COSMIC-RAY PROPAGATION. I. CHARGED PARTICLES IN A RANDOM MAGNETIC FIELD

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ABSTRACT

A description of charged-particle motion in an irregular magnetic field is obtained. The magnetic field is taken to be the superposition of a constant field $B_0\hat{e}_z$ and a smaller fluctuating component $\boldsymbol{B}_1$ which is a homogeneous random function of position with zero mean. The Fokker-Planck coefficients, describing the evolution of the particle distribution in pitch angle and position, are derived explicitly in terms of the two-point correlation function of $\boldsymbol{B}_1$, or, alternatively, in terms of the Fourier spectrum of the irregular field. The time evolution of the distribution is found to depend on the irregular field in two ways: (1) A particle is scattered by those irregularities which are seen by the particle to be in resonance with its cyclotron frequency. (2) A particle follows the random walk of a field line in the X,Y-plane as it moves along the Z-axis. The diffusion limit of the Fokker-Planck equation, suppressing the pitch-angle dependence, is then considered and a diffusion tensor is derived. The application to spacecraft magnetic field observations is discussed, and the Fokker-Planck coefficients are related to the observed power spectrum of interplanetary magnetic-field fluctuations.

I. INTRODUCTION

A detailed understanding of cosmic radiation depends on an accurate treatment of the effects of interplanetary and interstellar magnetic fields on the motion of charged particles. Ultimately, this must be done through measurement of the irregular magnetic field and a determination of particle motion through the observed field. Since the magnetic irregularities are presumably a consequence of turbulence, a statistical treatment is necessary.

Previous treatments of charged-particle motion in an irregular field have been confined to phenomenological discussions based on the concept of magnetic scattering centers (Morrison 1956; Parker 1956, 1965). In analogy to ordinary diffusion, the particles random walk through the magnetic field, being scattered at the assumed scattering centers. The resulting motion depends on a diffusion coefficient, or mean step length, which remains an undetermined parameter. In the present paper a statistical description of the particle motion is obtained in terms of directly observable properties of the fluctuating field, rather than in terms of an unknown mean free path. The particles are taken to move in a time-independent[1] magnetic field $\boldsymbol{B}$ which is a random function of position. The Fokker-Planck coefficients which describe the mean motion of the particle and its scattering in pitch angle are derived from the particle equations of motion and are expressed in terms of the two-point correlation function of the field, or, alternatively, in terms of the Fourier spectrum of the irregularities. The diffusion tensor is derived and, finally, application of the formalism to interplanetary magnetic field observations is discussed.

II. STATISTICAL SPECIFICATION OF THE MAGNETIC FIELD AND PARTICLE MOTION

The equations of motion of a charged particle in a time-independent magnetic field $\boldsymbol{B}(\boldsymbol{r}) = (\gamma m_0 c/Ze)\boldsymbol{\omega}(\boldsymbol{r})$ may be written in the form

$$\frac{d\boldsymbol{V}}{dt} = \boldsymbol{V} \times \boldsymbol{\omega} , \tag{1}$$

[1] Any effects due to changes with time are thus excluded. This is perhaps reasonable for energetic particles since irregularities move at about the Alfvén speed V_A and electric field intensities are therefore of the order of $V_A B/c$, where c is the velocity of light. Thus, if the particle velocity $V \gg V_A$, effects due to temporal changes in the magnetic field should be relatively unimportant.

where V is the particle velocity, $\gamma = (1 - V^2/c^2)^{-1/2}$ and m_0 and Ze are the particle's mass and charge, respectively. Suppose that $\boldsymbol{\omega}(\boldsymbol{r})$ is the superposition of a uniform component $\omega_0 \hat{e}_Z$ and a fluctuating component $\boldsymbol{\omega}_1(\boldsymbol{r}) = \omega_X \hat{e}_X + \omega_Y \hat{e}_Y + \omega_Z \hat{e}_Z$, where ω_X, ω_Y, and ω_Z will be taken to be homogeneous random functions of position with zero mean. That is, $\boldsymbol{\omega}_1(\boldsymbol{r})$ is random in the sense that a specification of all mth order product mean values

$$\langle \boldsymbol{\omega}_1(\boldsymbol{r}_1)\boldsymbol{\omega}_1(\boldsymbol{r}_2) \ldots \boldsymbol{\omega}_1(\boldsymbol{r}_m) \rangle \, , \tag{2}$$

as a function of $\boldsymbol{r}_1, \boldsymbol{r}_2 \ldots \boldsymbol{r}_m$, constitutes a statistically complete specification of the magnetic field (Yaglom 1962). Under the assumption of homogeneity, the mean values defined by equation (2) are invariant under translations along any axis. Also, since the mean field $\omega_0 \hat{e}_z$ defines the only characteristic direction, $\boldsymbol{\omega}_1$ will be assumed to be statistically invariant under rotations about the Z-axis and under reflections through any point. The means are to be taken over an ensemble of systems having the same macroscopic boundary conditions, although, because of the homogeneity, this can be assumed to coincide with an average over space. This specification of the magnetic field is completely analogous to the usual specification of the velocity field in a turbulent fluid (Batchelor 1960). A statistical specification is required since the fluctuating field is presumably a result of plasma turbulence and, for any given initial conditions, the magnetic-field fluctuations vary randomly in phase and amplitude.

A complete statistical description of the particle motion, to all orders in $\boldsymbol{\omega}_1$, would involve all product mean values as defined by equation (2). However, if it is assumed that $\langle \boldsymbol{\omega}_1^2 \rangle^{1/2} \ll \omega_0$, terms involving the third or higher powers in $\boldsymbol{\omega}_1$ can be neglected and the particle motion is well described in terms of the two-point correlation. For later reference, consider the following properties of the two-point correlation tensor.

Since $\boldsymbol{\omega}_1$ is a homogeneous random function, the two-point correlation tensor depends only on the vector between the points. As a consequence of the assumed axial symmetry and because $\nabla \cdot \boldsymbol{\omega}_1 = 0$, the two-point correlation tensor must be of the form (Batchelor 1946)

$$\begin{aligned} R_{ij}(\eta,\psi,\zeta) &= \langle \omega_{1i}(X,Y,Z)\omega_{1j}(X+\eta, Y+\psi, Z+\zeta) \rangle \\ &= \begin{pmatrix} a\eta^2 + b & a\eta\psi & a\eta\zeta + d\eta \\ a\eta\psi & a\psi^2 + b & a\psi\zeta + d\psi \\ a\eta\zeta + d\eta & a\psi\zeta + d\psi & a\zeta^2 + b + c + 2d\zeta \end{pmatrix} , \end{aligned} \tag{3}$$

where a, b, and c are even functions of $\rho = (\eta^2 + \psi^2)^{1/2}$ and ζ, and d is even in ρ and odd in ζ. If $\tilde{\boldsymbol{\omega}}_1(\boldsymbol{k})$ is the Fourier transform of $\boldsymbol{\omega}_1(\boldsymbol{r})$, then it can be shown that

$$\langle \tilde{\omega}_{1i}(\boldsymbol{k}_1)\tilde{\omega}_{1j}(\boldsymbol{k}_2) \rangle = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3 r R_{ij}(\boldsymbol{r}) \, e^{i\boldsymbol{k}_1 \cdot \boldsymbol{r}} \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2) \, . \tag{4}$$

This provides a relation between $R_{ij}(\boldsymbol{r})$ and the spectrum of the fluctuating field.

Turning to the particle motion, we note that since the unperturbed orbit is the usual helix, it will prove convenient to define the complex quantities $X_+ = X + iY$, $V_+ = V_X + iV_Y$, and $\omega_+ = \omega_X + i\omega_Y$. The equation of motion can then be written

$$\ddot{Z} = -\frac{1}{2} i \, (\omega_+ V_+^* - \omega_+^* V_+) \tag{5}$$

$$\ddot{X}_+ = i[V_Z \omega_+ - (\omega_0 + \omega_Z) V_+] \, , \tag{6}$$

where ω_+ and ω_Z may be regarded as functions of Z and X_+ and the superscript * indicates complex conjugate. Clearly, the correlation coefficients may be readily defined in terms of ω_+ and ω_Z.

The unperturbed particle orbit is then given by $Z_u = Z_0 + V_0 t$ and $X_{+u} = X_{+0} + i(V_{+0}e^{-i\omega_0 t}/\omega_0)$. Now consider the perturbations about this orbit due to the fluctuating field $\boldsymbol{\omega}_1$. Set

$$Z(t) = Z_u(t) + Z_1(t) + Z_2(t) + \ldots, \tag{7}$$

$$X_+(t) = X_{+u}(t) + X_{+1}(t) + X_{+2}(t) + \ldots, \tag{8}$$

where Z_1 and X_{+1} are linear in ω_1, and Z_2 and X_{+2} are of second order in ω_1. Substituting equations (7) and (8) into the equations of motion (5) and (6), one obtains for $Z_1(t)$ and $X_{+1}(t)$,

$$\ddot{Z}_1 = -\frac{i}{2}\left[\omega_+(Z_u, X_{+u})V_{+0}{}^* e^{i\omega_0 t} - \omega_+{}^*(Z_u, X_{+u})V_{+0}e^{-i\omega_0 t}\right], \tag{9}$$

$$\ddot{X}_{+1} + i\omega_0 \dot{X}_{+1} = iV_{Z0}\omega_+(Z_u, X_{+u}) - iV_{+0}e^{-i\omega_0 t}\omega_Z(Z_u, X_{+u}) \tag{10}$$

and similar, though more complex equations for Z_2 and X_{+2}. It will be found unnecessary, however, to consider the latter in any detail.

Now, we are interested in a statistical description of the particle motion and so will consider the average change in the parameters X_+, Z, and $\mu = V_Z/V$ as a function of time. Since the phase of the particle gyration about ω_0 is not of interest, all quantities will be averaged over the gyration. Let $n(\mu, \boldsymbol{r}, t)d\mu d\boldsymbol{r}$ be the probability of finding the particle in $\boldsymbol{r}$ to $\boldsymbol{r} + d\boldsymbol{r}$, and in μ to $\mu + d\mu$ at time t. A particle travels along the Z-axis at a velocity μV and, as it interacts with the magnetic irregularities, it is scattered in pitch angle and its orbit shifted. If these changes are small in a correlation length of the fluctuating field, the evolution of n is governed by a Fokker-Planck equation (Chandrasekhar 1943). The problem is to compute the Fokker-Planck coefficients $\langle(\Delta\mu)^2\rangle/\Delta t$, etc., in terms of the fluctuating magnetic field.

Consider first the scattering in pitch angle. From the definition of μ and equation (9),

$$\begin{aligned}\langle(\Delta\mu)^2\rangle &= \left\langle \frac{1}{V^2}\dot{Z}_1{}^2\right\rangle \\ &= -\frac{1}{4V^2}\left\langle \int_0^{\Delta t} d\tau \int_0^{\Delta t} d\tau' \left[\omega_+(Z_u, X_{+u})V_{+0}{}^* e^{i\omega_0\tau} - \omega_+{}^*(Z_u, X_{+u})\right.\right. \\ &\quad \left.\left. \times V_{+0}e^{-i\omega_0\tau}\right]\left[\omega_+(Z_u, X_{+u})V_{+0}{}^* e^{i\omega_0\tau'} - \omega_+{}^*(Z_u, X_{+u})V_{+0}e^{-i\omega_0\tau'}\right]\right\rangle,\end{aligned} \tag{11}$$

correct to second order in $\boldsymbol{\omega}_1$. Define

$$\zeta = V_{Z0}(\tau' - \tau) \tag{12}$$

and

$$\rho_{+1} = X_{+u}(\tau') - X_{+u}(\tau) = i\frac{V_{+0}}{\omega_0}e^{-i\omega_0\tau}(e^{-i\omega_0\zeta/\mu V} - 1). \tag{13}$$

Then, because of the symmetry,

$$\langle\omega_+[Z_u(\tau), X_{+u}(\tau)]\omega_+{}^*[Z_u(\tau'), X_{+u}(\tau')]\rangle = 2b(\zeta, \rho_1) + \rho_1{}^2 a(\zeta, \rho_1) \tag{14}$$

and

$$\langle\omega_+[Z_u(\tau), X_{+u}(\tau)]\omega_+[Z_u(\tau'), X_{+u}(\tau')]\rangle = a(\zeta, \rho_1)\rho_{+1}{}^2, \tag{15}$$

where

$$\rho_1(\zeta) = |\rho_{+1}| = \left[2\,\frac{(1-\mu^2)V^2}{\omega_0{}^2}\left(1-\cos\frac{\omega_0}{\mu V}\,\zeta\right)\right]^{1/2}. \tag{16}$$

Substituting these relations into equation (11), and remembering that $\langle\omega_1\rangle = 0$, one eventually obtains

$$\langle(\Delta\mu)^2\rangle = \frac{1-\mu^2}{|\mu|V}\int_0^{\Delta t} d\tau \int_{-V_{Z0}\tau}^{V_{Z0}(\Delta t-\tau)} d\zeta \\ \times \left\{b[\zeta,\rho_1(\zeta)]\,e^{-i\omega_0\zeta/\mu V} + \frac{(1-\mu^2)V^2}{2\omega_0{}^2}\,a[\zeta,\rho_1(\zeta)](1-e^{-2i\omega_0\zeta/V\mu})\right\}. \tag{17}$$

If we further make the usual assumption that $V_{Z0}\Delta t$ is much greater than the correlation length along Z, the ζ-integration can be taken from $-\infty$ to $+\infty$ and the integrand with respect to τ becomes independent of τ. Thus

$$\frac{\langle(\Delta\mu)^2\rangle}{\Delta t} = \frac{1-\mu^2}{|\mu|V}\int_{-\infty}^{\infty} \\ \times d\zeta\left\{b[\zeta,\rho_1(\zeta)]\,e^{i\omega_0\zeta/\mu V} + \frac{(1-\mu^2)V^2}{2\omega_0{}^2}\,a[\zeta,\rho_1(\zeta)](1-e^{-2i\omega_0\zeta/\mu V})\right\}. \tag{18}$$

To compute $\langle\Delta\mu\rangle/\Delta t$ explicitly, a similar analysis would have to be made for $Z_2(t)$. However, this may be circumvented if we note that the scattering in pitch angle must lead to isotropy (n independent of μ); that is, scattering must cause simple diffusion in pitch angle. If scattering did not tend toward isotropy, a spatially uniform, isotropic distribution would relax toward anisotropy, in violation of Liouville's theorem. Thus the coefficients $\langle\Delta\mu\rangle/\Delta t$ and $\langle(\Delta\mu)^2/\Delta t$ must be related so that

$$\tfrac{1}{2}\,\frac{\partial^2}{\partial\mu^2}\left(\frac{\langle(\Delta\mu)^2\rangle}{\Delta t}\,n\right) - \frac{\partial}{\partial\mu}\left(\frac{\langle\Delta\mu\rangle}{\Delta t}\,n\right) = \tfrac{1}{2}\,\frac{\partial}{\partial\mu}\left(\frac{\langle(\Delta\mu)^2\rangle}{\Delta t}\,\frac{\partial n}{\partial\mu}\right), \tag{19}$$

and knowledge of $\langle(\Delta\mu)^2\rangle/\Delta t$ is sufficient. Note from equation (18) that scattering is caused primarily by those irregularities seen by the particle at its cyclotron frequency.

Similarly, using equation (10) the motion in the X,Y-plane is given by

$$V_{+1}(t) = e^{-i\omega_0 t}\int_0^t [iV_{Z0}\omega_+(Z_u,X_{+u})\,e^{i\omega_0\tau} - iV_{+0}\omega_Z(Z_u,X_{+u})]\,d\tau \tag{20}$$

and

$$X_{+1}(t) = -\frac{1}{\omega_0}\int_0^t [V_{Z0}\omega_+(Z_u,X_{+u})\,e^{i\omega_0\tau} - V_{+0}\omega_Z(Z_u,X_{+u})][e^{-i\omega_0 t} - e^{-i\omega_0\tau}]\,d\tau\,. \tag{21}$$

The mean motion of the particle may be represented by the mean motion of the guiding center $\boldsymbol{r}_{gc} = \boldsymbol{r} - (\boldsymbol{\omega}\times\boldsymbol{V})/|\omega|^2$. Consider first the guiding center motion in the X,Y-plane: define the complex quantity $r_{+g} = X_{gc} + iY_{gc} = r_{+gu} + r_{+1}$, where r_{+1} contains terms of first order in $\boldsymbol{\omega}_1$. Then

$$r_{+1}(t) = X_{+1}(t) + \frac{i}{\omega_0{}^2}[\omega_+ V_{Z0} - V_{+1}\omega_0 + V_{+u}\omega_Z]. \tag{22}$$

The motion of the guiding center may be obtained from equation (22) in a manner analogous to that used above to calculate $\langle(\Delta\mu)^2\rangle/\Delta t$. Thus, again making use of the assumed symmetry and casting out cyclic terms which vanish when averaged over the orbit, one finds for the mean guiding center motion,

$$\frac{\langle (\Delta X)^2\rangle}{\Delta t}=\frac{\langle (\Delta Y^2)\rangle}{\Delta t}=\frac{\langle \Delta r_{+1}\Delta r_{+1}^{*}\rangle}{2\Delta t}$$

$$=\frac{1}{2\omega_0^2|\mu|V}\int_{-\infty}^{\infty}d\zeta\Big[\!\!\Big[\mu^2V^2\{2b[\zeta,\rho_1(\zeta)]+\rho_1^2a[\zeta,\rho_1(\zeta)]\}$$

$$+(1-\mu^2)V^2R_{ZZ}[\zeta,\rho_1(\zeta)]\,e^{-i\omega_0\zeta/\mu V} \tag{23}$$

$$+\frac{2i\mu(1-\mu^2)V^3}{\omega_0}\{a[\zeta,\rho_1(\zeta)]\zeta+d[\zeta,\rho_1(\zeta)]\}\,e^{-i\omega_0\zeta/\mu_V}\Big]\!\!\Big],$$

$$\frac{\langle \Delta X\Delta Y\rangle}{\Delta t}=\tfrac{1}{2}\left[\frac{\langle (\Delta r_{+1})^2\rangle}{\Delta t}-\frac{\langle (\Delta r_{+1}^{*})^2\rangle}{\Delta t}\right]=0 \tag{24}$$

and $\langle \Delta X\rangle/\Delta t = \langle \Delta Y\rangle/\Delta t = 0$.

Proceeding further, one can show that $\langle \Delta\mu\Delta r_{+1}\rangle/\Delta t = 0$, so that $\langle \Delta\mu\Delta X\rangle/\Delta t = \langle \Delta\mu\Delta Y\rangle/\Delta t = 0$. The motion in the X,Y-plane is thus a simple random walk. Note that, in addition to the resonant terms, the guiding center motion normal to the field depends on simple integrals of a and b. These merely give the particle motion due to a possible net inclination of the fluctuating field with respect to the Z-axis; the terms may be physically visualized as due to the guiding center following the random walk of a given field line in the X,Y-plane as the particle moves in the Z-direction.

Finally, consider the motion along the Z-axis. This is qualitatively different from the motion normal to the field since μ is assumed to change only a small amount in a correlation length. Thus, the particle simply moves along the Z-axis at a rate μV and only gradually does this rate change. The quantity $\langle \Delta Z\rangle/\Delta t$ is then simply μV, and $\langle (\Delta Z)^2\rangle$ can be shown to be of second order in Δt. Similarly, $\langle \Delta Z\Delta\mu\rangle = 0$, and the complete Fokker-Planck equation for $n(r,\mu,t)$ can be written

$$\frac{\partial n}{\partial t}=-\mu V\frac{\partial n}{\partial Z_1}+\tfrac{1}{2}\frac{\partial}{\partial\mu}\left[\frac{\langle (\Delta\mu)^2\rangle}{\Delta t}\frac{\partial n}{\partial t}\right]+\tfrac{1}{2}\frac{\langle (\Delta X)^2\rangle}{\Delta t}\left(\frac{\partial^2 n}{\partial X^2}+\frac{\partial^2 n}{\partial Y^2}\right), \tag{25}$$

where the various coefficients are now given in terms of the correlation coefficient of the random field by equations (18), (19), (23), and (24). Clearly, rather complete knowledge of the correlation tensor is necessary to accurately compute the coefficients. If the magnetic structure is moving at velocity V_1 relative to the observer (as, e.g., in the case of irregularities carried by the solar wind) a term $V_1\cdot\nabla n$ must be added to the right side of equation (25).

III. THE DIFFUSION LIMIT

In many cases not all the information contained in equation (25) is of interest. In particular, if the variation of n with Z is small over the distance in which a particle is scattered appreciably in pitch angle, the pitch-angle distribution may be taken to be isotropic and we may approximate the Z motion as diffusion with a mean free path λ_Z, of the order of this scattering distance. Since the motion in the X,Y-plane is already simple diffusion, this leads to a diffusion tensor D_{ij} for the particle motion. D_{ZZ} may be derived directly from equation (25) as follows. Consider only the Z-motion, which is governed by

$$\frac{\partial n}{\partial t}+\mu V\frac{\partial n}{\partial Z}=\tfrac{1}{2}\frac{\partial}{\partial\mu}\left[\frac{\langle (\Delta\mu)^2\rangle}{\Delta t}\frac{\partial n}{\partial\mu}\right], \tag{26}$$

where $\langle(\Delta\mu)^2\rangle/\Delta t$ is given by equation (18); n will be taken to be very nearly isotropic so that $n = n_0 + n_1(\mu)$ with $n_1 \ll n_0$. The diffusion limit here is equivalent to the assumption that $\partial n/\partial t \ll \mu V \partial n/\partial Z$. That is, the time rate of change of n is due to diffusive motion, which is much slower than individual particle motion. Then, with neglect of the $\partial n/\partial t$ term, integration of equation (26) readily yields the diffusion equation

$$n_0 V\langle\mu\rangle = -D_{ZZ}\frac{\partial n}{\partial Z}, \tag{27}$$

where

$$D_{ZZ} = V^2\int_{-1}^{1}\mu_1\Big[\int^{\mu_1}\frac{(1-\mu^2)}{\langle(\Delta\mu)^2\rangle/\Delta t}\,d\mu\Big]d\mu_1. \tag{28}$$

This assumes, of course, that there is enough scattering that the integrals are defined. Now, to determine the remaining coefficients of the diffusion tensor, note that in § II $\langle(\Delta X)^2\rangle/\Delta t$, and $\langle(\Delta Y)^2\rangle/\Delta t$ have been determined as functions of μ. Since the pitch-angle distribution of the particles is here assumed to be very nearly isotropic, the components of the diffusion tensor which give diffusion normal to the field are simply

$$D_{XX} = D_{YY} = \tfrac{1}{2}\int_0^1\frac{\langle(\Delta X)^2\rangle}{\Delta t}\,d\mu \tag{29}$$

$$D_{XY} = D_{YX} = 0\,. \tag{30}$$

In addition, since the mean motions along the field and normal to the field are uncorrelated, set $D_{ZX} = D_{XZ} = D_{ZY} = D_{YZ} = 0$ to complete the specification of the diffusion tensor. The diffusive motion of the charged particles is then described by the diffusion equation

$$\frac{\partial n}{\partial t} = D_{XX}\frac{\partial^2 n}{\partial X^2} + D_{YY}\frac{\partial^2 n}{\partial Y^2} + D_{ZZ}\frac{\partial^2 n}{\partial Z^2}. \tag{31}$$

Again, if the irregularities are being convected past the observer at a velocity V_1 as in the solar wind, a term $V_1\cdot\nabla n$ must be added to the right-hand side of equation (31). We therefore regain the diffusion picture which has been much used in the theory of cosmic-ray propagation, except that now the diffusion tensor is explicitly related to the fluctuating magnetic field.

IV. RELATION TO OBSERVED INTERPLANETARY MAGNETIC-FIELD FLUCTUATIONS

In the preceding sections the motion of charged particles in a random magnetic field has been formally related to the correlation tensor or Fourier spectrum of the field. As an example of a possible application of the above formalism, consider now the problem of determining cosmic-ray motion in the interplanetary magnetic field from space-probe magnetic-field observations. For purposes of illustration, consider a model in which the magnetic field is statistically isotropic and time-independent in a frame moving with the solar wind. Then the correlation tensor is as in equation (3), except that $d = c = 0$ and a and b are functions only of $r = \sqrt{(\rho^2 + \zeta^2)}$. As the irregularities are blown past the satellite, the magnetometer observes a varying field $B(t)$. Since all statistical properties are isotropic, we can without loss of generality let the wind velocity V_w define the Z-axis. Then $B(Z) = B(V_w t + Z_0)$ and if B_{av} is the average field, the functions b and a are defined by

$$b(\zeta) = \frac{Z^2e^2}{\gamma^2 m_0^2 c^2}\langle[B_X(Z) - B_{X\,av}][B_X(Z+\zeta) - B_{X\,av}]\rangle, \tag{32a}$$

$$a(\zeta)\zeta^2 + b(\zeta) = \frac{Z^2e^2}{\gamma^2 m_0^2 c^2}\langle[B_Z(Z) - B_{Z\,av}][B_Z(Z+\zeta) - B_{Z\,av}]\rangle. \tag{32b}$$

Alternatively, if the power spectrum $P_{ij}(f)$ gauss²/cycle/sec of the field is observed, then a and b may be obtained from equation (32) and the relation

$$P_{ij}(f)=\frac{\gamma^2 m_0^2 c^2}{Z^2 e^2}\int_{-\infty}^{\infty} R_{ij}(\zeta,0)\, e^{-2\pi i f \zeta/V_w}\frac{d\zeta}{V_w}. \tag{33}$$

A further simplification is possible if the cyclotron radius $r_c = V/\omega_0$ is small compared with the correlation distance of the fluctuating field. Then $R_{ij}(r)$ may be expanded in powers of $\rho_1 \leq 2r_c$. Note that $\partial R_{ij}/\partial\rho|_{\rho=0} = 0$. The Fokker-Planck coefficients may then be written in terms of the integrals

$$\int_{-\infty}^{\infty} R_{XX}(\zeta,0)\, e^{-i\omega_0\zeta/\mu V} d\zeta\,, \tag{34}$$

$$\int_{-\infty}^{\infty} R_{ZZ}(\zeta,0)\, e^{-i\omega_0\zeta/\mu V} d\zeta\,, \tag{35}$$

and

$$\int_{-\infty}^{\infty} R_{XX}(\zeta,0)\, d\zeta \tag{36}$$

correct to lowest order in ρ_1 or V/ω_0. In this approximation the particle samples only irregularities on the Z-axis. Then, from equation (33) and the results of § II we have the Fokker-Planck coefficients directly in terms of the observed power spectrum:

$$\frac{\langle(\Delta\mu)^2\rangle}{\Delta t}=\frac{(1-\mu^2)}{|\mu|V}\frac{Z^2 e^2 V_w}{\gamma^2 m_0^2 c^2} P_{XX}\left(\frac{V_w\omega_0}{2\pi\mu V}\right), \tag{37}$$

$$\frac{\langle(\Delta X)^2\rangle}{\Delta t}=\frac{\langle(\Delta Y^2)\rangle}{\Delta t}=\frac{|\mu|V}{2\omega_0^2}\frac{V_w Z^2 e^2}{\gamma^2 m_0^2 c^2}\left[2P_{XX}(0)+\frac{1-\mu^2}{\mu^2}P_{ZZ}\left(\frac{V_w\omega_0}{2\pi\mu V}\right)\right]. \tag{38}$$

It is evident from equations (37) and (38) that scattering is due to power near the frequency $f_0 \sim V_w\omega_0/2\pi V$ which corresponds to irregularities with scale sizes equal to the particle cyclotron radius being carried past the satellite. The term in equation (38) involving $P_{XX}(0)$ represents the effect of the random walk of field lines in the X,Y-plane.

Clearly, simplifications also result if r_c is much greater than the correlation length.

Now consider the application of these results to interplanetary magnetic field observations. Coleman (1966) has reported power spectra in the frequency range 10^{-5}–10^{-2} sec^{-1} obtained from Mariner II data. The spectra fall off with increasing frequency slightly more rapidly than f^{-1} and have no significant peaks. The spectra are consistent with the assumptions used above to derive equations (37) and (38), so it is reasonable to construct a diffusion tensor from the data. It is sufficient for present purposes to approximate Coleman's spectra in the region of frequencies $f = 10^{-5}$–10^{-3} sec^{-1} by $P_{XX}(f) \sim \delta/f$ Gauss² sec, $P_{ZZ} \sim a/f$ Gauss² sec, where $\delta \sim 1.4 \times 10^{-10}$, $a \sim 1.2 \times 10^{-10}$. Then, from equations (37) and (38) and the results of § III, one finds for the $\sim$50 – 500 MeV particles that are affected by this range of frequencies,

$$\begin{aligned} D_{ZZ} &= \frac{V^2}{3\pi}\left(\frac{\gamma m_0 c}{Ze}\right)^2\frac{\omega_0}{\delta} \\ &\sim 3\frac{A}{Z}V^2 \end{aligned} \tag{39}$$

and, neglecting the power at zero frequency,

$$D_{XX} = D_{YY} = \frac{\pi}{3}\left(\frac{ZeV}{\gamma m_0 c \omega_0}\right)^2 \frac{a}{\omega_0}$$
$$\sim 0.1\,\frac{A}{Z}V^2\,. \tag{40}$$

Here Z/A is the charge to mass ratio and the mean interplanetary field B_0 has been set equal to 5×10^{-5} gauss. Clearly the diffusion is mainly along the average magnetic field. It is of interest that the ratio D_{XX}/D_{ZZ} can be written $r_c^2/\lambda^2 = V^2/\omega_0^2\lambda^2$, where $\lambda \simeq 3D_{ZZ}/V$ is roughly equivalent to the mean step length used in the scattering center approach. This ratio, which should not be sensitive to the assumed $P_{ij}(f)$, is the same as that obtained by Parker (1965) in the limit $\lambda \gg r_c$. We see that for 50–500-MeV protons $D_{ZZ} \sim 10^{21}$ cm^2 sec^{-1}, in agreement with the values commonly obtained from cosmic ray observations. (Parker 1965). Further, a definite dependence of D_{ij} on particle velocity and charge to mass ratio is predicted. A subsequent paper will discuss in detail the application of these results to cosmic rays and present detailed comparison with observations.

V. DISCUSSION

The foregoing calculations present a statistical description of the motion of charged particles in a spatially random magnetic field. The time evolution of the particle distribution in pitch angle and position is described in terms of Fokker-Planck coefficients which are derived explicitly in terms of the correlation tensor or Fourier spectrum of the irregular field. It is felt that this approach represents an improvement over the traditional "diffusion" picture based on the concept of magnetic scattering centers; many features of the latter picture are regained, but the particle motion is now quantitatively related to the spectrum of magnetic irregularities. As discussed in § IV, application of the formalism to direct observations of the interplanetary magnetic field should make possible a quantitative discussion of cosmic-ray propagation and modulation in the solar system. Similarly, optical determination of the structure of the interstellar magnetic field (Kaplan 1966) may eventually permit a quantitative treatment of galactic cosmic-ray motion.

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