1966ApJ...145..505B

A CATALOGUE OF METHODS FOR STUDYING THE NORMAL MODES OF RADIAL PULSATION OF GENERAL-RELATIVISTIC STELLAR MODELS*

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Received February 16, 1966

ABSTRACT

A catalogue is presented of various methods now available for studying the normal modes of radial pulsation of stars in general relativity theory. The methods are classified according to the normal-mode properties which they investigate; and the relative merits of the methods are discussed.

I. INTRODUCTION

In the last two years astronomical discoveries have generated considerable interest in general-relativistic stellar models: The discovery of galactic X-ray sources has given impetus to theoretical research on neutron stars, and supermassive stars have been studied in connection with the quasi-stellar radio sources (QSRS's). One important aspect of relativistic stellar models is their behavior under radial perturbations. Hoyle, Narlikar, and Wheeler (1964), Cameron (1965*a*, *b*), and Finzi (1965) have independently argued that in supernovae the neutron star formed by the collapsed core may transfer energy into the expanding envelope by means of radial oscillations. On another front, the discovery of general-relativistic instabilities in supermassive stars (Chandrasekhar 1964*a*, Fowler 1964) has proved to be of crucial importance to the construction of certain models of QSRS's (Fowler 1964, 1965).

As a result of these and other developments, a number of theoreticians are now studying various relativistic stellar models and their behaviors under radial perturbations. However, in the study of the radial normal modes of these models, cumbersome methods are frequently used where simple but not widely known methods would be satisfactory. For this reason it may be useful to present here a catalogue and comparison of all methods of which we are aware for studying the properties of radial modes.

In this catalogue the methods will be classified according to the particular normal-mode properties they investigate, and the various methods which probe the same properties will be compared. An attempt will be made to give credit to the originators of each method and to refer to at least one investigation where each method has been used. Most of the methods have their analogues in the classical theory of stellar stability, which is comprehensively reviewed by Ledoux and Walraven (1958).

II. METHODS FOR CALCULATING EIGENFREQUENCIES AND EIGENFUNCTIONS

We are aware of two different methods for calculating the eigenfrequencies and eigenfunctions of the radial normal modes of a star. These methods are actually particular

* Supported in part by the U.S. Air Force Office of Scientific Research (AF 49-638-1545), in part by the Office of Naval Research (Nonr-220 (47)), and in part by the National Science Foundation (GP-5391).

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techniques for solving the same equation—the eigenequation of Chandrasekhar (1964*a*), which governs the normal modes. Before describing the two methods, let us briefly discuss the eigenequation which they solve.

If the Schwarzschild line element

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + sin^2\,\theta d\phi^2) \tag{1}$$

is used to describe the gravitational field of a star, and if the adiabatic motion of the star in its nth normal mode ($n = 0$ being the fundamental mode) is expressed in terms of an amplitude $u_n(r)$ by

$$\delta r(r,t) = r^{-2}\, e^{\nu/2}\, u_n(r) e^{i\omega_n t}\,, \tag{2}$$

then the eigenequation for $u_n(r)$ has the Sturm-Liouville form

$$(d/dr)(P du_n/dr) + (Q + \omega_n^2\, W)u_n = 0\,. \tag{3}$$

The functions $P(r)$, $Q(r)$, and $W(r)$ are expressed in terms of the equilibrium configuration of the star by

$$\begin{aligned} P &= e^{(\lambda+3\nu)/2}\, r^{-2}\, \gamma p^*\,, \\ Q &= -4e^{(\lambda+3\nu)/2}\, r^{-3}\, dp^*/dr - 8\pi e^{3(\lambda+\nu)/2}\, r^{-2}\, p^*(\rho^* + p^*) \\ &\quad + e^{(\lambda+3\nu)/2}\, r^{-2}\, (\rho^* + p^*)^{-1}\, (dp^*/dr)^2\,, \\ W &= e^{(3\lambda+\nu)/2}\, r^{-2}\, (\rho^* + p^*)\,. \end{aligned} \tag{4}$$

Here ρ^* and p^* are the density of total mass-energy and the pressure as measured by a local observer in geometrized units:[1]

$$\rho^* = (G/c^2)\rho[\mathrm{gm/cm^3}]\,, \qquad p^* = (G/c^4)p[\mathrm{dyne/cm^2}]\,; \tag{5}$$

and γ is the adiabatic index:

$$\gamma = (\rho^* + p^*)p^{*-1}\, (\partial p^*/\partial \rho^*)_{\text{constant entropy}}. \tag{6}$$

Acceptable solutions to the eigenequation (3) must satisfy certain boundary conditions at the center of the star ($r = 0$) and at the surface ($r = R$). The point $r = 0$ is a regular singular point of the eigenequation (3); and the demand that δr and $d\delta r/dr$ be finite there leads to the condition

$$u_n \sim r^3 \qquad \text{at} \qquad r = 0\,. \tag{7a}$$

Physically speaking, this boundary condition states that the fluid at the center of the star is not displaced in a radial oscillation.

At the surface of the star one imposes the physical conditions that the Lagrangian change in the pressure vanish,

$$\Delta p^* = -e^{\nu/2} r^{-2}\, \gamma p^* du_n/dr = 0 \qquad \text{at} \qquad r = R\,, \tag{7b}$$

and that δr and $d\delta r/dr$ be finite. As in Newtonian theory (Ledoux and Walraven 1958, pp. 459–461), so also in general relativity, the translation of the finiteness of δr and $d\delta r/dr$ at $r = R$ into a condition on u_n will generally depend upon the structure of the unperturbed stellar model. For most hot ($T > 0$) stellar models of physical interest, γ is finite and non-zero at $r = R$, and ρ^*/p^* has a pole of order 1 there ($\rho^*/p^* = a_1[R -$

[1] For a discussion of geometrized units see, e.g., Harrison, Thorne, Wakano, and Wheeler (1965), p. 12. Henceforth this reference will be cited as "HTWW."

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$r]^{-1} + a_2 + a_3[R - r] + \ldots$). In such a situation $r = R$ is a regular singular point of the eigenequation (3); and the regularity condition on δr and $d\delta r/dr$ there becomes

$$du_n/dr = \gamma^{-1} R^{-1}[4 + (M^*/R)e^{\lambda} + \omega_n^2 (R^3/M^*)e^{-\nu}]u_n \quad \text{at} \quad r = R . \tag{7c}$$

Here $M^* \equiv m^*(r = R)$ is the total mass of the star as measured gravitationally from outside. (Since $p^* = 0$ at $r = R$, boundary condition [7c] implies condition [7b].)

On the other hand, for certain zero-temperature stellar models (e.g., those of Harrison, Wakano, and Wheeler—see HTWW) ρ^* and γp^* are finite and non-zero at $r = R$, even though p^* has a zero of order 1 ($p^* = a_1 [R - r] + a_2 [R - r]^2 + \ldots$). In such cases $r = R$ is a regular point of the eigenequation (3), so that δr and $d\delta r/dr$ are automatically finite at $r = R$ for all solutions to equation (3). Hence, for these zero-temperature models the surface boundary condition is simply equation (7b), which is equivalent to $du_n/dr = 0$.

As in Newtonian theory (Ledoux and Walraven 1958, p. 461), so also here, for all stellar models of physical interest the frequency spectrum of the normal radial modes is discrete; the nth normal mode has n nodes between the center and the surface of the star; and the normal-mode eigenfunctions are orthogonal with respect to the weight function $W(r)$:

$$\int_0^R W u_j u_k = 0 \quad \text{if} \quad j \neq k . \tag{8}$$

Let us now turn from this discussion of the structure of the eigenproblem for the normal radial modes to a description of two methods for solving the eigenproblem.

a) Method 1-A: Numerical Integration of the Eigenequation

References.—The eigenequation (3) was originally derived from the Einstein field equations by Chandrasekhar (1964*a*) and was derived independently by others. Thorne in HTWW has shown that it can also be obtained by extremizing the difference between the kinetic and the potential energies associated with oscillations about an equilibrium configuration. Bardeen (1965) first used the method of numerical integration of the eigenequation to determine the eigenfrequencies of normal modes of massive stars and to study the relationship of binding energy and stability along sequences of models. Numerical integration has also been used by Meltzer and Thorne (1966) in studying the normal modes of certain neutron-star models.

Description of the method.—This method consists of iterative integrations of the differential equation (3) with various trial values for ω_n^2. The method for choosing the new trial value of ω_n^2 for each integration depends upon whether the surface of the star, $r = R$, is a singular point of equation (3).

If $r = R$ is a singular point of equation (3)—as is the case for all hot stellar models of physical interest—then a useful technique for choosing successive trial values of ω_n^2 is this: One first integrates equation (3) with some arbitrarily chosen trial value of ω_n^2, starting at $r = 0$ with $u_n \propto r^3$. Because $r = R$ is a singular point, the resultant trial function $u_n^{(t)}$ and, more dramatically, $du_n^{(t)}/dr$ will become infinite as $r \to R$. When $r = R$ is a *regular* singular point (cf. discussion between eqs. [7b] and [7c]), the surface blowup of $u_n^{(t)}$ has the form

$$u_n^{(t)} \propto (R - r)^{-s}, \qquad s = \left[\frac{d \ln \rho^*/dr}{d \ln (p^*/\rho^*)/dr}\right]_{r=R} . \tag{9}$$

The blowup of $u_n^{(t)}$, together with Sturm's oscillation theorem for Sturm-Liouville equations (see, e.g., Morse and Feshbach 1953), guarantees that, if the number of nodes in the trial function, $u_n^{(t)}$, exceeds n, then the trial value of ω_n^2 was larger than the true value; but, if the number of nodes in $u_n^{(t)}$ is less than or equal to n, then the trial value of ω_n^2

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was less than the true value. With this information as to whether the trial value of ω_n^2 was too small or too large, and with a knowledge of how near to $r = R$ $u_n^{(t)}$ begins to blow up as an indication of how far the trial value was from the correct value, one can make an educated choice of a new trial value of ω_n^2.

If $r = R$ is a regular point of equation (3)—as is the case for certain zero-temperature stellar models[2]—then the above method of choosing successive trial values of ω_n^2 breaks down. It also breaks down when $r = R$ is a regular singular point but the quantity s of equation (8) is very small compared to unity, since then the blowup of $u_n^{(t)}$ may occur too close to $r = R$ for numerical integration to follow it. In such situations one can integrate equation (3) from the surface of the star, where boundary condition (7b) or (7c) is imposed, to the center. The resultant trial function, $u_n^{(t)}$, will be well behaved at $r = 0$, but it will generally not satisfy the boundary condition $u_n \propto r^3$. The vanishing of the true eigenfunction at $r = 0$, together with Sturm's oscillation theorem, enables one to determine whether the trial value of ω_n^2 was too small or too large by counting the nodes of the trial function $u_n^{(t)}$. As before, if the number of nodes exceeds n, then the trial value of ω_n^2 was too large; if the number of nodes is less than or equal to n, then the trial value was too small.

The above iterative integration techniques enable one to determine quite rapidly the eigenvalue ω_n^2 to a desired degree of accuracy. However, the resultant eigenfunction is poorly determined near the singular endpoint of integration. To determine the eigenfunctions accurately everywhere, one can integrate simultaneously from the center of the star and from the surface, and require matching of the logarithmic derivative of u_n at an interior point.

b) Method 1-B: Rayleigh-Ritz Variational Technique

References.—The variational expression (10) was first derived by Chandrasekhar (1964*a*), and the relation of the variational expression to the kinetic and potential energy of the perturbed star was exhibited by Thorne (HTWW, Appendix B).[3] The method has been used by Meltzer and Thorne (1966) to calculate the eigenfrequencies of the lowest three modes of neutron-star models; and it was used in abbreviated form by Tsuruta, Wright, and Cameron (1965) (see also Tsuruta 1965) to calculate the frequencies of the fundamental modes for certain neutron-star models. For a discussion of this method in Newtonian theory see Ledoux and Walraven (1958, pp. 462–464).

Description of the method.—The eigenvalue problem presented in equations (3) and (7a)–(7c) can be re-expressed in a well-known variational form: the extremal values of the quantity

$$\omega^2 = \frac{\int_0^R [P(du/dr)^2 - Qu^2]\,dr}{\int_0^R Wu^2 dr} \tag{10}$$

are the eigenvalues, ω_n^2, of equations (3) and (7a)–(7c); and the functions $u(r)$ which give these extremal values are the corresponding eigenfunctions, $u_n(r)$. There is a technique due to Rayleigh and Ritz for using the variational expression (10) to calculate the N lowest eigenvalues, ω_n^2, and their corresponding eigenfunctions, $u_n(r)$:[4] One constructs

[2] For zero-temperature configurations of sufficiently high central density the point $r = R$ may appear to the computer to be a singular point of the equation (3) even though it is not so in reality. (For an example see Meltzer and Thorne 1966.) In such a situation the above method of choosing trial values of ω_n^2 can be very useful.

[3] Eqs. (B26) and (B27) of HTWW are incorrect; the factor $e^{\nu(0)/2}$ should be deleted from them.

[4] For more detailed and more precise discussions than we give here of the variational form of Sturm-Liouville eigenvalue problems and of the Rayleigh-Ritz technique, see, e.g , Morse and Feshbach (1953), pp. 720 ff. and 1117 ff.; also Goertzel and Tralli, pp. 215–224.

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a set of at least N trial functions $y_j(r)$ which satisfy the boundary conditions (7a)–(7c), which are orthonormal with respect to the inner product $W(r)$:

$$\int_0^R y_i y_j W dr = \delta_{ij}, \tag{11}$$

and which resemble the sought-after eigenfunctions $u_n(r)$ as closely as possible. Then the N lowest eigenvalues, a_n, of the matrix

$$A_{ij} = \int_0^R [P(dy_i/dr)(dy_j/dr) - Qy_i y_j] dr \tag{12}$$

are close approximations to the desired eigenvalues ω_n^2; and the inequality $\omega_n^2 \leq a_n$ necessarily holds for all $n \leq N$. Furthermore, if $a_i^{(n)}$ is the eigenvector of A_{ij} corresponding to a_n, then $\Sigma_i a_i^{(n)} y_i(r)$ resembles the eigenfunction $u_n(r)$ of the star more closely than any other linear combination of the trial functions $y_i(r)$.

c) *Comparison of the Methods*

In practice both of the above methods for calculating the eigenfunctions and eigenfrequencies of the radial modes of a star require extensive computer calculations. Method 1-A in particular requires an accurate routine for integrating simultaneous first-order differential equations. In selecting one method or the other for application to a particular problem, one must weigh the accuracy desired against the computer time required. Our experience has indicated that Method 1-B, the Rayleigh-Ritz technique, requires less computer time than Method 1-A, numerical integration of the eigenequation, if an accuracy of a few per cent in the lowest two or three eigenfrequencies is sufficient. However, if one requires an accuracy of better than a few per cent in the lowest eigenfrequencies, or if one desires accurate eigenfunctions, or if one wishes information about modes higher than the third or fourth, or if one is studying a configuration in which γ or $\log_{10} \rho^*$ varies widely from one region to another, then he will probably find Method 1-A more satisfactory than Method 1-B. Method I-A has been found by Bardeen (1965) to give ω_n^2 for hot stellar models to about one part in a thousand after ten trial integrations if the initial guess of the absolute value of ω_n^2 is within a factor 2 of the final result. This is true even if ω_n^2 is close to zero, in contrast to Method 1-B, where the accuracy is generally several per cent of the value which ω_n^2 assumes for configurations far from instability points.

A more detailed comparison of the two methods as applied to zero-temperature configurations is given in the following paper (Meltzer and Thorne 1966).

III. METHODS FOR DETERMINING THE NUMBER OF UNSTABLE MODES

For some purposes a detailed knowledge of the eigenfunctions and frequencies of the radial modes is not needed; it may be sufficient to know only how many modes are unstable. The methods of this section give such information quickly and accurately, without the excess baggage of the methods of § II.

a) *Method 2-A: The $M(R)$ Method for Sequences of Cold Stars*

References.—This method was originally devised by Wheeler (HTWW, pp. 60–66), but the precise conditions under which it is valid were first given by Thorne (1966). HTWW used this method to study the stability of Harrison-Wakano-Wheeler zero-temperature configurations. Meltzer and Thorne (1966) have applied it to Skyrme-Cameron-Saakyan neutron-star models. For an extensive discussion of the method and of several features not delineated here see Thorne (1966) §§ 4·2.2, 4·2.3, and 5·5.1.

Description of the method.—Corresponding to every equation of state for cold matter

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(matter at absolute zero temperature) there is a one-parameter family of non-rotating spherical equilibrium configurations. A useful parameter for distinguishing these configurations from one another is central density; for each value of the central density there is one and only one equilibrium configuration. By the following method one can calculate the number of unstable radial modes for each of the equilibrium configurations of any zero-temperature equation of state that satisfies one of these two conditions: (1) the adiabatic index appropriate to small radial motions is

$$\gamma = (\rho^* + p^*)p^{*-1}\,(dp^*/d\rho^*)_{\text{as given by equation of state}}\ ; \tag{13}$$

or (2) the equation of state represents matter catalyzed to the absolute end point of thermonuclear evolution.

One first calculates for each configuration its mass, M, as observed from outside (e.g., as measured by applying Kepler's laws to an orbiting satellite) and its radius, R (so defined that $4\pi R^2$ is the surface area of the star). One then constructs for the given equation of state a curve of M versus R (M upward, R to the right), which is parameterized by central density, ρ_c. From this curve the number of unstable modes of each configuration is determined as follows: We know from classical theory that the configurations of very low central density are stable. As we move along the curve of M versus R in the direction of increasing central density all configurations remain stable until the first extremal value (maximum or minimum) of M is reached. At the first extremal point the fundamental mode of radial oscillation becomes unstable. All configurations between the first extremal point and the second have one unstable radial mode. At the second extremal point—and also at each succeeding extremal point—one more mode of radial oscillation changes stability. The direction of stability change is determined by the way in which the $M(R)$-curve bends as it passes through the extremal point. If the direction of bend is *counterclockwise* (with increasing ρ_c), then one previously stable radial mode *becomes unstable* at the critical point; if the direction of bend is *clockwise*, then one previously unstable mode *becomes stable* at the critical point. This method of studying stability is illustrated in Figures 2 and 3 of the following paper (Meltzer and Thorne 1966). The manner in which the finite rates of nuclear reactions affect this method is discussed by Meltzer and Thorne (1966), § III, and in greater detail by Thorne (1966), § 5·5.1.

b) Method 2-B: The M(R) Method for Sequences of Hot, Isentropic Stellar Models

References.—This method was devised by Bardeen (1965) and has been used by him to study the properties of models with various simple equations of state. A precursor of this method was the rough argument of Fowler (1964) relating binding energy and stability for certain hot configurations. For a more detailed discussion of the method see Thorne (1966) § 4·2.4.

Description of the method.—The $M(R)$ technique of the last section is applicable only to sequences of zero-temperature stellar models. However, there is an $M(R)$ method very similar to that one, which is applicable to certain sequences of hot stellar models arranged according to increasing central density. The sequences to which this method can be applied are those for which (1) the entropy per baryon is independent of position in each model (isentropic configuration);[5] (2) the number of baryons in each model is the same; (3) all models in the sequence are constructed from material which obeys a single, given

[5] The method can be generalized to certain types of sequences of non-isentropic configurations (see Bardeen 1965). For example, it is sufficient if the change in entropy in going from one model to the next has the same sign at every mass shell, where mass shells are identified by the total number of baryons inside them. This generalization is of little practical value, however, because of the extremely artificial nature of such sequences.

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equation of state; (4) no thermodynamic processes, irreversible in a period of oscillation, are involved in going from one model to the next—in particular, no irreversible nuclear reactions may take place. The last condition enables one to define a rest mass per baryon, which in each configuration may be a function of radius, but which remains constant for each shell of matter (shell inside which a given number of baryons are contained) as one passes through the sequence of configurations.

If all of the above conditions are satisfied, then the relation of a plot of M versus R to the stability of the radial modes is the reverse of that for zero-temperature sequences.[6] As the central density increases from zero and the radius decreases from infinity, the total mass-energy M initially decreases if the fundamental mode of the low-density models is stable and initially increases if the fundamental mode is unstable. Subsequently one previously stable mode becomes *unstable* if the curve bends *clockwise* at an extremal point in M along the curve, while one previously unstable mode becomes *stable* if the curve bends *counterclockwise* at an extremal point. As a practical matter it is usually more convenient to plot upward the difference $M - M_0$ instead of M, where M_0 is the (conserved) sum of the rest mass-energies of the baryons in the star. Otherwise the variations in M will be just small perturbations on the constant M_0.

It should be noted that isentropic sequences of constant baryon number are difficult to construct unless the equation of state relating ρ^*, p^*, and the rest mass-energy density ρ_0^* has the form

$$\rho^*/\rho_0^* = f(p^*/\rho_0^*) \, . \tag{14}$$

This type of equation of state, like the relativistic equation of hydrostatic equilibrium, is invariant under changes in the scale of length. Consequently, one can construct models containing a given number of baryons A—or a given rest mass M_0—by integrating the equations of hydrostatic equilibrium for specified values of the dimensionless quantity $(p^*/\rho_0^*)_{\text{central}}$ and arbitrary values of $(\rho_0^*)_{\text{central}}$, and afterward adjusting $(\rho_0^*)_{\text{central}}$ with a scale transformation chosen so as to give the desired value of A or M_0. Note that the dimensionless quantity $(M - M_0)/M_0$ will be unaffected by the scale transformation. It is a function of $(p^*/\rho_0^*)_{\text{central}}$, but not of $(\rho_0^*)_{\text{central}}$.

c) Method 2-C: The Eigenequation Method for Individual Stellar Models

References.—The method was devised by Bardeen (1965) and has been used by him to study the relation between binding energy and stability in sequences of various types of hot stellar models.

Description of the method.—While Methods 2-A and 2-B are only applicable to special types of models, this method depends only on the existence of a discrete spectrum of eigenfrequencies. It consists of applying just the first step of method 1-A, an integration of the eigenequation (3) with the trial value of ω_n^2 set equal to zero. When the surface of the star, $r = R$, is a singularity of equation (3), one integrates from the center, where $u_n^{(t)} \propto r^3$, to the surface; but when the surface is a regular point, one integrates from the surface, where $du_n^{(t)}/dr = 0$ but $u_n^{(t)} \neq 0$, to the center. If, when the integration reaches its end point, the trial eigenfunction $u_n^{(t)}$ has gone through zero N times, then there are precisely N unstable modes of radial oscillation (modes 0 through $N - 1$). On the other hand, if, after going through N nodes, the integration reaches all the way to its end point with the boundary condition (7) satisfied there, then modes 0 through $N - 1$ are unstable and mode N is in neutral equilibrium.

[6] This reversal of the criteria for the direction of stability change is not disturbing in view of the totally dissimilar character of the sequences under consideration. In the zero-temperature case the entropy per baryon is held constant ($s \equiv 0$) and the total number of baryons is allowed to change as one passes through the sequence. In the case of hot configurations the total number of baryons is held fixed while the entropy per baryon is allowed to change.

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d) *Comparison of the Methods*

The first two methods require integration of the equations of hydrostatic equilibrium for a whole sequence of models before the stability of the radial modes of a single model can be known. However, since the stability determination involves no additional calculation, these methods are to be preferred over Method 2-C if there is some other reason for wanting to know the entire sequence of equilibrium configurations. There usually is such a reason in the zero-temperature case, but for most hot configurations Method 2-B is not applicable because the isentropic sequence required is not of independent physical interest. (To be sure, a hot isentropic sequence does correspond to a star in convective equilibrium undergoing Kelvin-Helmholtz contraction.)

IV. METHODS FOR DETERMINING WHETHER A CONFIGURATION IS STABLE AGAINST RADIAL PERTURBATIONS

It is generally assumed that a configuration is dynamically stable if and only if its fundamental mode of radial oscillation is stable. The stability of the fundamental mode can, of course, be determined by Methods 2-A, 2-B and 2-C, and in fact these are always the simplest methods. An additional method which has been frequently used is described below as Method 2-D.

a) *Method 2-D: Variational Calculation*

References.—The method was first formulated by Chandrasekhar (1964*a*), and the variational expression was connected with the potential energy of the perturbation by Thorne (HTWW, Appendix B; see note 3 above). The method has been used by Chandrasekhar (1964*b*), Misner and Zapolsky (1964), Tooper (1964, 1965), Gratton (1964), and others. Wright (1964) and Fowler (1965) have used a post-Newtonian version of the method to relate binding energy and stability for configurations with γ = constant.

Description of the method.—This method is based upon the fact that the fundamental radial mode is unstable if and only if there exists a function $u(r)$ which satisfies the boundary conditions (7a)–(7c) and which, when inserted into the numerator of expression (10), yields a negative value. One attempts to determine whether the numerator can be negative by calculating its value for several plausible fundamental-mode trial functions. One trial function which is frequently useful is a power series of the form

$$u(r) = b_1 r^3 + b_2 r^5 + \ldots , \tag{15}$$

where the b_j are adjustable constants. The first one or two terms in this trial function give a good indication of stability for models in which γ and $\log_{10} \rho$ are approximately constant over the inner 90 per cent of the star. However, in other situations they may give very misleading results (see, e.g., § III*b* and Appendix B of Meltzer and Thorne [1966]).

b) *Comparison of the Methods*

It cannot be overemphasized that Method 2-D (the variational method), although useful for theoretical analysis, in numerical calculations is more time-consuming, more money-consuming, and less reliable than Methods 2-A, 2-B, and 2-C. Although Method 2-D is satisfactory for models where γ and $\log_{10} \rho$ are approximately constant over the inner 90 per cent of the star, even with the simplest trial function, $u = r^3$, Method 2-D is no more convenient to use than Method 2-C. This is because in Method 2-C the eigenvalue equation can be solved right along with the equations of hydrostatic equilibrium in calculating a model. Thus we strongly recommend the use of Method 2-C, except in cases where Method 2-A or Method 2-B is made applicable by a previous knowledge of an entire sequence of equilibrium configurations.

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