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THE ANGULAR MOMENTUM OF THE SOLAR WIND*

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ABSTRACT

A steady-state model of the solar-wind flow in the equatorial plane including the effects of pressure gradients, gravitation, and magnetic forces is developed and solved for both the radial and azimuthal motions. The viscosity is taken to be zero, the electrical conductivity to be infinite, and the energy supply characterized by a polytropic index. The solution must pass through three critical points, whose significance is explained in terms of the characteristic velocities with which disturbances propagate. A numerical solution is obtained for typical parameters The magnetic field produces only a modest tendency toward co-rotation of the outer corona, but the magnetic stresses apply a torque to the Sun equal to that required to produce effective co-rotation out to the radial distance where the radial Alfvénic Mach number equals unity. For typical solar-wind values this will occur between 15 and 50 solar radii out, which implies a substantial loss in the angular momentum of the Sun.

I. INTRODUCTION

The magnetic field at the Sun is of the order of 1 gauss and can thus affect the motion of a highly conducting fluid such as the solar wind. While the magnetic-force terms in the radial equation of motion are small compared to the gravitational or pressure terms (at least for our Sun), they are dominant in the rotational motion and have to be included even in the radial motion to give a complete and proper solution. Our purpose is to devise a model simple enough to be readily solved and at the same time complete enough to give a sound understanding of the azimuthal motion of the solar wind, the extent to which there is co-rotation, and the rate of loss of angular momentum from the Sun. If the model is to be readily applicable to other stars with higher magnetic fields and higher rotation rates, it should be reasonably complete.

II. BASIC POSTULATES AND ASSUMPTIONS

The motion of the solar wind will be described by means of the magnetohydrodynamic equations for a fluid with an infinite conductivity, no viscosity, and a scalar pressure. The Sun is assumed to have a general magnetic field that depends only on latitude. The local irregularities in the field, the polarity reversals and wind velocity fluctuations of the sector structure, and the waves superimposed on the smooth field in interplanetary space are all unessential in treating the basic spiral magnetic pattern and the average angular momentum in the solar wind. Accordingly, we examine a steady-state model with complete axial symmetry in which in the equatorial plane of the Sun the field is combed out by the solar wind and has no component normal to this plane. There will be a flux return at other latitudes, but we concentrate our attention on the equatorial plane. Thus in spherical polar coordinates, r, the distance from the center of the Sun, is the only independent variable, there being no ϕ-dependence. It is further assumed that in the steady-state solar wind the velocities and magnetic fields as well as their derivatives are continuous, smooth functions of position, i.e., no shocks exist anywhere.

III. SOLAR-WIND EQUATIONS

The solar wind has a velocity

$$v = ue_r + v_\phi e_\phi \tag{1}$$

and a magnetic field

$$B = B_r e_r + B_\phi e_\phi \,. \tag{2}$$

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Conservation of mass requires that

$$\rho u r^2 = \text{const.}\,, \tag{3}$$

where ρ is the mass density in gm cm^{-3}.

The solar wind is a perfect conductor, thus $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}/c$, in Gaussian units, and in a steady state we obtain from Maxwell's equations

$$c\,(\nabla \times \boldsymbol{E})_\phi = \frac{1}{r}\frac{d}{dr}[\,r\,(uB_\phi - v_\phi B_r)\,] = 0\,. \tag{4}$$

But in a perfectly conducting fluid $\boldsymbol{v}$ is parallel to $\boldsymbol{B}$ in a frame that rotates with the Sun, and thus we obtain

$$r(uB_\phi - v_\phi B_r) = \text{const.} = -\Omega r^2 B_r\,, \tag{5}$$

where Ω is the angular velocity of the roots of the lines of force in the Sun. Also, since div $\boldsymbol{B} = 0$,

$$r^2 B_r = \text{const.} = r_0{}^2 B_0\,, \tag{6}$$

where the subscript 0 refers to an arbitrary reference level, $r = r_0$.

This model has ϕ-symmetry, and thus the momentum term and the magnetic-force term are the only terms that enter the steady-state ϕ-equation of motion

$$\rho\,\frac{u}{r}\frac{d}{dr}(\,r v_\phi) = \frac{1}{c}(\boldsymbol{J} \times \boldsymbol{B})_\phi = \frac{1}{4\pi}[\,(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}\,]_\phi = \frac{B_r}{4\pi r}\frac{d}{dr}(\,rB_\phi)\,. \tag{7}$$

But

$$\frac{B_r}{4\pi\rho u} = \frac{B_r r^2}{4\pi\rho u r^2} = \text{const.}\,, \tag{8}$$

which allows us to integrate the azimuthal equation of motion immediately, getting

$$r v_\phi - \frac{B_r}{(4\pi\rho u)}\,rB_\phi = \text{const.} = L\,. \tag{9}$$

The first term in the equation is the ordinary angular momentum per unit mass and the second term represents the torque associated with the magnetic stresses. Their sum must be a constant, the total angular momentum carried away from the Sun per unit mass loss.

It is convenient to introduce a new variable, M_A, which we call the radial Alfvénic Mach number and which is defined by

$$M_A{}^2 = \frac{4\pi\rho u^2}{B_r{}^2}\,. \tag{10}$$

Now solve equations (5) and (9) for the azimuthal velocity, getting

$$v_\phi = \Omega r\,\frac{(M_A{}^2 L r^{-2}\Omega^{-1} - 1)}{(M_A{}^2 - 1)}\,. \tag{11}$$

The radial Mach number M_A is much smaller than 1 near the surface of the Sun, but at 1 a.u. M_A is approximately 10. Thus there exists a point between the Sun and Earth where $M_A = 1$; let the radius and radial velocity at this point be called r_a and u_a, respectively. This point will be called the Alfvénic critical point. The denominator of expression (11) will go to zero at this point; in order to keep the expression for v_ϕ finite, we require that the numerator vanish identically at the same point. Thus the magnetic field must arrange itself so that the constant L has the value

$$L = \Omega r_a{}^2\,. \tag{12}$$

But from equations (3), (6), (8), and (10), we see that M_A^2/ur^2 is a constant which may be evaluated at the critical point to give

$$M_A^2 = \frac{u r^2}{u_a r_a^2} = \frac{\rho_a}{\rho} . \tag{13}$$

This reduces equation (11) to

$$v_\phi = \frac{\Omega r}{u_a} \frac{u_a - u}{1 - M_A^2} . \tag{14}$$

The azimuthal magnetic field is thus given by

$$B_\phi = -B_r \frac{\Omega r}{u_a} \frac{r_a^2 - r^2}{r_a^2 (1 - M_A^2)} . \tag{15}$$

It should be noted that the constant L is not determined by the value of B_ϕ or v_ϕ at the sun. Instead, L is determined from conditions at the Alfvénic critical point and this fixes B_ϕ for all smaller r. The values of B_ϕ and u at the sun fix v_ϕ there.

The asymptotic behavior of these functions v_ϕ and B_ϕ can now be obtained. For $r \gg r_a$, the radial velocity, u, in the usual solutions is almost a constant and thus $M_A \propto r$ and both v_ϕ and B_ϕ vary as $1/r$. For $r \ll r_a$, where $u \ll u_a$, equation (15) gives us for the azimuthal field

$$B_\phi = -B_r \frac{r\Omega}{u_a} \left[1 - \frac{r^2}{r_a^2} \left(1 - \frac{u}{u_a} \right) + \ldots \right] , \tag{16}$$

whereas

$$v_\phi = \Omega r \left[1 - \frac{u}{u_a} \left(1 - \frac{r^2}{r_a^2} \right) + \ldots \right] . \tag{17}$$

Near the surface of the Sun most of the angular-momentum loss is due to the torque exerted by the magnetic fields. As we go farther away from the Sun, the azimuthal fluid velocity increases and the magnetic stress decreases until at large distances the relative contributions to the angular-momentum loss are $[1 - (u_a/u_\infty)]$ and u_a/u_∞, respectively.

If we can find, or are given, either u or ρ as a function of r, we can immediately determine B_ϕ and v_ϕ. It might be expected that for the Sun it would be a reasonable approximation to use any of the known solutions for the radial motion, neglecting B_ϕ and v_ϕ. However, it will be more satisfying to include these terms, and for some stars this may be essential. Thus we take as the radial momentum equation

$$\rho u \frac{d u}{d r} = -\frac{d}{d r} p - \rho \frac{G M_\odot}{r^2} + \frac{1}{c} (J \times B)_r + \rho \frac{v_\phi^2}{r} , \tag{18}$$

where p is the pressure, G is the universal gravitational constant, and $M_\odot$ is the mass of the Sun.

Since in a fully ionized gas of pure hydrogen the effective particle mass is only half the hydrogen mass, m, the equation of state is

$$p = \frac{2kT}{m} \rho , \tag{19}$$

where T is the temperature of the gas (assumed equal for ions and electrons), and k is Boltzmann's constant. However, instead of determining the temperature from specified sources of energy and the energy equation, we make the approximation that is usual when the sources are poorly known and use the polytrope law

$$p = p_0 \left(\frac{\rho}{\rho_0} \right)^\gamma = p_a \left(\frac{\rho}{\rho_a} \right)^\gamma , \tag{20}$$

where γ is the polytropic index.

The magnetic force can be written as

$$\frac{1}{c}(J\times B)_r = -\frac{1}{4\pi r}B_\phi\frac{d}{dr}(rB_\phi), \tag{21}$$

which substituted into expression (18) results in

$$\frac{d}{dr}\left\{\tfrac{1}{2}u^2+\frac{\gamma}{\gamma-1}\frac{p_a}{\rho_a}\left(\frac{\rho}{\rho_a}\right)^{\gamma-1}-\frac{GM_\odot}{r}\right\}=\frac{v_\phi{}^2}{r}-\frac{1}{8\pi\rho r^2}\frac{d}{dr}(rB_\phi)^2. \tag{22}$$

This equation with the right-hand side equal to zero will be recognized as Parker's (1958) equation of motion for the solar wind. The terms on the right-hand side are the additional terms which are present due to the inclusion of the magnetic force and the azimuthal velocity. Now express ρ, v_ϕ, and B_ϕ in terms of u and r by means of equations (13)–(15). Algebraic manipulation then gives

$$\begin{aligned}\frac{du}{dr}=\frac{u}{r}&\left\{\left(\frac{2\gamma p_a}{\rho_a M_A{}^{2(\gamma-1)}}-\frac{GM_\odot}{r}\right)(M_A{}^2-1)^3\right.\\&\left.+\Omega^2 r^2\left(\frac{u}{u_a}-1\right)\left[(M_A{}^2+1)\frac{u}{u_a}-3M_A{}^2+1\right]\right\}\\&\times\left[\left(u^2-\frac{\gamma p_a}{\rho_a M_A{}^{2(\gamma-1)}}\right)(M_A{}^2-1)^3-\Omega^2 r^2 M_A{}^2\left(\frac{r_a{}^2}{r^2}-1\right)^2\right]^{-1}.\end{aligned} \tag{23}$$

This is an equation in only the two variables u and r when $M_A{}^2$ is expressed by means of equation (13). It will be noticed immediately that the Alfvénic critical point $r = r_a$ is also a critical point of the radial equation. As we shall see below, this is a singularity of higher order than a simple "X"-type singularity. We shall also see that there are two more critical points.

The radial equation of motion (23) can be integrated to give F, the total energy flux per steradian, which is a constant for our solution

$$F=\rho u r^2\left\{\frac{u^2}{2}+\frac{\gamma}{\gamma-1}\frac{p_a}{\rho_a}M_A{}^{-2(\gamma-1)}-\frac{GM_\odot}{r}+\frac{\Omega^2 r_a{}^4}{2r^2}\left[1+\frac{(2M_A{}^2-1)(r^2-r_a{}^2)^2}{r_a{}^4(M_A{}^2-1)^2}\right]\right\}. \tag{24}$$

The first term in the equation is the kinetic energy associated with the radial velocity, the second the sum of the enthalpy and the energy transported by thermal conduction, magnetic heating, etc. (this is the result of using a polytropic law), the third is the gravitational energy, and the fourth term is the sum of the magnetic and rotational energies. To show its nature more clearly, we can rewrite this term using equations (14) and (15) for the azimuthal velocity and magnetic field and obtain

$$F_{\text{rot+mag}}=\rho u r^2\left(\frac{v_\phi{}^2}{2}-\frac{B_\phi B_r}{4\pi\rho}\frac{\Omega r}{u}\right). \tag{25}$$

The first term is the kinetic energy flux associated with the azimuthal velocity. The second term represents the energy transported out by the magnetic field and is the Poynting energy flux. The total energy flux as given in equation (24) is constant. If there are energy sources or sinks in the interplanetary medium which we have not included in our model, either F must be regarded as a function of r or additional terms must be inserted in equation (24).

IV. TOPOLOGY OF THE SOLUTION

In the neighborhood of the origin, the asymptotic forms for the radial velocity u are

$$u = a_0 r^{(3-2\gamma)/(\gamma-1)}(1+a_1 r - a_2 r^3 + \ldots) \tag{26}$$

and

$$u = \frac{b_0}{r^{1/2}}[1 + b_1 r - b_2 r^{(5-3\gamma)/2} + \ldots]. \tag{27}$$

Equation (26) shows clearly that in this model, as well as in Parker's model, there is no solution in which u approaches zero for small r when $\gamma > \frac{3}{2}$. As r tends to zero, the density increases as $r^{-1/(\gamma-1)}$ for the case of equation (26) and as $r^{-3/2}$ for the case of equation (27). The constants a_0, a_1, a_2, b_0, b_1, b_2 depend on the initial conditions and the parameters entering equation (24). If the solutions are required to run through the

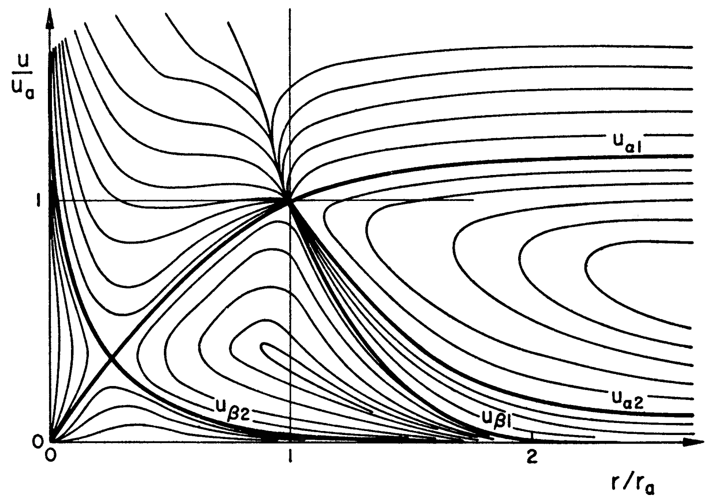

FIG. 1.—Family of solutions of eq. (23) for a given γ and r_a. The solutions passing through the critical points are designated as u_{a1} and u_{a2} (with zero pressure at infinity) and $u_{\beta 1}$ and $u_{\beta 2}$ (with non-zero pressure at infinity).

critical points, these constants depend on the properties there. The asymptotic behaviors at large distances are given by

$$u_a = a_0 \left[1 + a_1 \frac{1}{r^{2(\gamma-1)}} + a_2 \frac{1}{r} + a_3 \frac{1}{r^2} + \ldots\right] \tag{28}$$

and

$$u_\beta = \beta_0 / r^2 \left[1 - \frac{1}{r^2}(\beta_1 - \beta_2 / r) + \ldots\right], \tag{29}$$

where again the values of the a's and β's are determined in the same way as the constants of equations (26) and (27). It is of interest to determine in particular the asymptotic behavior of those solutions which pass through the critical points. If the constants are evaluated for these solutions, we find that two branches behave like u_a and two like u_β. If we call these solutions u_{a1}, u_{a2}, $u_{\beta 1}$, and $u_{\beta 2}$, then we find that u_{a1} gives the behavior of a supersonic, super-Alfvénic wind at infinity, whereas u_{a2} will remain super-Alfvénic, but becomes subsonic again after passing through the critical points. If u_{a1} has a velocity at infinity of 425 km sec^{-1}, then the value for u_{a2} will be approximately 9 km sec^{-1}. For both u_{a1} and u_{a2} the pressure tends to zero as r becomes very large. The remaining two solutions $u_{\beta 1}$ and $u_{\beta 2}$ yield non-zero pressures at infinity.

The topology of the solution is shown in Figure 1, which appears to show two critical points only, a standard "X"-type singularity at $r = r_c$ and a much higher-order singularity at $r = r_a$. Actually two singularities, one higher-order singularity at $r = r_a$ and a simple "X"-type singularity slightly further out at $r = r_f$ nearly coincide, as shown by Figure 2, which is an enlargement of the small area around r_a of Figure 1. As usual, all these singularities are found at points where the flow velocity equals the velocity of a characteristic wave disturbance in the fluid. At r_a, u is equal to the radial Alfvén velocity. At $r = r_c$, the fluid velocity is slightly less than the pure sound velocity. This is just Parker's critical point displaced slightly, because the sound wave for this model is not a pure sound wave but a magneto-acoustic wave. The remaining critical point at $r = r_f$ is very close to but slightly outside r_a. At r_f the fluid velocity is very nearly equal to the Alfvén velocity $[(\boldsymbol{B}\cdot\boldsymbol{B})/4\pi\rho]^{1/2}$ which is slightly larger than the radial Alfvén velocity. Further discussion and a more detailed analysis of these velocities will be given below.

The four branches u_{a1}, u_{a2}, $u_{\beta 1}$, and $u_{\beta 2}$ are indicated on Figures 1 and 2. In order to have a solution for which u goes to zero for small r and has a value close to the observed one at 1 a.u. the solution has to pass through all three critical points.

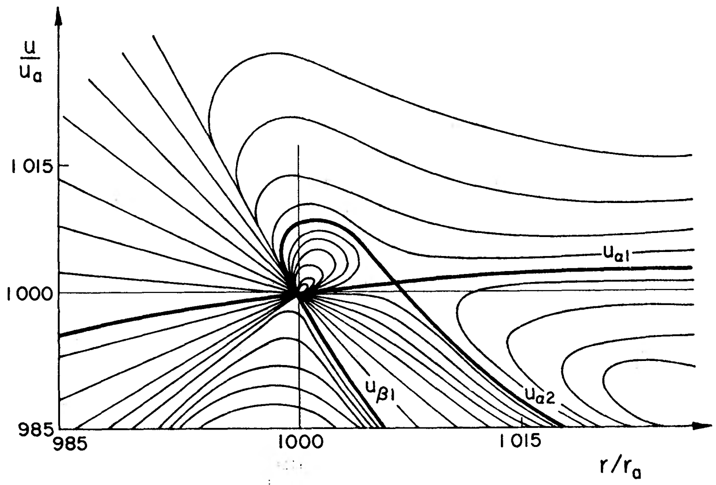

FIG. 2.—Enlargement of part of Fig. 1 near the Alfvénic critical point $r/r_a = 1$

V. PROPAGATION OF DISTURBANCES AND STABILITY

If magnetic fields are present, there exist other possible wavefronts besides those formed by sound waves. In magnetohydrodynamics, disturbances may travel in addition as Alfvén waves, and the structure of the characteristic manifold is therefore much more complicated. The direction of the magnetic field establishes a preferred axis and thus introduces anisotropy into the fluid. Friedrichs and Kranzer (1958) have shown that all possible speeds of the longitudinal wavefronts can be determined from the characteristic condition. The characteristic condition is

$$c^2[c^4 - (v_{AT}^2 + v_s^2)c^2 + v_s^2 v_{AN}^2] = 0\,, \tag{30}$$

where c is the velocity of the disturbance relative to the fluid; $v_s = (2\gamma kT/m)^{1/2}$ is the local sound velocity; $v_{AT} = [(B_r^2 + B_\phi^2)/4\pi\rho]^{1/2}$ is the local Alfvén velocity; and $v_{AN} = (B_r^2/4\pi\rho)^{1/2}$ is the local Alfvén velocity along the component of the magnetic field normal to the wavefront of interest. All these quantities vary with r. In the region between the surface of the Sun and the critical radius r_a, the angle $\delta(r)$ between the magnetic-field vector and the radius vector ranges from a very small value to approximately $\frac{1}{7}$ radian. Thus $B_\phi \cong -\delta B_r$, and to lowest order in δ we can solve equation (30) for the characteristic disturbances

$$c = 0\,, \quad \pm\, v_s\left[1 - \frac{v_{AN}^2}{2(v_{AN}^2 - v_s^2)}\,\delta^2\right], \quad \pm\, v_{AN}\left[1 + \frac{v_{AN}^2}{2(v_{AN}^2 - v_s^2)}\,\delta^2\right]. \tag{31}$$

This is valid as long as v_{AN} is not close to v_s, which is the case for the Sun. We can see from equation (31) that there are thus two separate wavefronts traveling with two different possible velocities, a "slow" wave and a "fast" wave.

The set of equations describing the motion of the disturbance is in this case as in Parker's (1963) case a first-order system of hyperbolic equations, from which the characteristics can be obtained as

$$\frac{dr}{dt} = u \pm c . \tag{32}$$

It is along these characteristics that all small-amplitude disturbances travel in the solar wind, except those which travel with the solar wind, i.e., which have $c = 0$. Figure 3 shows a sketch of the characteristics in the r–t plane.

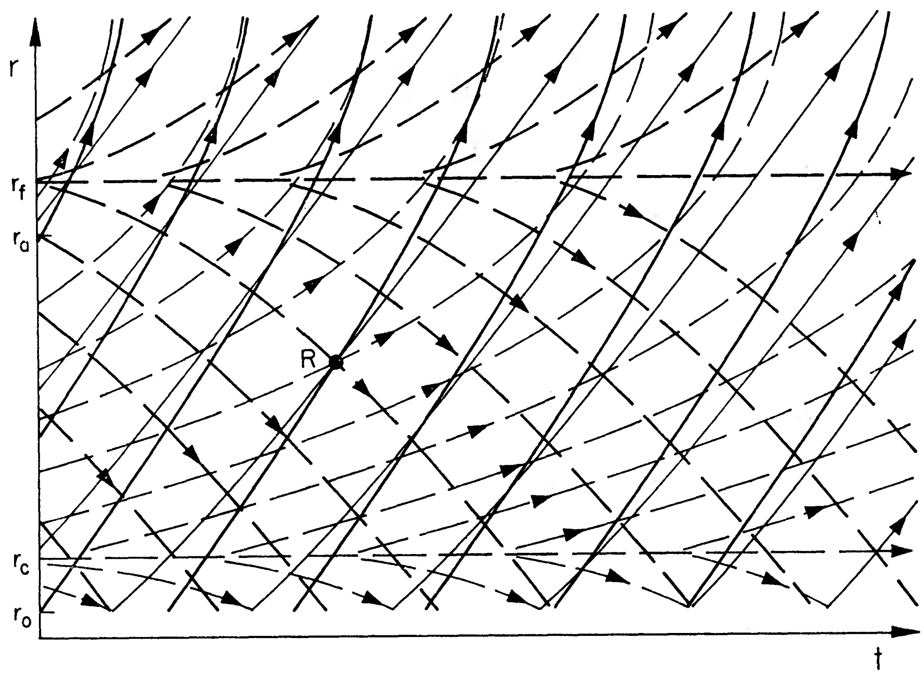

FIG. 3 —A sketch showing the characteristics in the r–t plane The characteristics for the "fast" wave are shown in heavy lines, and for the "slow" wave in light lines The solid lines refer to the solution of eq. (32) with the plus sign and the dashed lines refer to the minus sign; r_0 refers to the base of the corona.

At r_c, the fluid velocity is equal to the velocity of the "slow" wave, or the sonic wave, and at r_f, u is equal to the velocity of the "fast" wave. These are the two critical points besides r_a, which were discussed in § IV. Now it is quite clear that no small-amplitude hydromagnetic disturbance beyond r_f can be propagated back toward the Sun, i.e., all disturbances originating past r_f will be carried out of the region of interest by the wind and thus cannot grow and produce local instabilities.

All disturbances originating between r_c and r_f can be carried back toward the Sun by the "fast" wave, but the "slow" wave can carry them only away from the Sun. For $r < r_c$, disturbances can and will be carried back to the Sun by both the "fast" and the "slow" waves. If one considers a given point R, then all disturbances will be propagated along the four characteristics passing through this point. The amplitude of the disturbances can change along the characteristics due to changes in the local density, temperature, and magnetic-field strength, but no infinitely large growth in amplitude is possible due to the finite time which any disturbance spends in the region between R and

r_f. This model is thus as stable as Parker's model, for which Parker (1966) has shown that no intrinsic instabilities appear as the sole result of the existence of the solar wind.

VI. ILLUSTRATIVE CALCULATIONS

By integrating all the differential equations of motion, we have considerably simplified the problem, but it is still necessary to use machine computations to obtain a specific solution of the algebraic equations that remain. We found it desirable to make an illustrative calculation in order to get some order-of-magnitude values for the angular-momentum loss of the Sun and the angular velocity of the solar wind. In order to make these calculations it is necessary to determine values for the various parameters such as the total mass and energy fluxes, and the critical radius and velocity, r_a and u_a. In addition, boundary conditions are needed at r_0 to determine the radial magnetic field, pressure, and density at this point. Finally, we have to choose a value of γ which will then determine where and how much energy is supplied to the expanding solar atmosphere. A major difficulty is the determination of values for all these quantities that are mutually consistent, that lead to a physically possible solution passing through all three critical points, and that correspond to typical solar conditions. The boundary conditions could in principle be determined near the surface of the Sun, but we prefer to set $r_0 = 1$ a.u. and determine them at the orbit of Earth on the basis of observations from recent interplanetary probes. The subscript E rather than 0 is used for these parameters. The values we assign for our sample model are

$$\begin{aligned} u_E &= 400 \text{ km sec}^{-1}\,, & B_{rE} &= 5\ \gamma\,, \\ \rho_E &= 11.7 \times 10^{-24} \text{ gm cm}^{-3} \text{ (i.e., 7 protons cm}^{-3})\,, & T_E &= 2 \times 10^5\ {}^\circ\text{K}\,. \end{aligned} \tag{33}$$

Having thus specified the total mass flux, we cannot independently specify γ, since, to obtain the mass flux, a specific energy supply is required and this implies a specific, as-yet-unknown, value of γ.

This can also be seen from a purely mathematical point of view. We have specified $p_E/\rho_E = 4.7 \times 10^{18}$ ergs gm^{-1} and the constant $\rho u r^2 = 1.05 \times 10^{11}$ gm sec^{-1} sterad^{-1}; we take $\Omega = 3 \times 10^{-6}$ radians sec^{-1} and $GM_\odot = 1.33 \times 10^{26}$ dynes cm^2 gm^{-1}. By using equations (13) and (20), we get

$$\frac{2kT_a}{m} = \frac{p_a}{\rho_a} = \frac{p_E}{\rho_E}\left(\frac{\rho_a}{\rho_E}\right)^{\gamma-1} = \frac{2kT_E}{m} M_{AE}{}^{-2(\gamma-1)} = \frac{2kT_E}{m}\left(\frac{B_{rE}{}^2}{4\pi\rho_E u_E{}^2}\right)^{\gamma-1}. \tag{34}$$

To use equation (24) to get a solution we must find r_a, u_a, F, and γ. However, unless our solution passes through all three critical points, it cannot extend from very small to very large values of r. Thus there must be three pairs of values of (r,u), each of which makes both the numerator and denominator of equation (23) become equal to zero. The pair (r_a,u_a) of course does this, but we must find two more pairs, (r_c,u_c) and (r_f,u_f) that also do this. This gives us four equations, but we now have eight unknowns. In addition, the energy equation must be satisfied at $r = r_a, r_f, r_{c,2}$ and r_E, and we have thus the necessary eight equations. If we had tried to specify γ, i.e., the power supply, the system of eight equations would be overdetermined unless we regarded u or ρ at $r_0 = r_E$ as unknown. Since the eight equations are not linear, there might be more than one solution. We have proceeded by an iterative procedure of machine calculations to find one set which appears to be satisfactory and have assured ourselves that there is no other physically acceptable solution.

The essential dimensionless parameters on which the character of the solution depends are γ, $f = F/(u_a{}^2 u\rho r^2)$, $S_T = 2kT_a/(mu_a{}^2)$, $S_G = GM_\odot/(r_a u_a{}^2)$ and $S_\Omega = \Omega^2 r_a{}^2/u_a{}^2$. If these are the same for two models, one will find that u/u_a is the same function of r/r_a in both. Thus, if these essential parameters are held constant, ρ_a, u_a, and r_a may be varied independently to generate a three-parameter family of equivalent solutions. It is difficult to replace r_a and γ by more directly observable quantities, and thus there is no easy way to predict whether two different sets of observed boundary conditions at $r_0 \gg r_a$ correspond to essentially the same solution without working out one solution in detail.

The solution corresponding to equation (33) yields $\gamma = 1.221$, $r_a = 24.3\, r_\odot$, $u_a = 3.32 \times 10^7$ cm sec^{-1}, $r_c = 3.0\, r_\odot$, $u_c = 1.82 \times 10^7$ cm sec^{-1}, $r_f = 24.6\, r_\odot$, $u_f = 3.33 \times 10^7$ cm sec^{-1}, and $F/\rho u r^2 = 9.02 \times 10^{14}$ ergs gm^{-1} sec, where $r_\odot$ is the solar radius. The determination of u as a function of r, and from this all other properties of the solution, follows by direct numerical solution of the algebraic equation (24). As might have been expected, the dependence of the radial velocity and the density on r resembles Parker's solution so closely that there seems no point in showing figures for them. The spiral magnetic-field pattern also resembles the Archimedes spiral very closely and shows no anomaly at any of the critical points. For example, a line of force spirals 62° in solar longitude between the surface of the Sun and 1 a.u. This corresponds to a time lag of 4.65 days between central meridian passage of the photospheric foot of the line of force, assuming the model is valid clear to the photosphere, and passage of the line of force past

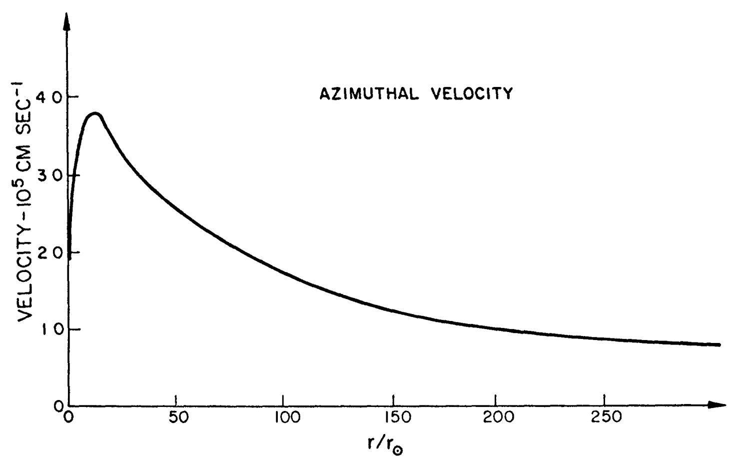

FIG. 4.—Azimuthal velocity of the solar wind

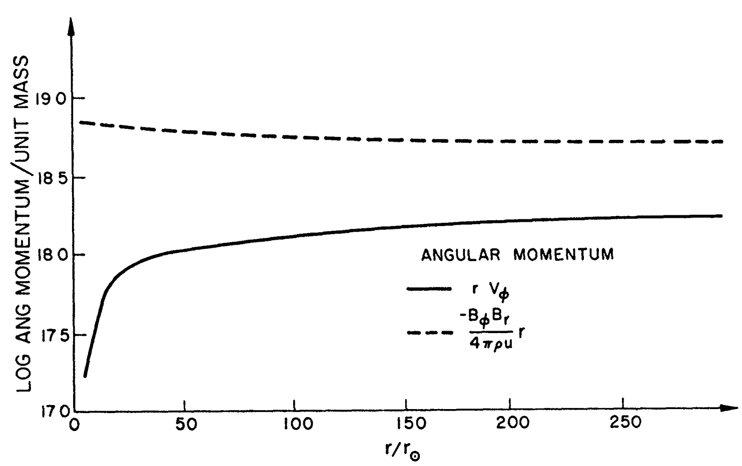

FIG. 5.—Angular momentum and magnetic torque in the solar wind

Earth. This may be compared to the characteristic time of (1 a.u.)/(400 km/sec) = 4.38 days. The dependence of azimuthal velocity on radius is shown in Figure 4. It matches the Sun's velocity at the surface, increases to a maximum nearly twice as high at 11.5 $r_\odot$, and then declines. The increase in azimuthal velocity is never as rapid as though there were strict co-rotation with the Sun, the angular velocity at 11.5 $r_\odot$ being about 0.16 that at the surface. Even by changing the parameters of the model we have not been able to obtain solar-wind data with $u = 300$ km sec^{-1}, $\rho_E = 3$ particles cm^{-3} and a $v_\phi \cong 15$ km sec^{-1} which Hundhausen, Asbridge, Bame, and Strong (1966) have measured on the Vela satellite. The angular momentum convected by the solar wind increases monotonically with radius as shown in Figure 5 but never produces nearly as large an effect as the torque due to the magnetic field. It is well to note, however, that the sum of the angular-momentum term rv_ϕ and the magnetic-torque term $-B_\phi B_r/4\pi\rho u$ is exactly equal to $L = \Omega r_a^2$. Any increase in the plasma angular momentum is exactly balanced by a decrease in the torque, even though this is not immediately apparent from Figure 5, which shows these quantities on a logarithmic scale. The final value for rv_ϕ will be $(1 - u_a/u_\infty)L$.

VII. SUMMARY

The model which we have presented here makes it possible to understand the large-scale properties of the solar wind and of the angular-momentum flux in the solar wind. The model is plagued by the same difficulties which have been experienced by other, simpler models which use the polytropic approximation to the energy equation, namely, that it is impossible to reproduce the physical conditions both at Earth and at the Sun. This model uses the boundary conditions at Earth, with the consequence that at the Sun the radial wind velocity is still very high, the temperature is 2.7×10^6 ° K, and the density is only 3×10^7 particles cm^{-3}. Values resembling more closely those known to exist at the Sun can be obtained by using a slowly varying polytropic index γ between the Sun and the critical point. But this is just another way of saying that the energy supply given by any constant γ is incorrect. A sounder but less-well-defined approach is to use the actual energy-flux equation and put a source term in it which accounts for all the heating due to waves, turbulence, and thermal conduction, and which can then be adjusted to give the proper energy supply needed to match what is known of conditions both near the Sun and at 1 a.u.

Even though there is no rigid co-rotation of the gas with the Sun, the gas does convect away a substantial amount of angular momentum and the torque due to the magnetic field is even more effective in decelerating the angular motion of the Sun. To treat this quantitatively, we need only to evaluate L. We assume that the calculations made for the equatorial region apply to the entire surface, except that the factor sin θ is inserted. We can then write for the total rate of change of $J_\odot$, the angular momentum of the Sun,

$$\frac{dJ_\odot}{dt} = \tfrac{2}{3}\Omega r_a^2 \frac{dM_\odot}{dt} = -\frac{J_\odot}{\tau}, \tag{35}$$

where τ is a characteristic time, which for our particular case turns out to be about 7×10^9 years. Thus the solar wind should have a substantial influence on the angular momentum of the entire Sun.

It may not be safe to use the value of τ found above to estimate the angular momentum of the Sun soon after its formation both because B_r and the mass flux in the solar system may have changed, and also because, if Ω were substantially larger, the centrifugal and magnetic terms would play a much more important role in the solution of the radial equation of motion and could easily substantially change the value of r_a.

If the observed differential rotation of the Sun's surface is regarded as evidence that the surface layers, including the convection zone, can slip over the inner core, the large torque exerted by the magnetic fields in the solar wind would imply that all the angular momentum in the surface layers would be removed in a time very short compared to the lifetime of the Sun. This would seem to imply that there must be balance between the torque due to the solar wind and that due to the action of the lower regions on the surface layers.

It is easy to estimate the change in r_a caused by reasonable changes in the solar-wind parameters as observed near Earth. From these parameters M_A can be calculated at Earth's orbit, and r_a can then be estimated from the fact that equation (13) shows M_A to be proportional to $u^{1/2}r$. A reasonable approximation, which is a lower limit, is obtained by assuming u to be constant. For example, using our parameters, $M_A = 9.6$ at the orbit of Earth and hence $r_a > 15.5 \times 10^6$ km, which is a good approximation to the result obtained. Using typical solar-wind data, we estimate that the critical Alfvénic radius may lie between 15 and 50 $r_\odot$.

This model provides an understanding of the coupling between the magnetic field and the plasma motion and especially the effect on the motion of the plasma in the azimuthal direction. While this model reproduces essentially Parker's (1963) radial solution for the Sun or a sunlike star, it allows the calculation of the properties of stellar winds of stars which have such a high rotation rate or such large magnetic fields that the corresponding terms in the radial equation of motion are important. In addition, the information on the angular-momentum loss of stars due to their expanding coronas can be obtained.

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