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GENERAL-RELATIVISTIC FLUID SPHERES

III. A STATIC GASEOUS MODEL

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ABSTRACT

It is not difficult to find static spherically symmetric distributions of matter which satisfy the general-relativistic field equations. The situation is, however, rather more difficult if (i) the sphere is to be "gaseous" (i.e., the density is to vanish at the boundary), and (ii) if one requires that the various field quantities, and more especially the equation of state of the medium constituting the sphere, be exhibited in simple, closed form. A model is here presented in detail for which the equation of state is

$$\varrho = (1 + k^*)\sqrt{p} - k^*p ,$$

where ϱ and p are the ratios of density and pressure to their central values, and k^* is a parameter which can be freely chosen in the range $0 \leqq k^* \leqq k'$, with $k' = \frac{5}{7}$ or 1, depending on certain physical conditions imposed upon the sphere.

I. INTRODUCTION

In one sense the problem of finding static spherically symmetric distributions of matter which satisfy the field equations of the general theory of relativity is almost trivial. Thus, if canonical coordinates be adopted, so that

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu(r)}dt^2 , \tag{1.1}$$

it was shown in an earlier paper (Buchdahl 1959; hereinafter referred to as "I") that one merely has to solve the equation

$$(1 - 2xw)\zeta_{,xx} - (w_{,x} + w)\zeta_{,x} - \tfrac{1}{2}w_{,x}\zeta = 0 , \tag{1.2}$$

where $\zeta = e^{\nu/2}$, $x = r^2$, and $3w$ is the "mean density" interior to r. Equation (1.2) is linear in w if ζ be prescribed and linear in ζ if w be prescribed. One may in fact choose one of these functions at will: more particularly in such a way that equation (1.2) then has a "simple" solution (e.g., § 7 of I). However, as soon as one wishes to consider gaseous spheres one must require that the pressure and the density of the sphere vanish together at the boundary, and then the situation is far less simple than might be apparent from the remarks above, especially if one demands that the solution be exhibited in closed form, *and* that the equation of state of the medium of the sphere also have a simple closed form. In fact the only solution of this kind of which I know is the analogue to the polytrope of index 5 presented earlier (Buchdahl 1964; hereinafter referred to as "E"), but even this is not regular in the sense of I, § 3*b* insofar as its radius is infinite. It is true that even with the simple choice $w = \frac{1}{2}MR^{-5}(5R^2 - 3x)$, for instance, one can arrive at the description of a family of gaseous spheres; yet, although ζ is an elementary transcendental function of x, it is so complicated a function that this solution presents a bewildering appearance, not least where the equation of state is concerned.

The aim of the present work, then, is to obtain the equations of a regular *gaseous* sphere (i.e., one whose boundary density is zero) such that the gravitational potentials, the pressure, the density, and particularly the equation of state all appear in simple closed form (see eq. [5.3]). The motivation is to have an easily surveyable model available (i) which is at least more realistic than the Schwarzschild interior solution (which is in any case certainly physically unrealizable since the speed of sound in it exceeds the speed

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of light everywhere—in fact it is infinite); (ii) which may be helpful in arriving at some explicit, physically not too unrealistic, non-static solutions.

Two remarks on methods and results achieved are appropriate at this stage. First, whereas in work of this kind it is usual to normalize the coordinate system at the outset, i.e., one chooses static, orthogonal coordinates in which g_{22} is taken to be either $-r^2$ or $r^2 g_{11}$, this is not done here. In fact, subsequent transformation to canonical coordinates ($g_{22} = -r^2$) shows the complicated appearance of the solution in these coordinates (§ VII). Second, the equation of state of the medium constituting the sphere involves a parameter k which may be freely chosen within a certain range. In particular, as $k \rightarrow 1$ the sphere becomes a classical polytrope of index 1, whereas when $k = \frac{2}{3}$ the sphere effectively has a "Schwarzschild core," surrounded by an envelope which becomes polytropic ($n = 1$) as the boundary is approached.

II. SOLUTION OF THE FIELD EQUATIONS

Let the metric be taken in the form

$$ds^2 = -e^{\lambda(r)}dr^2 - r^2 e^{\mu(r)}(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu(r)}dt^2 \,. \tag{2.1}$$

Since λ, μ, and ν are functions of r only, one can always set

$$\lambda = f(\mu) \,, \qquad \nu = g(\mu) \,, \tag{2.2}$$

and then

$$p = e^{-f}\left[\left(\tfrac{1}{2}\frac{dg}{d\mu} + \tfrac{1}{4}\right)\left(\frac{d\mu}{dr}\right)^2 + \frac{1}{r}\left(\frac{dg}{d\mu} + 1\right)\frac{d\mu}{dr} + \frac{1}{r^2}(1 - e^{f-\mu})\right], \tag{2.3}$$

$$\begin{aligned} p = e^{-f}\Big\{\tfrac{1}{2}\left(\frac{dg}{d\mu}+1\right)\frac{d^2\mu}{dr^2} + \tfrac{1}{4}\left[2\frac{d^2g}{d\mu^2} + \left(\frac{dg}{d\mu}\right)^2 - \frac{df}{d\mu}\frac{dg}{d\mu} + \frac{dg}{d\mu} - \frac{df}{d\mu} + 1\right]\left(\frac{d\mu}{dr}\right)^2 \\ + \frac{1}{2r}\left(\frac{dg}{d\mu} - \frac{df}{d\mu} + 2\right)\frac{d\mu}{dr}\Big\}, \end{aligned} \tag{2 4}$$

$$\rho = -e^{-f}\left[\frac{d^2\mu}{dr^2} + \left(\tfrac{3}{4} - \tfrac{1}{2}\frac{df}{d\mu}\right)\left(\frac{d\mu}{dr}\right)^2 + \frac{1}{r}\left(3 - \frac{df}{d\mu}\right)\frac{d\mu}{dr} + \frac{1}{r^2}(1 - e^{f-\mu})\right], \tag{2 5}$$

where the value 1 has been assigned to the speed of light and to Newton's constant, and a factor 8π has been absorbed in both p and ρ for the time being. Keeping in view the aim that p and ρ should vanish together at the boundary one will now try so to arrange things that p and ρ have a common factor. One way of doing this is to choose $f(\mu) = \mu$ (isotropic coordinates) with the consequence that the last terms on the right-hand sides of equations (2.3) and (2.5) disappear. One then concludes without difficulty that p and μ have a common factor if

$$\frac{d^2f}{d\mu^2} + \tfrac{1}{2}\left(\frac{df}{d\mu}\right)^2 + \tfrac{1}{4}\frac{df}{d\mu} = 0 \,. \tag{2.6}$$

It turns out eventually that one is thus led just to the solution forming the subject of E.

When $f \neq \mu$, equations (2.3) and (2.5) give

$$\begin{aligned} -\tfrac{1}{2}\left(\frac{dg}{d\mu}+1\right)(p+\rho) = e^{-f}\Big[\tfrac{1}{2}\left(\frac{dg}{d\mu}+1\right)\frac{d^2\mu}{dr^2} - \tfrac{1}{4}\left(\frac{dg}{d\mu}+1\right)\left(\frac{df}{d\mu} + \frac{dg}{d\mu} - 1\right) \\ \times\left(\frac{d\mu}{dr}\right)^2 - \frac{1}{2r}\left(\frac{dg}{d\mu}+1\right)\left(\frac{df}{d\mu} + \frac{dg}{d\mu} - 2\right)\frac{d\mu}{dr}\Big]. \end{aligned} \tag{2.7}$$

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One can evidently arrange p and ρ to have a common factor by identifying the factors multiplying $(d\mu/dr)^2$ in equations (2.4) and (2.7), respectively, and likewise the factors multiplying $d\mu/dr$. The latter thus require $df/d\mu + dg/d\mu = 0$ (since $dg/d\mu \neq 0$ in general) and then the factors of $(d\mu/dr)^2$ lead to the requirement

$$\frac{d^2 g}{d\mu^2}+\left(\frac{dg}{d\mu}\right)^2+\tfrac{1}{2}\frac{dg}{d\mu}=0\,.$$

Hence now

$$e^g = be^{-\mu/2} - a\,, \qquad e^f = ce^{-g}\,, \tag{2.8}$$

where a, b, and c are constants of integration. By using equations (2.8) and equating the two expressions (2.3) and (2.4) for p we obtain the following equation for μ results:

$$(\tfrac{1}{2}b\,e^{-\mu/2}-a)\frac{d^2\mu}{dr^2}+\tfrac{1}{4}b\,e^{-\mu/2}\left(\frac{d\mu}{dr}\right)^2+\frac{2}{r^2}(c\,e^{-\mu}-b\,e^{-\mu/2}+a)=0\,. \tag{2.9}$$

The substitution $\mu = 2\ln[(\xi + b)/2a]$ reduces this to

$$r^2\left[\xi\frac{d^2\xi}{dr^2}-\left(\frac{d\xi}{dr}\right)^2\right]-\xi^2=-\beta^2\,, \tag{2.10}$$

where β is the positive square root of $b^2 - 4ac$. This equation has the solutions

$$\xi = \pm\beta W\,, \qquad W=\begin{cases}\sin(Ar+B)/Ar\\ \sinh(Ar+B)/Ar\\ 1+B/r\,,\end{cases} \tag{2.11}$$

where A and B are constants of integration.

III. THE ADMISSIBLE SOLUTION

1. To isolate the specific (interior) solution of equation (2.10) which is of physical interest it is appropriate first to write down various quantities which appear above as functions of ξ. Using equation (2.10) one finds

$$p=-\frac{a\xi}{c(b+\xi)^2}\left(\frac{d^2\xi}{dr^2}+\frac{2}{r}\frac{d\xi}{dr}\right),$$

$$\rho=-\frac{a(2b-3\xi)}{c(b+\xi)^2}\left(\frac{d^2\xi}{dr^2}+\frac{2}{r}\frac{d\xi}{dr}\right), \tag{3.1}$$

whence

$$\chi\equiv\frac{3p}{\rho}=\frac{3\xi}{(2b-3\xi)}\,; \tag{3.2}$$

while

$$e^\lambda=\frac{c(b+\xi)}{a(b-\xi)}\,,\qquad e^\mu=\left(\frac{b+\xi}{2a}\right)^2,\qquad e^\nu=\frac{a(b-\xi)}{b+\xi}\,. \tag{3.3}$$

The third of the solutions (2.11) causes both ρ and p to vanish everywhere. This singular integral of equation (2.10) must therefore be the Schwarzschild exterior solution. To obtain the Lorentz metric at infinity one has to take

$$a=k\,,\qquad b=k+1\,,\qquad c=1\,,\qquad \beta=k-1\,, \tag{3.4}$$

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where k is a constant. The correct asymptotic behavior of the metric as $r \rightarrow \infty$ is insured by setting

$$B=\frac{4k}{(k^2-1)M}, \tag{3.5}$$

where M is the invariant field producing mass of the sphere. In fact now

$$e^{-\lambda}=e^{\nu}=\frac{1-k\phi}{1+\phi}, \qquad e^{\mu}=(1+\phi)^2, \tag{3.6}$$

with

$$\phi=\frac{2M}{(k+1)r}.$$

Note that the substitution

$$r=\bar{r}-\frac{2M}{k+1} \tag{3.7}$$

leads to the usual (canonical) form of the exterior solution.

2. As regards the other two solutions (2.11),

$$\frac{d^2\xi}{dr^2}+\frac{2}{r}\frac{d\xi}{dr}$$

in both cases differs from ξ only by a constant factor. Equations (3.1) then show that the boundary of the sphere corresponds to a zero of ξ, and so

$$(e^{\lambda})_b=\frac{c}{a}, \qquad (e^{\mu})_b=\left(\frac{b}{2a}\right)^2, \qquad (e^{\nu})_b=a. \tag{3.8}$$

When $B \neq 0$, ξ has a pole at the origin, and then $\chi_c = -1$, according to equation (3.2). This conclusion is at variance with the restriction that p and ρ be not negative. B must therefore be taken as zero. In that case, however, since $\sinh Ar/Ar$ has no real zeros, the second of the solutions (2.11) must be rejected and only the first (i.e., $W = \sin Ar/Ar$) needs henceforth to be considered. From equation (3.2) it follows that ξ and b must have the same sign. Again $(e^{\nu})_c/(e^{\nu})_b < 1$, so that the upper sign must be adopted in the first member of expression (2.11). Hence

$$\xi=\beta\,\frac{\sin Ar}{Ar}, \tag{3.9}$$

and b must be positive. In view of expression (3.8) this is true also of a and c, so that altogether

$$a>0, \qquad b>0, \qquad c>0. \tag{3.10}$$

The general form of the physically admissible (interior) solution is thus determined. It may be noted that equation (3.9) is effectively the Lane-Emden function of index 1.

IV. THE BOUNDARY CONDITIONS

At the boundary of the sphere the gravitational potentials and their first derivatives must be continuous in the problem under study. A given interior solution does not, of course, fully determine the explicit form of the exterior solution, for having found one such solution there still exists a set of transformations of the radial variable which leaves the boundary conditions and the asymptotic conditions ($r \rightarrow \infty$) satisfied. However, in the present situation one will naturally adopt the exterior solution in the form (3.6), and this is to be fitted to equations (3.3) and (3.9) at $r = R$.

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When we recall equations (3.8), and bear expressions (3.10) in mind, the continuity of λ, μ, and ν at once gives

$$a=\frac{1-k\phi_1}{1+\phi_1}\,,\qquad b=2(1-k\phi_1)\,,\qquad c=1\,, \tag{4.1}$$

with $\phi_1 = 2M/[(k+1)R]$. Since the exterior and the interior solutions both have $d\lambda/dr = -d\nu/dr$, continuity of $d\lambda/dr$, $d\mu/dr$, and $d\nu/dr$ gives only two further conditions, viz.,

$$\beta=\frac{b\phi_1}{1+\phi_1}\,,\qquad \beta=\frac{b(k+1)\phi_1}{2(1+\phi_1)(1-k\phi_1)}\,. \tag{4.2}$$

These give at once $\phi_1 = (1-k)/2k$, and $\beta = 1-k$. Accordingly the various constants all become simple functions of k, viz.,

$$a=k\,,\qquad b=1+k\,,\qquad c=1\,,\qquad \beta=1-k\,; \tag{4.3}$$

and k itself is related to M/R as follows:

$$M/R=\frac{1-k^2}{4k}\,. \tag{4.4}$$

It may be noted that certainly

$$0<k\leq 1\,. \tag{4.5}$$

With (4.3), equations (3.6), (3.3), and (3.9) give the explicit expressions for the components of the metric tensor satisfying the boundary and asymptotic conditions and representing a regular gaseous sphere.

V. THE EQUATION OF STATE

If f be any function of r, write $f = f/f_c$. Then from equations (3.1) and (4.3) one infers easily that

$$p=\frac{4W^2}{[(1+k)+(1-k)W]^2}\,,\qquad \varrho=\frac{4W[2(1+k)-3(1-k)W]}{(5k-1)[(1+k)+(1-k)W]^2}\,, \tag{5.1}$$

where

$$W=\frac{\sin(\pi r/R)}{(\pi r/R)}\,. \tag{5.2}$$

In equations (5.1) W may be eliminated and so one finally gets the equation of state of the medium of the sphere (see also eq. [5.10]):

$$\varrho=\frac{4}{5k-1}\sqrt{p}-\frac{5(1-k)}{5k-1}\,p\,. \tag{5.3}$$

The physically admissible range of values of k can now be narrowed down beyond expression (4.5), in the sense that the smallest value k can assume is greater than 0, depending on the kind of condition one imposes. Thus, certainly, since $\varrho \geqq 0$,

$$k>\tfrac{1}{5}\,. \tag{5.4}$$

However, ϱ must not increase outward, and from equations (5.1) one finds that $\rho' \leqq 0$ provided $\xi \leqq \frac{1}{4}b$, i.e.,

$$k\geqq\tfrac{3}{5}\,. \tag{5.5}$$

Finally, since the velocity of sound v is given by $v^2 = dp/d\rho$, one has

$$v^2=\frac{(1-k)\sqrt{p}}{2-5(1-k)\sqrt{p}}\,. \tag{5.6}$$

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The maximum of v occurs at the center:

$$v_c^2 = \frac{1-k}{5k-3}\,. \tag{5.7}$$

Thus condition (5.5) insures that v is everywhere real. On the other hand any physically realizable situation surely requires that $v_c \leqq 1$, and therefore

$$k \geqq \tfrac{2}{3}\,. \tag{5.8}$$

It may be noted that when $k = \frac{3}{5}$

$$\varrho = 2\sqrt{p} - p\,. \tag{5.9}$$

In this case the sphere virtually has a core of constant density, surrounded by an envelope which is polytropic ($n = 1$) near the boundary. On the other hand, as $k \rightarrow 1$, one has a classical polytrope of index 1. In conclusion it may be noted that equation (5.3) may be written

$$\varrho = (1 + k^*)\sqrt{p} - k^* p\,, \tag{5.10}$$

where

$$k^* = \frac{5(1-k)}{5k-1} = \frac{v_c^2}{v_c^2 + \frac{2}{5}}\,. \tag{5.11}$$

VI. SOME PARAMETERS OF THE SPHERE

A quantity of considerable interest is

$$\Delta = (g_{44})_b\,, \tag{6.1}$$

which plays a prominent part in I. One must, of course, not fall into the error—from force of habit—of setting $\Delta = 1 - 2M/R$, for this relation only applies in canonical coordinates. In fact here

$$\Delta = k \tag{6.2}$$

in view of equations (3.8) and (4.3), whence by equation (4.4)

$$\Delta = \left(1 + \frac{4M^2}{R^2}\right)^{1/2} - \frac{2M}{R}\,. \tag{6.3}$$

Of course one may write $\Delta = 1 - 2M/\bar{R}$, where $\bar{R}$ is the canonical coordinate radius. Then, in view of equation (3.7)

$$\bar{R} = R + \frac{2M}{k+1}\,, \tag{6.4}$$

and in this way one gets back to equation (6.2) or (6.3).

The last remark is very relevant to the calculation of the "mass concentration" δ (cf. I[2.16]) the definition of which relates to canonical coordinates. In fact, by I (2.17),

$$w_b = M\bar{R}^{-3} = 2R^{-2}k^2(1-k)^2(1+k)^{-2}\,, \tag{6.5}$$

using equations (6.4) and (4.4). Also, when one bears in mind that a factor 8π is contained in ρ,

$$w_c = \tfrac{1}{6}\rho_c = \frac{\pi^2}{24R^2}\,k(5k-1)(1-k)\,. \tag{6.6}$$

Hence

$$\delta = \frac{48k}{\pi^2(1+k)^2(5k-1)}\,, \tag{6.7}$$

which correctly goes over into the classical result $3/\pi^2$ as $k \rightarrow 1$.

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Finally, from equation (3.2) it follows that χ is monotonically decreasing with r. The quantity β introduced in § 2 of I (which must not be confused with the constant β above) may therefore be taken as χ_c, i.e.,

$$\beta = \frac{3(1-k)}{5k-1} \left(= \tfrac{3}{5}k^{*} \right). \tag{6.8}$$

It is a little unfortunate that for any physically realizable sphere $\beta \leqq \frac{3}{7}$.

VII. TRANSFORMATION TO CANONICAL COORDINATES

In conclusion, the explicit form of the transformation which leads from the present radial coordinate r to the canonical radial coordinate $\bar{r}$ will be written down. For this purpose one need only observe that according to equations (3.3) and (3.9)

$$\sqrt{(-g_{22})} = \frac{(1+k)}{2k} r + \frac{(1-k)}{2kA} \sin Ar ,$$

and this must be the same as $\bar{r}$. To express r in terms of $\bar{r}$ from this is a well-known problem whose solution is

$$r = \frac{2k}{(1+k)} \bar{r} + \sum_{n=1}^{\infty} h_n \sin nz , \tag{7.1}$$

where

$$h_n = 2(-1)^n n^{-1} J_n(ne) , \qquad z = \frac{2\pi k \bar{r}}{(1+k)R}, \qquad e = \frac{1-k}{1+k}, \tag{7.2}$$

and J_n denotes the Bessel coefficient of order n. Then, for instance,

$$g_{44} = k + (1+k) \bar{r}^{-1} \sum_{n=1}^{\infty} h_n \sin nz ; \tag{7.3}$$

but the expression for g_{11} is rather more involved. The expression for w is, of course, correspondingly complicated, a feature which brings out very clearly the disadvantages which may arise from normalizing coordinates to be, say, canonical at the outset when dealing with the kind of problem here investigated.

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