GENERAL RELATIVISTIC FLUID SPHERES

II. GENERAL INEQUALITIES FOR REGULAR SPHERES

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ABSTRACT

Certain inequalities relating to regular spheres, i.e., spherically symmetric static distributions of matter whose central density and radius are finite, whose pressure and density are nowhere negative, and whose density does not increase outwards, were derived in an earlier paper. These were generally much too weak, either in the Newtonian limit, or in the limit of infinite mass concentration ($\delta = 0$), or both. In particular the upper limit established for the gravitational potential energy Ω tended to ∞ as $\delta \rightarrow 0$. These various defects are remedied here and some fairly sharp inequalities derived which do not become nugatory in any situation. In particular it is shown that if Δ is the boundary value of g_{44} then

$$\frac{(\beta+3)^2}{9(\beta^2+6\beta+1)}\left[1+\frac{4\beta}{(\beta+1)^2}\,\delta^{\kappa}\right] \leqq \Delta \leqq 1-\frac{4q_c(2q_c+3)}{9(q_c+1)^2}\,\delta^{1/3},$$

where q is the ratio of pressure to mean density, β the maximum value of q, and $\kappa(\frac{1}{3} \leqq \kappa \leqq 4 - 2\sqrt{2})$ a certain constant which depends on β only. The various inequalities relating to Δ and Ω are illustrated by means of specific numerical examples.

I. INTRODUCTION

Some years ago I derived (Buchdahl 1959)[1] some general inequalities governing certain quantities which characterize any "regular sphere" in general relativity, i.e., a static spherically symmetric distribution of matter whose radius and central density are finite, whose pressure and density are nowhere negative, and whose density does not increase outward. It was evident at the time that some of these inequalities were too weak. In particular, when $\delta(\leqq 1)$ is small, the right-hand member of formula I(3.10) does not tend to unity in the "Newtonian limit" $\beta \rightarrow 0$, while formula I(6.13) becomes in effect empty for sufficiently small δ. This feature is the result of the weakness of formula I(6.10), in the sense that the latter does not limit the minimum value of y sufficiently strongly (cf. the discussion following eq. I[6.3]).

More recently Bondi (1964) has considered massive spheres in general relativity by somewhat different methods; and his results show incidentally that formula I(6.10) is in fact unnecessarily weak under some circumstances. I therefore return now to the results obtained in Paper I with the intention of effecting a substantial improvement of some of them, using methods similar to those used earlier. Indeed, to some extent echoing the somewhat unsystematic procedure of Paper I, § 6, I focus attention from the outset on functions which are known to be constant when the density of the sphere is constant.

The derivatives of three such functions are considered in § II, and this leads quickly to a lower limit for Δ in § III with the property that for any fixed $\delta \neq 0$ one has generically

$$\Delta \geqq 1 - 0(\beta), \quad \beta \rightarrow 0 . \tag{1.1}$$

In § IV it is shown that Δ is not less than a certain constant which depends solely on β and whose value is never less than $\frac{1}{9}$ and which tends to 1 as $\beta \rightarrow 0$. This result in turn leads to a stronger inequality for Δ which has the property (1.1) for *all* values of δ (in-

[1] All references to the 1959 paper will be distinguished by the letter "I," and its terminology and notation are here presupposed.

cluding zero). A certain upper limit for Δ is established in § V. Finally, after considering in § VI the smallest value that y can take, an upper limit is established in § VII for the gravitational potential energy. The inequality in question does not become nugatory as $\delta \to 0$, unlike the corresponding inequality in Paper I. The various general results obtained are illustrated throughout by numerical examples.

II. AUXILIARY FUNCTIONS AND THEIR DERIVATIVES

The Schwarzschild interior solution is characterized by the constancy of ρ, or equivalently, the constancy of w. When w is constant one easily finds from formulae I(2.9, 12) that

$$\zeta = A - By\,, \quad q = (3By - A)/(A - By)\,, \tag{2.1}$$

where A and B are constants of integration, and

$$q = P/w\,. \tag{2.2}$$

In this case the quantities

$$C = (q+3)\zeta\,, \quad D = (q+1)\zeta/y \tag{2.3}$$

are therefore constant throughout the sphere. When w is not necessarily constant, the definitions (2.3) may stand, and then

$$\frac{C_{,x}}{C} = -\frac{[(1-u)q+u]w_{,x}}{(q+3)y^2 w}\,, \quad \frac{D_{,x}}{D} = -\frac{q w_{,x}}{(q+1)w}\,, \tag{2.4}$$

where

$$u = xw\,. \tag{2.5}$$

When w is constant, any function of C and D is of course also constant. Of particular interest is the case where this does not involve ζ explicitly, and so one is led to define the function

$$F = (q+1)/[(q+3)y]\,. \tag{2.6}$$

Then, in general,

$$\frac{F_{,x}}{F} = \frac{[(q^2+6q+1)u - 2q]w_{,x}}{y^2(q+1)(q+3)w}\,. \tag{2.7}$$

It may be remarked that the function ψ that occurs in § 6 of Paper I has F as a factor.

III. LOWER LIMIT FOR Δ (FIRST TYPE)

Since $q \leqq \beta$ and $w_{,x} \leqq 0$, it follows from equations (2.4) that $Dw^{\beta/(\beta+1)}$ is a nonincreasing function of x. Hence, comparing central and boundary values, one has

$$\zeta_c \geqq (q_c+1)^{-1}\delta^{\beta/(\beta+1)} \geqq (\beta+1)^{-1}\delta^{\beta/(\beta+1)}\,. \tag{3.1}$$

Recalling the inequality I(3.8), viz.,

$$\Delta \geqq \tfrac{1}{9}(1+2\zeta_c)^2\,, \tag{3.2}$$

one therefore now has

$$\Delta \geqq \tfrac{1}{9}\left[1 + \frac{2}{\beta+1}\,\delta^{\beta/(\beta+1)}\right]^2. \tag{3.3}$$

The imposition of the requirement that the trace T of the energy momentum tensor be nowhere negative is equivalent to setting $\beta = 1$, and then

$$\Delta \geqq \tfrac{1}{9}(1+\sqrt{\delta})^2\,, \tag{3.4}$$

which may be compared with the inequality I(3.11).

It may be remarked that using results yet to be derived one can also determine a constant α (depending on β) such that Cw^{α} is a non-increasing function of x. It appears, however, that the resulting inequality is never stronger than (3.3).

The inequality (3.3) is, in general, much stronger than I(3.10). It has the great advantage over the latter that as long as δ is finite the right-hand member tends to unity as $\beta \rightarrow 0$, as it should. However, a certain weakness still remains. Even though in the classical limit β may be regarded as of order c^{-2}, it is nevertheless finite. If, therefore, one contemplates a classical sphere with given q_c but sufficiently small δ (e.g., an Emden polytrope of index just less than 5) the inequality (3.3) will merely give $\Delta \geqq \frac{1}{9}$ again. In the context of such extreme situations the inequality (3.3) is therefore not good enough.

By way of illustrating the inequality (3.3), consider the explicit example of § 7 of Paper I. Let $\Delta_{-\text{I}}$ denote the right-hand member of the inequality I(3.10) and $\Delta_{-\text{II}}$ that of (3.4). Table 1 then gives the values of the relevant quantities of interest for selected values of β.

TABLE 1

VALUES OF $\Delta_{-\text{I}}$, $\Delta_{-\text{II}}$, AND Δ

β	$\Delta_{-\text{I}}$	$\Delta_{-\text{II}}$	Δ
∞	0 1111	0 1111	0 1574
3	0 1736	0 1870	0 2500
1	0 2888	0 3530	0 4183
0 3	0 5475	0 6715	0 6896
$\rightarrow 0$	$\sim 1-\frac{68}{27}\beta$	$\sim 1-\frac{4}{3}\beta$	$\sim 1-\frac{4}{3}\beta$

IV. LOWER LIMIT FOR Δ (SECOND TYPE)

a) The second type of lower limit for Δ to be considered is such that in the classical limit it yields

$$\Delta \geqq 1 - 0(\beta) \qquad (\beta \rightarrow 0)\,, \tag{4.1}$$

whatever the value of δ may be. One general inequality of this kind may be deduced directly from equation (2.7). If ϕ denotes the expression in brackets in the numerator on the right of equation (2.7), one has $\phi_c = -2q_c < 0$ and $\phi_b = u_b > 0$, so that ϕ has at least one zero in $0 < x < x_b$. Suppose for the moment that the sphere does not have a constant density core, i.e., that there is no finite neighborhood of the center in which w is constant. Then $w_{,x} < 0$ in the range $0 < x \leqq x_b$, so that F is an increasing function in a neighborhood of the origin, while it is a decreasing function near the boundary. It therefore attains its largest value at some point x_0, say; and here ϕ vanishes. At x_0, therefore, F takes the value

$$F_0 = (q_0 + 3)^{-1}\,(q_0^2 + 6q_0 + 1)^{1/2}\,. \tag{4.2}$$

Since $F \leqq F_0$ throughout, one has in particular at the boundary

$$3\sqrt{\Delta} \geqq (q_0 + 3)\,(q_0^2 + 6q_0 + 1)^{-1/2}\,,$$

from which it follows in turn that

$$\Delta \geqq \frac{(\beta + 3)^2}{9(\beta^2 + 6\beta + 1)} \qquad (= \gamma^2,\ \text{say})\,. \tag{4.3}$$

Though inequality (4.3) is rather weak in some respects (and it will be improved upon in due course) it has the desired property (4.1), independently of δ. In particular, when $T \geqq 0$ (i.e., $\beta = 1$), $\Delta \geqq \frac{2}{9}$, whereas previously (the value of δ being unknown) one could only say that $\Delta \geqq \frac{1}{9}$.

It remains to contemplate the possibility that the sphere has a constant density core, $w =$ const. in $0 \leqq x \leqq x_1$, say. (Such a situation is quite unphysical and might well be ignored.) Then if $F_b \geqq F_c$, there must be a maximum of F in the range $x_1 < x < x_b$, and expression (4.3) again applies. On the other hand, if $F_b < F_c$ two possibilities arise, namely, if F^* is the largest value of F attained in the range $x_1 \leqq x \leqq x_b$, then either $F^* \leqq F_c$ or $F^* > F_c$. In the first case (recalling that $F = F_c$ in $0 \leqq x \leqq x_1$) $F \leqq F_c$ throughout, and so $\Delta \geqq j$ (in the notation of eq. I[3.13]). However,

$$\gamma^2 = \frac{(\beta+1)^2 j}{\beta^2 + 6\beta + 1} \leqq j \,, \tag{4.4}$$

so that when inequality (4.4) is satisfied, the inequality (4.3) certainly is. Finally, when $F^* > F_c$, the maximum of F occurs at a zero of ϕ, and inequality (4.3) follows as before. All together, then, it has been shown that inequality (4.3) holds under all circumstances.

b) The virtue of the procedure adopted above is that it leads so directly to the general limitation (4.3), while it yields at the same time the inequality $F \leqq 1/3\gamma$, a result which will be used later. On the other hand, inequality (4.3) is evidently rather weak for moderate values of β and δ, and some improvement is desirable. How this is to be obtained is, however, far from obvious, and it may well be that a relatively trivial modification of the following procedure might yield even better results.

When w is constant, any function of F and w, say $H(F,w)$ is of course also constant, and in the spirit of the preceding work one should consider the derivative of some suitable function H. How is this to be selected? To begin with, observe that equation (2.7) may be written as

$$F_{,x} = \tfrac{1}{2}\chi_1 F(F^2 - \chi_2) w_{,x}/w \,, \tag{4.5}$$

where

$$\chi_1 = (q+3)/(q+1) \,, \quad \chi_2 = (q^2 + 6q + 1)/[(q+1)(q+3)] \,. \tag{4.6}$$

Now, for instance, if one replaces the factors multiplying $F^3 w_{,x}/w$ and $F w_{,x}/w$ by their smallest and largest values, respectively, one obtains an inequality which may be integrated; and in this way one is led to the choice

$$H = (1 - aF^{-2}) w^{-s} \,, \tag{4.7}$$

where a and s are positive constants yet to be chosen. (The "relatively trivial modification" referred to above would consist in taking in place of eq. [4.7] a similar, more favorable function.) Using equation (4.5) one now gets

$$H_{,x} = [(a\chi_1 - s) + a(s - \chi_2)F^{-2}] w^{-s-1} w_{,x} = G w^{-s-1} w_{,x} \,, \tag{4.8}$$

say. Now $\chi_2 \leqq \kappa$, where

$$\kappa = \begin{cases} \dfrac{\beta^2 + 6\beta + 1}{(\beta+1)(\beta+3)}, & \beta \leqq 1 + 2\sqrt{2}\,(=\beta^*, \text{ say}), \\ 4 - 2\sqrt{2}\,, & \beta > \beta^* . \end{cases} \tag{4.9}$$

Hence with the choice $s = \kappa$ the factor multiplying F^{-2} in G is never negative. If we recall that $F \leqq 1/3\gamma$, it follows that

$$\begin{aligned} G &\geqq a\chi_1 - \kappa + 9\gamma^2 a(\kappa - \chi_2) \\ &= a\left[\frac{9(1-\gamma^2) - (9\gamma^2 - 1)(6q + q^2)}{(q+1)(q+3)} + 9\gamma^2 \kappa\right] - \kappa \,. \end{aligned}$$

Since $\frac{1}{3} \leqq \gamma \leqq 1$ the expression in brackets in the foregoing equation attains its least value when $q = \beta$ and thus

$$G \geqq (9\gamma^2 a - 1)\kappa \,.$$

Hence choosing $a = 1/9\gamma^2$ one has $G \geqq 0$, i.e., $H_{,x} \leqq 0$. Comparing now the boundary value of H with its central value, one has

$$(1 - 9a\Delta) \leqq \left[1 - a\left(\frac{q_c + 3}{q_c + 1}\right)^2\right]\delta^s \leqq \left[1 - a\left(\frac{\beta + 3}{\beta + 1}\right)^2\right]\delta^s ,$$

whence follows the required inequality

$$\Delta \geqq \gamma^2\left[1 + \frac{4\beta}{(\beta + 1)^2}\,\delta^\kappa\right], \tag{4.10}$$

where κ is given by equation (4.9). In particular, when $\beta = 1$,

$$\Delta \geqq \tfrac{2}{9}(1 + \delta), \tag{4.11}$$

the right-hand member of which exceeds that of (3.4) by $\frac{1}{9}(1 - \sqrt{\delta})^2$. For moderate values of β and δ expression (4.10) evidently represents a substantial improvement over the earlier result (4.3). Table 2 again refers to the example of Paper I, § 7, where Δ_- now stands for the right-hand member of inequality (4.10).

TABLE 2

VALUES OF Δ AND Δ_-

β	Δ	Δ_-
∞	0.1574	0.1111
3.	0.2500	0.1906
1.	0.4183	0.3582
0.3.	0.6896	0.6728
$\rightarrow$0.	$\sim 1 - \frac{4}{3}\beta$	$\sim 1 - \frac{4}{3}\beta$

V. UPPER LIMIT FOR Δ

Though one must always have $\Delta \leqq 1$, one may aim to derive a sharper inequality than this when the values of some suitable parameters associated with the sphere be regarded as given. Such a result will be analogous to that derived in § 4 of Paper I, according to which

$$(g_{44})_c \leqq (q_c + 1)^{-2} \,.$$

Returning to equation (4.8), one easily convinces oneself that the choice $s = \frac{1}{3}$, $a = \frac{1}{9}$, insures that $H_{,x} \geqq 0$ everywhere. Comparing central and boundary values of H, one finds at once that

$$\Delta \leq 1 - \frac{4q_c(2q_c + 3)}{9(q_c + 1)^2}\,\delta^{1/3} . \tag{5.1}$$

In Table 3, Δ_+ (i.e., the right-hand member of expression [5.1]) is compared with the actual values of Δ for the case of the example of Paper I, § 7. Note that if $q_c = \beta$, as is most often the case, expressions (4.10) and (5.3) sandwich the value of Δ for a given sphere between Δ_- and Δ_+. In particular one has then in the Newtonian limit

$$\tfrac{2}{3}\delta^{1/3} q_c \leqq M/R \leqq (\tfrac{8}{3} - 2\delta^{1/3}) q_c \,. \tag{5.2}$$

VI. LOWER LIMIT FOR y

To obtain the inequality (4.10) the central value of H was compared with its boundary value. In place of the latter one may take the value of H at any point x, and then

$$y^2 \geqq \gamma^2\left[1+\frac{4\beta}{(\beta+1)^2}\left(\frac{w}{w_c}\right)^{\kappa}\right], \tag{6.1}$$

of which the inequality (4.10) is a special case. When $\beta = 1$, one infers that

$$y^2 \geqq \frac{1+2w_c x}{1+9w_c x}, \tag{6.2}$$

but for other values of β one cannot get such a simple result. Even inequality (6.2) proves to be awkward, for instance, when one wishes to calculate an upper limit for the physical radius R^* of the sphere (cf. eq. I[9.1]): one is then led to an elliptic interval.

TABLE 3

VALUES OF Δ AND Δ_+

β	Δ	Δ_+
∞	0 1574	0 3249
3 . ..	0.2500	0 4048
1 ...	0 4183	0 5283
0 3 .	0 6896	0.7372
$\to$0.	$1-\frac{4}{3}q_c$	$1-\frac{4}{3}q_c$

VII. UPPER LIMIT FOR Ω

The negative gravitational potential energy Ω (cf. eqs. I[6.2], [6.3]) is always positive. In the Newtonian limit it is an extremely small fraction of the total energy of the sphere. Let the fraction Ω/M be denoted by h. Then it is of interest to obtain a reasonably sharp inequality limiting the values which h can attain in the general case of an arbitrary regular sphere.

To this end it suffices to replace w by w_b in expression (6.1). (This step is admittedly rough, but if one wishes to deal with reasonably tractable expressions one can hardly avoid it.) The right-hand member of inequality (6.1) then becomes identical with that of (4.10), and it will be denoted by y_0^2. Now (Paper I, § 6*c*)

$$M_0 = \int_0^R (r^3 w' + 3r^2 w)\, y^{-1} dr, \tag{7.1}$$

and the integrand is never negative, as a consequence of the initial hypothesis that the density cannot be negative. If one now replaces y by y_0 (again a rather rough step), one has at once

$$h < y_0^{-1} - 1. \tag{7.2}$$

In particular if one requires that $\beta = 1$, then

$$h < 3[2(1+\delta)]^{-1/2} - 1,$$

and so certainly

$$h < 3/\sqrt{2} - 1 \approx 1.1213; \tag{7.3}$$

while for any regular sphere whatever

$$h < 2 . \tag{7.4}$$

General results such as these were of course absent from § 6 of Paper I the inequalities of which became nugatory as $\delta \rightarrow 0$, that is to say, extremely weak for small values of δ.

It may in fact be appropriate to show that expression (7.2) reproduces the general trend of Ω quite well in the case of Emden polytropes (cf. Paper I, § 6*c*). In the Newtonian limit expression (7.2) becomes

$$\Omega \leqq \tfrac{2}{3}(4 - 3\delta^{1/3})\beta M . \tag{7.5}$$

If Ω_+ denotes the right-hand member of this, one has the numerical results for selected values of the polytropic index n shown in Table 4. Even for quite moderate values of n

TABLE 4

VALUES OF Ω_+/Ω

n	Ω_+/Ω	n	Ω_+/Ω
0	1.667	3	3 652
1	2 643	4	4 050
2	3.210	5	4 257

(say, $n \gtrsim 1.5$) inequality (7.5) is much stronger than I(6.14); e.g., when $n = 3$ the latter gives $\Omega_+/\Omega \approx 294$; and, moreover, as $n \rightarrow 5$ formula I(6.13) gives the empty result $\Omega_+/\Omega < \infty$ in place of the value $128/9\pi$ in Table 4. Of course, when $n = 5$, one does not have a regular sphere since then $R = \infty$. Since M is finite, one may, however, regard this case as the limit of a sequence of regular spheres. In this sense one may also contemplate, for the purposes of illustration, certain spheres which resemble polytropes of index 5, studied previously (Buchdahl 1964); for one has a simple closed expression for Ω in this case. In fact one finds that

$$\Omega_+/\Omega = [16(\beta + 1)/3\pi\beta][3(\beta + 3)^{-1}(\beta^2 + 6\beta + 1)^{1/2} - 1] , \tag{7.6}$$

and this decreases monotonically with increasing β from its largest value $128/9\pi$ already quoted above to its least value $32/3\pi$. This is quite a satisfactory result, especially in view of the unfavorable situation of δ being zero for all values of β.

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