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CLASSIFICATION OF GENERAL RELATIVISTIC WORLD MODELS

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Summary

General relativistic world models are classified by means of the deceleration parameter q_0 and the density parameter σ_0. The evolution of the different models is illustrated by ' evolution-curves ' in the (q, σ) plane. The age of the universe is computed for the exploding models. A discussion of the particle horizon and the event horizon in the different models is given in the last section.

1. *Introduction.* With the natural assumption that the universe is homogeneous and isotropic, the metric can be written in the form given by Robertson and Walker:

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{1}$$

where k is equal to $+1$, -1 or 0, and R is a scale-factor with dimension length. From Einstein's field equations (**1**) we then get two non-trivial equations:

$$8\pi G\rho = \frac{3}{R^2}(kc^2 + \dot{R}^2) - \lambda, \tag{2}$$

$$\frac{8\pi G}{c^2}p = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{kc^2}{R^2} + \lambda. \tag{3}$$

Here G is the gravitational constant, λ is the cosmical constant, ρ is the mean density in the universe and p is the pressure. A dot indicates differentiation with respect to time.

2. *Robertson's classification of world models.* By using equations (2) and (3), H. P. Robertson was able to give an extensive classification of relativistic world models in his famous paper of 1933 (**2**). In order to compare this classification with the one presented in this paper, we give here a brief summary of Robertson's discussion in the special case $p = 0$.

The time dependence of R is one of the most interesting features in the different world models. Robertson was able to establish an expression for $\dot{R}$, from which he could find the time dependence of R in a qualitative way. The curve $\dot{R} = 0$ is demonstrated in Figs. 1 and 2 in the (R, λ) plane. The type of curve is seen to be different in the two cases $k = 1$ and $k = 0, -1$. Noting that λ is a constant, we see that the evolution of the different models is represented by straight lines parallel to the R-axis. Below the curve $\dot{R} = 0$, $\dot{R}^2$ is negative, and the only admissible values of λ and R are those points lying on or above this curve.

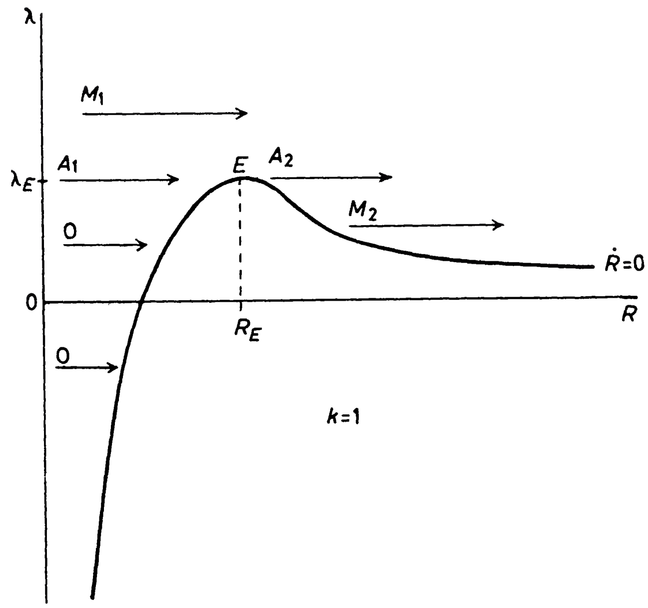

FIG. 1. *The different models in the* (λ, *R*) *plane for* $k = +1$.

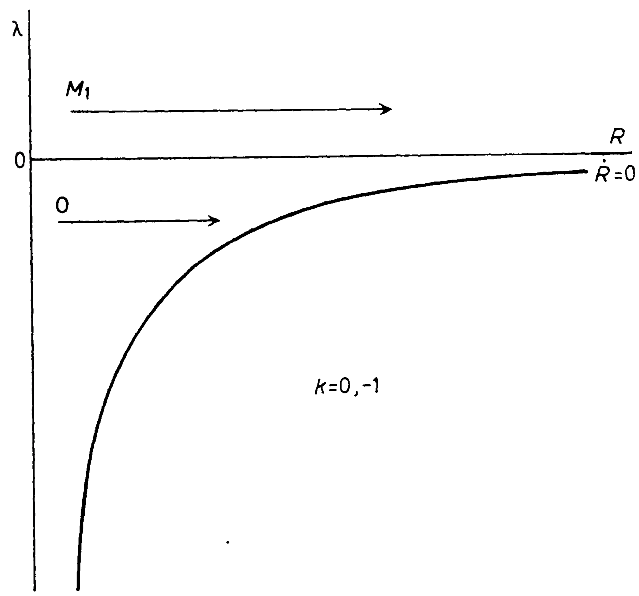

FIG. 2. *The different models in the* (λ, *R*) *plane for* $k = -1, 0$.

In the case $k = 1$ the curve $\dot{R} = 0$ has a maximum value of λ equal to λ_E, the corresponding value of R being R_E, and we have the following world models:

$\lambda > \lambda_E$: monotonic universes of the first kind, denoted by M_1. These models are of an ever-expanding type which proceeds from some singular state at $R \geqslant 0$ to the final state of an empty de Sitter universe when $R \to \infty$.

$\lambda = \lambda_E$, $R < R_E$: an asymptotic universe of the first kind, denoted by A_1. This model starts expanding from a singular state at $R < R_E$ and asymptotically approaches the condition of a static Einstein universe.

$\lambda = \lambda_E$, $R = R_E$: the static Einstein universe, denoted by E.

$\lambda = \lambda_E$, $R > R_E$: an asymptotic universe of the second kind, denoted by A_2. This model can be regarded as having asymptotically started from the stati

Einstein universe at an infinite time in the past, and expands into the state of an empty de Sitter universe.

$0<\lambda<\lambda_E$, $R>R_E$: monotonic universes of the second kind, denoted by M_2. These models have been expanding continuously from some point $R>R_E$ at the critical curve, where a reversal in the direction of motion from a preceding phase takes place. Universes of this type also expand into an empty de Sitter universe.

$\lambda<\lambda_E$, $R<R_E$: oscillating universes, denoted by O. These models expand from a singular state at $R<R_E$ to a maximum value of R when the critical curve is reached. The subsequent contraction will bring the universe back to the singular state, from which an expansion starts again.

From equations (2) and (3) it is seen that the singular state is at $R=0$. However, for small values of R the assumption $p=0$ is not valid and the singular states actually appear at $R>0$.

In the case $k=-1$ or 0 two world models are possible:

$\lambda\geqslant 0$: the universe is of type M_1.

$\lambda<0$: the universe is of type O.

3. *Classification of world models by means of q_0 and σ_0.* Today it is often more convenient to describe the world models by the deceleration parameter q and the density parameter σ defined by

$$q=-\frac{\ddot{R}}{RH^2}, \tag{4}$$

and

$$\sigma=\frac{4\pi G\rho}{3H^2}, \tag{5}$$

where $H=\dot{R}/R$ is the Hubble parameter. When $p=0$, equations (2) and (3) reduce to

$$\lambda=3H^2(\sigma-q)=3H_0^2(\sigma_0-q_0), \tag{6}$$

and

$$kc^2=H^2R^2(3\sigma-q-1)=H_0^2R_0^2(3\sigma_0-q_0-1), \tag{7}$$

where the subscript zero indicates the present value of the variables. It is also seen that

$$\rho R^3=\rho_0 R_0^3=\text{const.} \tag{8}$$

Inserting equations (5), (6), (7) and (8) into equation (2) we get

$$\dot{R}^2=\dot{R}_0^2\left\{2\sigma_0\frac{R_0}{R}+(\sigma_0-q_0)\frac{R^2}{R_0^2}+q_0+1-3\sigma_0\right\}\equiv F(R). \tag{9}$$

This equation is seen to be analogous to equation (7.1) in Robertson's paper and to equations (9.16) and (10.13) in Bondi's book (3), and it can be solved in terms of elliptic functions. Like Bondi we shall here only give a qualitative solution of equation (9).

$\sigma_0-q_0<0$, i.e. $\lambda<0$. From an investigation of equation (9) it is seen that F is positive when R is less than a certain value R_c, and negative when $R>R_c$. In order that $\dot{R}^2$ can never be negative, R cannot be larger than R_c, and we thus have an oscillating universe.

$\sigma_0 - q_0 \geqslant 0$ and $3\sigma_0 - q_0 - 1 \leqslant 0$ i.e. $\lambda \geqslant 0$ and $k \leqslant 0$. From equation (9) it is seen that F is always positive. Due to the fact that $\dot{R}_0 > 0$, $\dot{R}$ has always been and will forever be positive, i.e. we have universes of type M_1, except in the special case $q_0 = -1$, $\sigma_0 = 0$ which is de Sitter's universe.

$\sigma_0 - q_0 \geqslant 0$ and $3\sigma_0 - q_0 - 1 > 0$, i.e. $\lambda \geqslant 0$ and $k = 1$. In this case it cannot be seen directly whether $F = 0$ for any positive values of R. Differentiating equation (9) with respect to R it is seen that F has a minimum value

$$F_m = \dot{R}_0^2[q_0 + 1 - 3\sigma_0 + 3\sigma_0^{2/3}(\sigma_0 - q_0)^{1/3}] \tag{10}$$

at

$$R_m = R_0\left(\frac{\sigma_0}{\sigma_0 - q_0}\right)^{1/3}. \tag{11}$$

When $F_m > 0$, $\dot{R}$ can never change sign and the universe must be in a monotonous state of expansion, i.e. we have world models of type M_1.

When $F_m < 0$, F is negative in an interval around $R = R_m$. We thus have two different types of models depending on the sign of $(R_m - R_0)$, i.e. the sign of q_0.

$F_m < 0$, $R_m > R_0$, i.e. $q_0 > 0$. The universe expands until $R = R_{\max} < R_m$, and contraction starts. We have a universe of type O.

$F_m < 0$, $R_m < R_0$, i.e. $q_0 < 0$. The universe contracted from an infinite time in the past until $R = R_{\min} > R_m$, where expansion started. The final state of this model, which is of type M_2, is de Sitter's universe.

When $F_m = 0$, we get from equation (10) with $\dot{R}_0^2 \neq 0$:

$$\sigma_0^2 - \frac{1}{3}(q_0 + 1)^2\sigma_0 + \frac{1}{27}(q_0 + 1)^3 = 0,$$

or

$$\sigma_0 = \tfrac{1}{6}(q_0 + 1)[q_0 + 1 \pm \sqrt{(q_0 + 1)(q_0 - \tfrac{1}{3})}] = \sigma_c. \tag{12}$$

From equation (12) we must have $q_0 \leqslant -1$ or $q_0 \geqslant \frac{1}{3}$.

$q_0 \leqslant -1$. In order to get a non-negative value of σ_0 we must choose the minus sign before the square root in equation (12). This solution separates universes of type M_1 and M_2 and thus represents models of type A_2.

$q_0 \geqslant \frac{1}{3}$. We have earlier discussed the case $q_0 \geqslant \sigma_0$, so that we are now only interested in the case $q_0 \leqslant \sigma_0$. It can then be shown that we must choose the positive sign before the square root in equation (12), and that we must have $q_0 \geqslant \frac{1}{2}$. This solution separates models of type M_1 and O and thus represents models of type A_1.

The representation of the A_1 and the A_2 models in the model diagram, Fig. 3, we call the A-curve. In addition to the A-curve it is only the straight lines $\lambda = 0$ and $k = 0$ that separate the different models. The models are presented in Table I and in Fig. 3.

A basic quantity in cosmological calculations is

$$\omega = \int_t^{t_0} \frac{c\,dt}{R} = \int_R^{R_0} \frac{c\,dR}{R\sqrt{F(R)}}. \tag{13}$$

An exact solution of this integral can be obtained when $q_0 = \sigma_0$, $\sigma_0 = 0$ or when $F(R) = 0$ has a double root. This has recently been discussed by McVittie (**6**). From the way in which σ_c was determined it is immediately seen that $F(R) = 0$ has a double root when and only when $\sigma_0 = \sigma_c$. We thus see that exact solutions

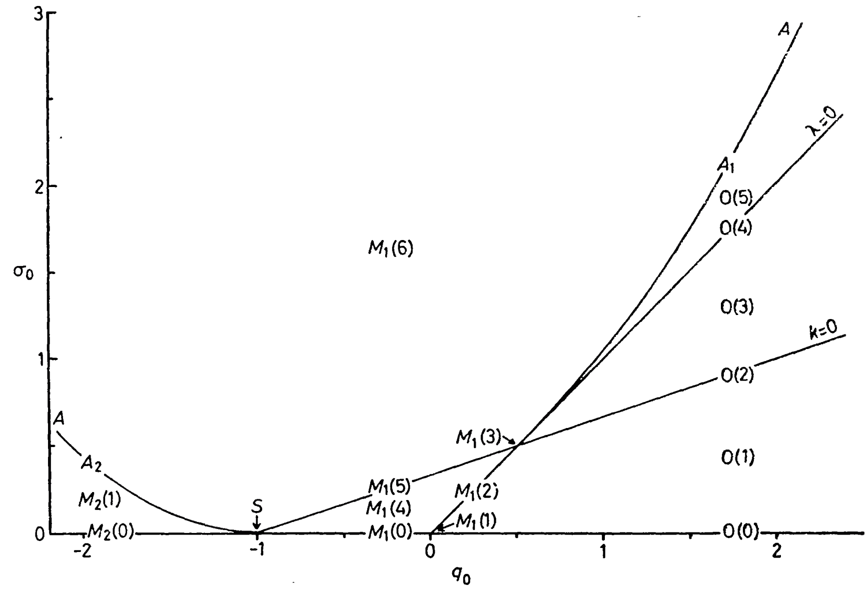

FIG. 3. *The different models in the* (q_0, σ_0) *plane.*

TABLE I

Relativistic world models

$\sigma_0 - q_0$	$3\sigma_0 - q_0 - 1$	σ_0	q_0	Type	Remarks
>0	>0	0	<-1	$M_2(0)$	Lanczos' form $\to S$
>0	>0	$>0,\ <\sigma_c$	<-1	$M_2(1)$	$\to S$
>0	>0	σ_c	<-1	A_2	Lemaitre's case $\to S$
>0	>0	$>\sigma_c$	$\leqslant -1$	$M_1(6)$	$\to S$
>0	>0	>0	$-1<q_0<\frac{1}{2}$	$M_1(6)$	$\to S$
>0	>0	$>\sigma_c$	$>\frac{1}{2}$	$M_1(6)$	$\to S$
>0	>0	σ_c	$>\frac{1}{2}$	A_1	$\to E$
>0	>0	$<\sigma_c$	$>\frac{1}{2}$	$O(5)$	
$+1$	0	0	-1	S	de Sitter's universe
>0	0	>0	$-1<q_0<\frac{1}{2}$	$M_1(5)$	$\to S$
>0	<0	>0	$-1<q_0<\frac{1}{2}$	$M_1(4)$	$\to S$
>0	<0	0	$-1<q_0<0$	$M_1(0)$	$\to S$
0	>0	>0	$>\frac{1}{2}$	$O(4)$	Investigated by Einstein
0	0	$\frac{1}{2}$	$\frac{1}{2}$	$M_1(3)$	Einstein–de Sitter universe (ES)
0	<0	>0	$0<q_0<\frac{1}{2}$	$M_1(2)$	$\to M_1(1)$
0	<0	0	0	$M_1(1)$	$R=ct$
<0	>0	>0	$>\frac{1}{2}$	$O(3)$	
<0	0	>0	$>\frac{1}{2}$	$O(2)$	
<0	<0	>0	>0	$O(1)$	
<0	<0	0	>0	$O(0)$	
>0	>0	$\sigma_c=\infty$	∞	E	Einstein's static case

can be obtained for the A_1 and A_2 models, and also for a class of oscillating models for which $\sigma_0=\sigma_c$ and $q_0>\sigma_0$. We shall later discuss the evolution of the different models and illustrate this evolution by curves in the model diagram, Fig. 4 The evolution curve that is closest to the $\sigma=0$ axis in Fig. 4 represents this class of oscillating models which can be solved exactly. This evolution curve has a minimum value of σ equal to $\frac{1}{4}$ when $q=\frac{1}{2}$.

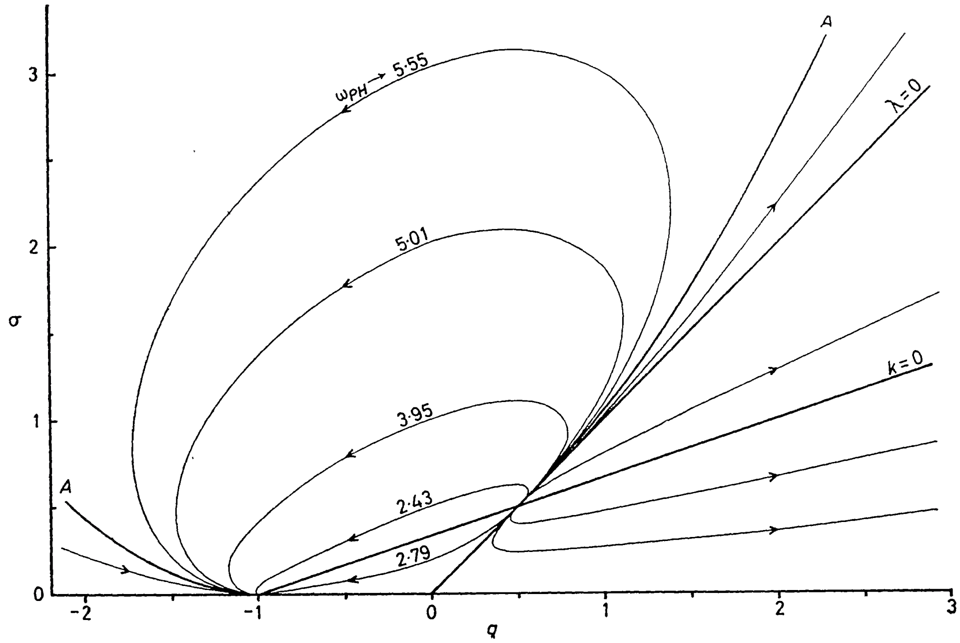

FIG. 4. *Evolution-curves in the* (q, σ) *plane.*

4. *Variation of q and* σ. We shall now investigate the variation of q and σ in the different models. From equations (5) and (8) we get

$$\sigma = \sigma_0 \frac{H_0{}^2 R_0{}^3}{\dot{R}^2 R}. \tag{14}$$

Inserting the expression for $\dot{R}^2$ given by equation (9) we get the dependence of σ on R:

$$\sigma = \frac{\sigma_0}{2\sigma_0 + (q_0 + 1 - 3\sigma_0)y + (\sigma_0 - q_0)y^3}, \tag{15}$$

where

$$y = \frac{R}{R_0}.$$

In a similar way we get the dependence of q on R from equations (6) and (15):

$$q = \frac{\sigma_0 + (q_0 - \sigma_0)y^3}{2\sigma_0 + (q_0 + 1 - 3\sigma_0)y + (\sigma_0 - q_0)y^3}. \tag{16}$$

From equations (15) and (16) σ and q can be calculated for various values of y and the evolution of the different models can be demonstrated by evolution-curves in the (q, σ) plane, see Fig. 4. As expected, the evolution-curves do not intersec the A-curve or any of the straight lines $k = 0$ or $\lambda = 0$.

The empty $O(0)$ model starts at the origin in the (q, σ) diagram and follow the positive q-axis. When R reaches its maximum value, $H = 0$ and q has a infinite value. The model returns along the same line. All the other O model start at $q = \frac{1}{2}$, $\sigma = \frac{1}{2}$. When R reaches maximum, q and σ have infinite values It can be seen, however, that the ratio σ/q has a finite value.

For $O(1)$ models we find minimum values of q and of σ, with all the σ-minim at $q = \frac{1}{2}$ (Appendix A).

The A_1 model starts at $q=\frac{1}{2}$, $\sigma=\frac{1}{2}$, follows the A-curve and asymptotically approaches the Einstein universe, q and σ tending to infinite large values.

The empty $M_1(0)$ model starts at the origin, $q=0$, $\sigma=0$, and follows the negative q-axis until de Sitter's universe is reached at $t=\infty$.

The empty $M_1(1)$ model is static in the diagram, q and σ being constant and equal to zero.

The $M_1(2)$ model starts from $q=\frac{1}{2}$, $\sigma=\frac{1}{2}$ and follows the line $\lambda=0$ to the empty state at the origin, $q=0$, $\sigma=0$.

The $M_1(3)$ model, usually designated the Einstein–de Sitter universe, is static in the (q, σ) diagram, q and σ being constant and equal to $\frac{1}{2}$. The $M_1(4)$, $M_1(5)$ and $M_1(6)$ models all start at $q=\frac{1}{2}$, $\sigma=\frac{1}{2}$ and ultimately enter the state of de Sitter's universe. The $M_1(6)$ models have maximum as well as minimum values of q and maximum values of σ at $q=\frac{1}{2}$ (Appendix A). The time derivative of H can be written $\dot{H}=-H^2(q+1)$ so that H has a minimum when $q=-1$. Differentiating equation (6) it is seen that $d\sigma/dq=1$ in these minimum points. The A_2 model follows the left branch of the A-curve, starting from infinitely large values of $-q$ and σ (Einstein's static universe) at $t=-\infty$. The final state of this model is the de Sitter universe. The $M_2(1)$ models start at $q=-1$, $\sigma=0$ when $t=-\infty$. When R reaches its minimum value, $-q$ and σ have infinite values. The models return along the same curves.

The behaviour of the $M_2(0)$ model is similar to the $M_2(1)$ models, but follows the negative q-axis. Both these types of models approach the de Sitter universe when $t\to\infty$.

5. *The age of the universe.* Choosing $t=0$ when $R=0$, the age of the universe is equal to t_0. The concept of age has no meaning in the M_2, the S and the A_2 models. Inserting the variable y in equation (9) we find:

$$t_0=\int_0^{R_0}\frac{dR}{\dot{R}}=H_0^{-1}\int_0^1\frac{dy}{\sqrt{(\sigma_0=q_0)y^2+2\sigma_0 y^{-1}+q_0+1-3\sigma_0}}. \qquad (17)$$

Following Tomita & Hayashi (4), this integral can be calculated numerically for different values of q_0 and σ_0 and the results are presented in Table II and in Fig. 5 where curves of constant values of t_0H_0 are drawn. When we compare Figs. 4 and 5, it is seen that several evolution-curves cross the same age-curve twice, so that

TABLE II

The age of the universe in units of H_0^{-1}

q_0 \ σ_0	0·00	0·05	0·15	0·30	0·50	1·00	2·00	3·00	5·00
−3·0							1·052	0·775	0·595
−2·0					1·631	0·986	0·733	0·630	0·525
−1·5			1·994	1·290	1·055	0·833	0·668	0·588	0·500
−1·0		1·436	1·156	0·994	0·883	0·745	0·620	0·555	0·479
−0·5	1·246	1·069	0·949	0·857	0·785	0·683	0·584	0·528	0·461
0·0	1·000	0·910	0·835	0·771	0·717	0·637	0·554	0·505	0·446
0·5	0·870	0·812	0·758	0·709	0·667	0·601	0·529	0·486	0·432
1·0	0·785	0·743	0·701	0·662	0·627	0·571	0·508	0·469	0·419
2·0	0·676	0·649	0·620	0·593	0·567	0·524	0·473	0·440	0·398
3·0	0·605	0·585	0·564	0·543	0·522	0·488	0·445	0·417	0·380
5·0	0·514	0·503	0·489	0·475	0·460	0·435	0·403	0·381	0·351

$t_0 H_0$ has the same value at different stages of the evolution. This may seem to be a paradox, but it is easily explained, because H is not constant along an evolution-curve.

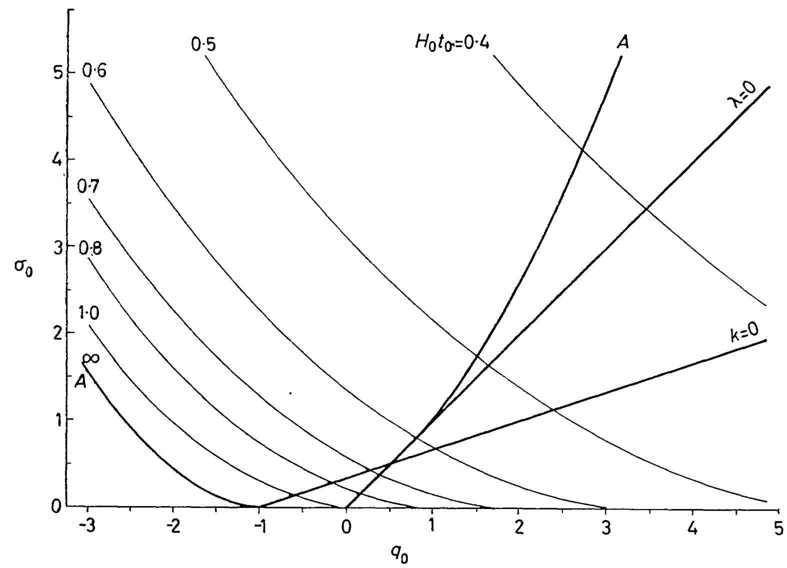

FIG. 5. *The age of the universe in units of H_0^{-1}.*

6. *Particle-horizons and event-horizons.* Rindler (5) has defined the particle-horizon as follows: 'A particle-horizon, for any given fundamental observer A and cosmic instant t_0 is a surface in the instantaneous 3-space $t = t_0$, which divides all fundamental particles into two non-empty classes: those that have already been observable by A at time t_0 and those that have not'. Restricting ourselves to exploding models, the necessary and sufficient condition for a particle-horizon to exist in a given model (5) is the convergence of the integral

$$\int_0^{t_0} \frac{dt}{R(t)}.$$

The particle-horizon at time t_0 is the surface

$$\omega = \omega_{PH} = \int_0^{t_0} \frac{c\,dt}{R(t)} = |1 + q_0 - 3\sigma_0|^{1/2} \int_1^\infty \frac{dv}{\sqrt{2\sigma_0 v^3 + (1 + q_0 - 3\sigma_0)v^2 + \sigma_0 - q_0}}, \quad (18)$$

where

$$v = \frac{R_0}{R}.$$

This integral has been computed for different values of q_0 and σ_0, and the results are illustrated in Fig. 6 where curves of constant ω_{PH} are drawn. The particle-horizon does not exist for the empty models.

The event-horizon may be defined as follows, (5): 'An event-horizon, for a given fundamental observer A, is a (hyper-) surface in space-time which divides all events into two non-empty classes: those that have been, are, or will be observable by A, and those that are forever outside A's possible powers of observation'. The necessary and sufficient condition for an event-horizon to exist in a given model(5) is the convergence of the integral

$$\int_{t_0}^\infty \frac{dt}{R(t)}.$$

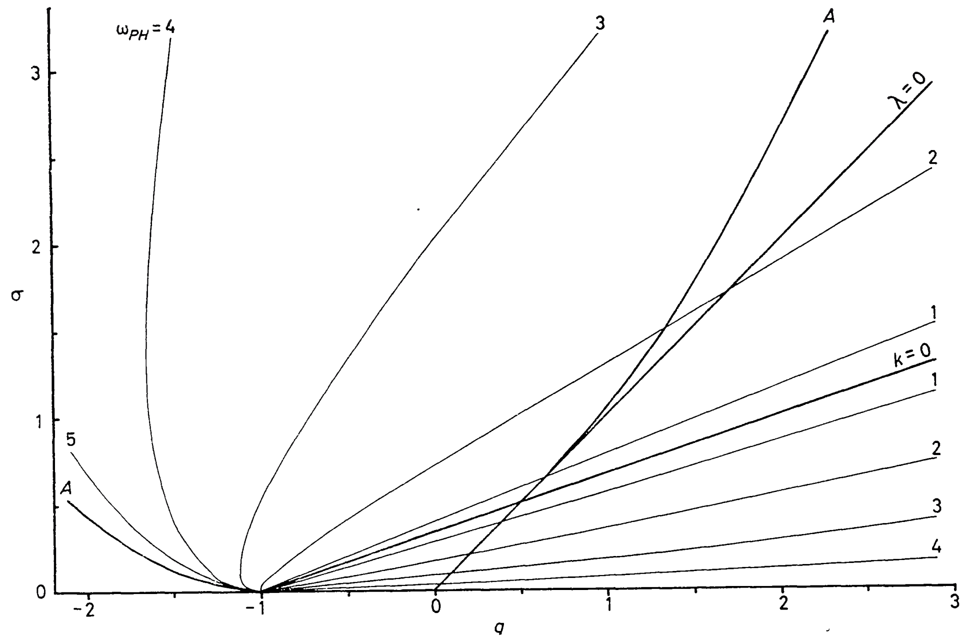

FIG. 6. *The particle-horizon in the different models.*

The event-horizon at time t_0 is the surface

$$\omega=\omega_{EH}=\int_{t_0}^{\infty}\frac{c\,dt}{R(t)}=|1+q_0-3\sigma_0|^{1/2}\int_0^1\frac{dv}{\sqrt{2\sigma_0 v^3+(1+q_0-3\sigma_0)v^2+\sigma_0-q_0}}. \quad (19)$$

This integral has been computed for different values of q_0 and σ_0 and the results are demonstrated in Fig. 7 where curves of constant ω_{EH} are drawn. The oscillating models and the models for which $\lambda=0$ have no event-horizon.

The absolute horizon, as defined by Rindler, is easily seen to be the surface $\omega=\omega_{AH}=2(\omega_{PH}+\omega_{EH})$. The absolute horizon ω_{AH} is constant along an

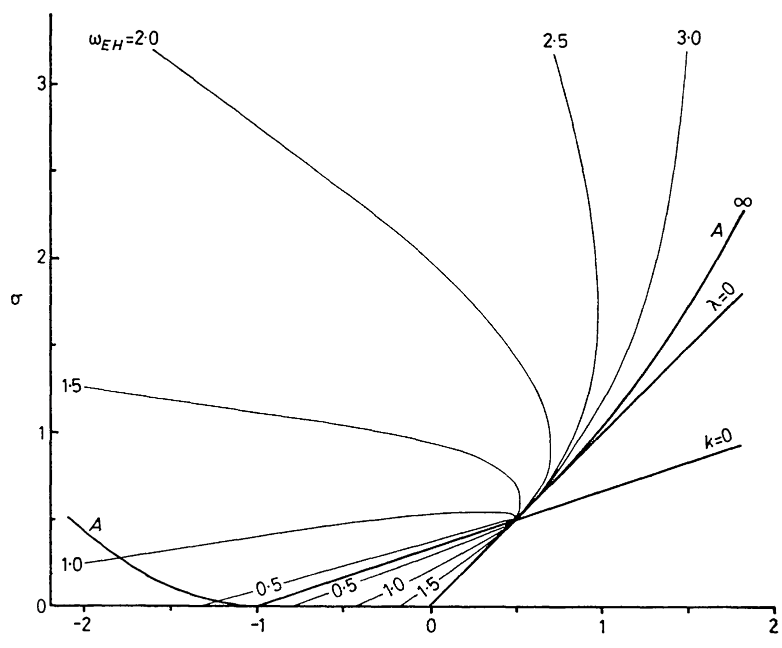

FIG. 7. *The event-horizon in the different models.*

evolution-curve. When $t \to \infty$, $\omega_{PH} \to \frac{1}{2}\omega_{AH}$, and it is thus seen that ω_{PH} tends to a finite limit in some of the M_1 models when $t \to \infty$. This limiting value of ω_{PH} is given for some of the evolution-curves in Fig. 4, and we are thus able to find out for these M_1 models how far out in space we can be able to observe in the ultimate future.

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Appendix A

From equations (5) and (8) it is seen that

$$\sigma = \frac{4\pi G\rho_0 R_0^3}{3} \frac{1}{R\dot{R}^2} \equiv \frac{\text{const.}}{R\dot{R}^2}\,.$$

Hence,

$$\dot{\sigma} = \frac{-\text{const.}}{R^2\dot{R}^4}(\dot{R}^3 + 2R\dot{R}\ddot{R}) = \frac{\text{const.}}{R^2\dot{R}}(2q-1) = \sigma H(2q \quad 1).$$

We thus see that $\dot{\sigma}$ is zero when $q = \frac{1}{2}$.