative leptonic charge requires $\phi > 0$, we have $\epsilon_f > -\beta$. Thus, for β not much larger than unity, ϵ_f does not become very negative.

The quantities α, γ, ν defined in equations (15) are easily expressed in terms of the derivatives defined in this appendix.

The astrophysical units are chosen so as to make the radius and mass of the Sun equal to 1. Further, one chooses the gravitational constant G as unity or the luminosity of the Sun as unity. The two alternatives yield different units of time. The following relations between the astrophysical system and the CGS system result:

$$T_\odot = E_\odot(M_\odot/M_{\mathrm{H}}) = 2.29\times 10^7\,^\circ\mathrm{K} = 3.155\times 10^9 \text{ erg/k} ,$$

$$M_\odot = 1.99\times 10^{33} \text{ gm} , \qquad R_\odot = 6.96\times 10^{10} \text{ cm} ,$$

$$\rho_\odot = M_\odot/R_\odot^3 = 5.9 \text{ gm/cm}^3 , \qquad u_\odot = E_\odot/M_\odot = 1.905\times 10^{15} \text{ erg gm}^{-1}$$

$$p_\odot = E_\odot/R_\odot^3 = 1.124\times 10^{16} \text{ ergs cm}^{-3} , \qquad m_e c^2/kT_\odot = 259 .$$

C. DETAILS OF NUCLEAR REACTION RATES AND OF NEUTRINO PROCESSES

At temperatures at which oxygen burning occurs in massive stars, $T_9 \sim 1.6$–2.2, the nuclei O^{17}, F^{17}, and Ne^{20} undergo fast photodisintegration, while S^{32} starts to photodisintegrate at somewhat higher temperatures. This situation favors the production of S^{32} and its neighbors over production of lighter isotopes.

The over-all process of oxygen burning can be represented by (Fowler and Hoyle 1964)

$$2\,O^{16} \rightarrow S^{32} + 16.54 \text{ MeV} .$$

The lifetime in seconds is given by

$$\log \tau(O^{16}) = -38.0 - \log \rho x_{16} + \tfrac{2}{3}\log T_9 + \frac{59.04}{T_9^{1/3}}(1+0.080\,T_9)^{1/3} \qquad \text{(C1)}$$

and the rate of energy production in ergs gm^{-1} sec^{-1} is

$$\log \epsilon_{00} = 55.7 + \log \rho x_{16}^2 - \tfrac{2}{3}\log T_9 - \frac{59.04}{T_9^{1/3}}(1+0.080\,T_9)^{1/3} . \qquad \text{(C2)}$$

It should be mentioned that at $T_9 \sim 2.2$, the lifetime of O^{16} to the $O^{16}(\gamma,\alpha)C^{12}$ reaction is $\sim 10^7$ sec. This time is much longer than the time for oxygen burning obtained in our calculation (about 10^5 sec from oxygen ignition at $T_9 \sim 1.6$ to dynamical instability at $T_9 \sim 2.1$). Photodisintegration of oxygen can therefore be neglected.

Neutrino losses were assumed to consist of pair annihilation ($e^+ + e^- \rightarrow \nu + \bar{\nu}$) and photoneutrino ($\gamma + e^- \rightarrow e^- + \nu + \bar{\nu}$) processes. In all cases, the non-degenerate approximation was used.

Neutrino energy losses due to pair annihilation are given by (Fowler and Hoyle 1964)

$$\epsilon_\nu = 0.325\times 10^{21}\left(\frac{1}{\beta}\right)^3 \left(2\beta K_1 K_2 + 5K_2^2 + 2K_1K_3 + \frac{8}{\beta}K_2K_3\right) \text{erg gm}^{-1}\text{ sec}^{-1} \qquad \text{(C3)}$$

where $\beta = 5.93/T_9$ and $K_\nu(\beta)$ are the modified Bessel functions of order ν.

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In the non-relativistic approximation, the rate of neutrino energy generation from photoneutrino processes is (Chiu 1961; Chiu and Stabler 1961)

$$\epsilon_\nu = \frac{10^8}{\mu_e} T_9{}^8 \text{ erg gm}^{-1} \text{ sec}^{-1} . \tag{C4}$$

For $T_9 > 0.75$ the neutrino losses due to pair annihilation are much greater than those due to photoneutrino and hence above this temperature photoneutrino losses were neglected. For $T_9 < 0.75$, both processes were taken into account.

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