ON THE NATURE OF THE HORIZONTAL BRANCH. I*†

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ABSTRACT

The correlation of certain horizontal-branch characteristics with observed metal abundance is discussed, and the hypothesis that stars survive the helium flash without mixing is re-examined. It is shown that metal content, particularly as it affects the strength of a hydrogen-burning shell, controls the structure of a double-energy-source model, leading qualitatively to the observed correlation. Models are constructed for two values of the envelope helium content, 10 per cent and 35 per cent by mass. Quantitative agreement at present suggests a preference for the latter figure, which may have cosmological or cosmogonical significance.

I. INTRODUCTION

The color-magnitude diagrams of globular clusters are among the most useful observational tools in the study of stellar evolution. During the past ten years there has been a growing interest in the systematics of the diagrams. Arp (1955, 1958) brought together data on seven globular clusters, and Sandage and Wallerstein (1960) published a table comparing the characteristics of sixteen clusters. One of the more intriguing results of these studies concerned an apparently strong correlation between the horizontal-branch properties of a cluster and the metal content of its members. The present paper presents a theoretical understanding of one form which this correlation takes.

The correlation noted by these authors was essentially the following. Counts of the numbers of stars on the red and blue sides of the RR Lyrae gap correlated with metal abundance. Clusters of high metal content were found to have a predominantly red horizontal branch, while clusters of low metal content had essentially blue horizontal branches. As a rough guide, high and low metal content are of the order of one tenth and one hundredth of the solar value, respectively.

Arp's earliest paper already suggested that the counts implied a decreasing density function toward the red end of the branch, while Sandage and Wallerstein referred to total number counts on both sides of the RR Lyrae gap. We would like to call attention to a description of the horizontal branch in terms of an actual gap which occurs there, a gap whose existence and length are correlated with metal abundance. Because we wish to emphasize this particular gap, we shall discard the phrase "RR Lyrae gap" for the remainder of this paper.

Clusters of high metal content possess short, densely populated horizontal branches continuous with the subgiant branch. An extreme example is 47 Tuc (Wildey 1961; Tifft 1963); others of this type include NGC 6171 (Sandage and Katem 1964) and NGC 6356 (Sandage and Wallerstein 1960). In clusters of intermediate metal content such as M3 (Johnson and Sandage 1956) and M5 (Arp 1962) a significant development occurs. The horizontal branch extends considerably into the blue and at the same time a small gap becomes apparent between the red end and the subgiant branch. For example, in M3 the gap is only $\sim$0.1 mag. long, at $B - V \sim 0.6$. As the metals become even weaker, the

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blue end of the horizontal branch extends a little further, while remaining roughly similar in shape to that of the intermediate group. However, the gap at the red end becomes much wider, until it extends from the subgiant branch to $B - V \sim 0.4$ or less for extreme cases, e.g., M2 and M92 (Arp 1955; see Arp 1962 and the references there for the transformations that convert $m_{pg} - m_{pv}$ to $B - V$).

The horizontal-branch gap described in the preceding paragraph is taken as the main observational fact to be explained. We note here that the correlation with metal content may not be perfect. The most outstanding discrepancy occurs for M13 (Arp 1955), which seems to be metal-rich and yet possesses a horizontal branch similar to those of the weak-metal clusters. The explanation may lie in the other parameters that we shall show to enter the problem.

There are as yet no firm theoretical conclusions concerning the nature of the horizontal-branch stars. Indeed, it is fair to say that theoretical results have failed to live up to the expectations of ten years ago. By 1955 it was already known that an explosive situation would accompany central helium ignition at the tip of the giant branch (Mestel 1952). Hoyle and Schwarzschild (1955) proposed that the horizontal branch represented a post-giant phase. It was thought that helium ignition would be followed by evolution to the blue. The dynamical phases of the helium flash were not examined in detail, but attempts were nevertheless made to follow the subsequent evolution. Haselgrove and Hoyle (1958) assumed that no mixing of core and envelope would occur during the helium flash. The assumption led to apparent failure to explain the horizontal branch. Their model remained very close to the giant branch, slowly ascending as the central helium was consumed.

Following the apparent failure of the non-mixed models to explain the horizontal branch, it was suggested that stars might indeed totally mix during the violent helium ignition. Were this the case, a fairly rapid transition would occur to a helium-rich main-sequence position. The horizontal branch might then represent subsequent evolution to the red, a helium-rich analogue of the conventional migration from the main sequence to the subgiant branch.

Such a scheme, as a general explanation of the horizontal branch, has a rather obvious drawback. There is a complete absence of stars coming from the helium-rich main-sequence area to join the extremely stubby horizontal branch of 47 Tuc. Similar, though admittedly weaker, objections hold for the other clusters of the strong metal group. The objection is strengthened by further general considerations. One would expect a dense concentration of stars to occur at the helium-rich main sequence in all the clusters. Completely mixed helium-rich models still spend the major part of their remaining lives in the new main-sequence phase (Larson 1965). Thus the horizontal branch proper should contain only a small fraction of the number of stars on the helium-rich main sequence. This is contrary to observation. The absence of this effect presents difficulties for the tentative evolutionary scheme which Woolf (1964) proposed for M3. Observations of the differing spatial distributions of red giants and blue and yellow horizontal branch stars are used as evidence that mixing occurs initially to form the blue stars, followed by mass loss in the RR Lyrae region. M3 is distinguished, however, by a remarkably even distribution with color index of its horizontal-branch members. It would seem necessary to require complete mixing to be rare, and for selective mixing to occur. Stars left with a judiciously reduced helium core could be made equivalent to those in the conventional transition stages of slightly helium-enriched stars. However, the necessity to explain the apparently sharp cutoff to the amount of mixing must be considered a drawback in this scheme.

Schwarzschild and his collaborators (Schwarzschild and Selberg 1962; Schwarzschild and Härm 1962; Härm and Schwarzschild 1964) have attempted to evolve models up the giant branch and through the helium flash in order to determine whether mixing of the material will occur. These attempts have met with considerable computational difficulties. With simplifying assumptions in the equations, the answer to the mixing question appears marginally negative. No definitive answer is yet available.

The approach which has been adopted in this paper is as follows. We believe there are severe observational difficulties standing in the way of the complete mixing hypothesis. Accordingly, we have re-examined the hypothesis that no mixing occurs. A survey is made of the positions in the H-R diagram to which a model will move after ignition of the previously degenerate helium core. The survey shows how the correlation of horizontal-branch gap with metal content can arise. It seems likely that the horizontal branches as observed are indeed evidence for the absence of complete mixing during the helium flash. Rough estimates of evolutionary developments are then made, leading to a preliminary theory of the horizontal branch.

II. MODEL CONSTRUCTION

As indicated in the previous section it will be assumed that no mixing occurs during the helium flash and that the only effect of the flash is ignition of the helium core. Carbon produced during the flash will be ignored. Härm and Schwarzschild (1964) have shown that only some 5 per cent of the central helium is transformed into carbon during the dynamical stages of the flash. Accordingly our models will consist of a helium core, of mass determined by a preceding giant stage, surrounded by a hydrogen-rich envelope. For simplicity the change in composition will be taken as discontinuous. This will in any event be a close approximation to the situation reigning at the top of the giant branch (Hayashi, Hōshi, and Sugimoto 1962).

The only a priori assumption built into the structure will be that the temperature in the core is sufficiently high to maintain helium burning by the $3\alpha \rightarrow C^{12}$ process. Solution of the stellar-structure equations then determines the remaining features of the models. In the mass range to be studied, it turns out that there is a hydrogen shell source, and that the helium core consists of a central convecting region surrounded by a radiative zone. Relevant details of the model construction follow.

A. THE EQUATIONS

The standard equations of quasi-static equilibrium were employed in the form described elsewhere (Faulkner 1965). The surface boundary condition was replaced by conditions on the remaining variables at $T = 10^5\,^\circ$ K, as given by Faulkner, Griffiths, and Hoyle (1965*a*).

B. THE PHYSICAL ROUTINES

1. *The Equation of State*

It was assumed that ionization would be complete for $T > 10^5\,^\circ$ K, so that pressure was taken as the sum of radiative and perfect-gas contributions. The perfect-gas assumption was later checked against the models. It was found that electron degeneracy could contribute a few per cent to the total pressure at the centers of the smallest core masses investigated. For completeness, subsequent evolutionary work should allow for this possibility.

2. *The Opacity*

Simple formulae are fitted to the results of Keller and Meyerott (1955) for the bound-free and free-free contributions. (In the present work, conduction can be ignored.) The formulae are similar to those of Haselgrove and Hoyle (1959) but take on a slightly less involved form. The electron-scattering opacity is now taken to be 0.20 $(1 + X)$. The former value, 0.19 $(1 + X)$, has been noted by several authors (references are given, e.g., by Cox 1964) to arise from an incorrect use of stimulated emission in calculating the Rosseland mean. For the central temperatures here ($\sim 1.2 \times 10^8\,^\circ$ K), the relativistic correction to the electron scattering is about to become important (Sampson 1959).

However, central convection is so strong that the temperature drops to $\sim 8 \times 10^7$ ° K before radiation can take over. Thus the correction is not in practice too important for these models, although it may be included in future work when the present models are evolved.

The expressions used were therefore:

$$\begin{aligned} 0.1 \leq T_6 \leq 10\,, \quad & \kappa = 0.20\,(1+X) + A\rho\,(1+X)\,, \\ 10 \leq T_6 \leq 20\,, \quad & \kappa = 0.20\,(1+X) + A\rho\,(1+X)(2.0 - 0.1\,T_6) \\ & \qquad + B\rho\,(0.1\,T_6 - 1.0)\,, \\ 20 \leq T_6 \quad\;\;, \quad & \kappa = 0.20\,(1+X) + B\rho\,, \end{aligned} \tag{1a}$$

where

$$A = \frac{65000Z}{T_6^2 + 2.5T_6^4}\left[1 + \min\left(1, \frac{10\rho(1+X)}{T_6^3}\right)\right]^{-1} + \frac{165(X+Y)}{T_6^4}\left[1 + \min\left(1, \frac{3\rho(1+X)}{T_6^3}\right)\right]^{-1}, \tag{1b}$$

$$B = \frac{225 - 155X - 190Y}{T_6^{3.5}}.$$

Here T_6 is the temperature measured in units of 10^6 ° K.

3. *The Nuclear-Energy Generation*

Helium burning.—Using the data of Fowler and Hoyle (1964), the energy produced by the $3\alpha \rightarrow C^{12}$ process is given by

$$\epsilon_{3\alpha} = \frac{\rho^2 Y^3}{T_6^3} \exp\left(40.34 - \frac{4320}{T_6}\right). \tag{2}$$

Hydrogen burning.—In the current models hydrogen will be burning, if at all, in a shell source. It turns out that in the cases of interest the temperature there exceeds 2×10^7 ° K. Thus, for a reasonable CNO concentration, the CNO bi-cycle will be dominant and (Fowler and Hoyle 1964)

$$\epsilon_{\mathrm{CNO}} = 5.98\rho T_6^{-2/3} X Z_{\mathrm{CNO}} \exp\left(62.41 - \frac{152.3}{T_6^{1/3}}\right). \tag{3}$$

Energy generation in the *pp* chain is also computed, since it may be expected to grow in importance as Z_{CNO} decreases. In fact, it never dominates (in the range investigated here), as T increases in the shell when Z_{CNO} decreases. At these high temperatures one can consider an equilibrium abundance of He^3 to be built up instantaneously.

We define

$$\begin{aligned} S_{11} &= \exp\,(12.8 - 33.8/T_6^{1/3})\,, \qquad S_{33} = \exp\,(68.1 - 122.8/T_6^{1/3})\,, \\ S_{34} &= \exp\,(59.4 - 128.3/T_6^{1/3})\,, \end{aligned} \tag{4}$$

and use

$$\epsilon_{pp} = \rho T_6^{-2/3}\left[\!\left[3.20X^2 S_{11} + 3.08\,Y_{3e}^2 S_{33} + Y_{3e} Y_4 S_{34} \times \left\{11.92 + \frac{6.29}{1 + [X/(X+1)](T_6/21)^{12}}\right\}\right]\!\right]. \tag{5}$$

Equation (5) represents an approximation to the work of Parker, Bahcall, and Fowler (1964). It is here assumed that deuterium is burned only through the reaction $H^2(p,\gamma)He^3$. Parker *et al.* show that the competing reaction $H^2(He^3,p)He^4$ has a relative probability lower by a factor of $\lesssim 10^{-4}$ for $T_6 \gtrsim 20$. The use of the former mode alone reduces the cubic equation for Y_{3e}, the equilibrium abundance of He^3, to a quadratic. The solution is, in our notation,

$$Y_{3e} = -\frac{Y_4 S_{34}}{S_{33}} + \left[\left(\frac{Y_4 S_{34}}{S_{33}}\right)^2 + \frac{X^2 S_{11}}{S_{33}}\right]^{1/2}. \quad (6)$$

The final terms in equation (5) are a representation of the relative rates of Be^7 consumption by electron or proton capture. The latter mode will in fact dominate for most of the present models.

C. THE INTEGRATIONS

For a star of total mass M_s, the search for a solution begins by choosing four basic parameters u_i (e.g., P_c, T_c, L_s, T_e). Inward and outward integrations are made to a suitable point in the star. It is convenient to take the core-envelope interface as the fitting point. If x_κ ($\kappa = 1$–4) are the values of the dependent variables from an inward integration and y_κ similar values from an outward integration, it will in general be the case that $y_\kappa \neq x_\kappa$.

Let

$$z_\kappa = y_\kappa / x_\kappa \quad \text{and} \quad z_\kappa^{1/2} - z_\kappa^{-1/2} = \Delta_\kappa .$$

Then, if we vary u_i and use a summation convention for i, a Newton-Raphson iteration scheme requires the solution of

$$\tfrac{1}{2}(z_\kappa^{-1/2} + z_\kappa^{-3/2})\left(\frac{1}{x_\kappa}\frac{\partial y_\kappa}{\partial u_i} - \frac{y_\kappa}{x_\kappa^2}\frac{\partial x_\kappa}{\partial u_i}\right) du_i = -\Delta_\kappa, \quad \kappa = 1-4,$$

that is,

$$\left(\frac{1}{y_\kappa}\frac{\partial y_\kappa}{\partial u_i} - \frac{1}{x_\kappa}\frac{\partial x_\kappa}{\partial u_i}\right) du_i = \frac{2(x_\kappa - y_\kappa)}{x_\kappa + y_\kappa}, \quad \kappa = 1-4. \quad (7)$$

The symmetric set of equations (7) was found to lead to quite good convergence in most cases. The partial derivatives are evaluated by performing auxiliary integrations, varying the relevant u_i by some small fraction of itself. When the quantity $\Sigma_{\kappa=1}^4 |\Delta_\kappa|$ is reduced to some preassigned value, the equations are deemed to be solved.

The complete problem was coded for the IBM 7094 at the California Institute of Technology. In practice a suggested set of trial parameters for the first model was fed into the computer, together with the relevant physical data. At any subsequent stage the quantities M_s, M_c, Z, Z_{CNO}, and X_e could be varied. Projection from previously found parameters gave new starting values, enabling one to map the positions of a series of models in the H-R diagram.

III. MODEL PARAMETERS AND RESULTS

Models were constructed for two values of the envelope hydrogen concentration, $X_e = 0.90$ and $X_e = 0.65$. The former value, 0.90, has been a popular choice for typical Population II models for some time. However, there appears to be accumulating evidence that the older estimates for the composition may be too rich in hydrogen (O'Dell, Peimbert, and Kinman 1964). O'Dell *et al.* found K648, a planetary nebula in M15, to have $N_{He}/N_H = 0.18$, implying a hydrogen content of less than 0.60. This is, of course, a highly evolved object and might therefore be expected to possess an enhanced helium content. Nevertheless it is as well to realize that there is no direct observational evidence for a hydrogen content greater than ~ 0.80 in any of the clusters.[1] In addition, Christy (1965), in his attempts to explain the light-curves of RR Lyrae variables, prefers a hydrogen concentration ~ 0.70. Coupling this with the result of O'Dell *et al.* that there is evidence for a fairly widespread uniformity in $N_{\rm He}/N_{\rm H}$ at a ratio $\gtrsim 0.14$, we obtain our second value $X_e = 0.65$. This should in any event be an interesting case since it may shed some light on what to expect in Population I clusters such as M67 (Eggen and Sandage 1964). Such a value for X_e might also have cosmological or cosmogonical significance. It is close to the value necessarily found in an explosive ("big bang") universe, or in a galaxy which undergoes an early "massive object" phase, as a calculation for Hoyle and Tayler (1964) revealed.

The composition parameters are completed by specifying the metal content (helium then making up the remainder). A large range in $Z_{\rm CNO}$ was used, $Z_{\rm CNO} = 10^{-5}$, 10^{-4}, 10^{-3}, and 10^{-2}. The total heavy-element content, Z, was given by $Z/Z_{\rm CNO} = 2$ in the work reported here. For $Z_{\rm CNO} \lesssim 10^{-3}$ the ratio 2 for $Z/Z_{\rm CNO}$ is not critical. The opacity formula becomes insensitive to Z for this range, and the only other quantity which might be sensibly affected by changes in Z would be the surface boundary condition. However, most of the surfaces are radiative and therefore insensitive to the precise boundary condition. Even for those which are convective, a change in Z by a factor of 2 has little effect—a small adjustment in surface temperature will occur (Faulkner, Griffiths, and Hoyle 1965*a*). The results will be insensitive to the ratio $Z/Z_{\rm CNO}$ in the range $1 < Z/Z_{\rm CNO} \lesssim 4$. This conclusion was checked by computing explicit models with changes in $Z/Z_{\rm CNO}$, and was fully confirmed. Thus for $Z_{\rm CNO} \lesssim 10^{-3}$ our models will continue to be satisfactory should neon turn out to be as low in abundance as Gaustad (1964) has argued. From cosmic-ray data, Gaustad would obtain $Z/Z_{\rm CNO} \sim 1.2$. For $Z_{\rm CNO} = 10^{-2}$ our models will be sensitive to the ratio $Z/Z_{\rm CNO}$—but if the ratio is indeed to be revised in a downward direction, the range of variation in the models will still be fairly small as we shall see.

We now turn our attention to the core masses to be employed. Ideally, given the total mass and initial composition of a star, evolution up the giant branch should determine the core mass at the onset of the helium flash. As investigations of this phase are limited in number, we are forced to make estimates of the relevant core masses. Advantage can be taken of one simplifying feature. It has been shown (Faulkner, Griffiths, and Hoyle 1965*b*) that conditions at the composition discontinuity for given composition and core mass are virtually independent of the total mass of the model. Yet it is clearly these conditions which help determine the onset of the flash. We therefore take the core mass to be a function of composition only.

The argument is supported by examination of the models obtained by Härm and Schwarzschild (1964; here and subsequently, we refer to the models for stars A, B, and C from this paper). Models A and B have the same composition, $X_e = 0.90$, $Z = 10^{-3}$, but have masses 1.3 and 1.0 $M_\odot$, respectively. In spite of the 30 per cent difference in total mass, the core masses (0.596 $M_\odot$, 0.587 $M_\odot$) differ by only 1.5 per cent. A larger difference is to be expected by changing the metal content, since this affects the temperature of the shell-burning source. Model C differs from model A by having $Z = 10^{-2}$; the core mass, at 0.561 $M_\odot$ is ~ 6 per cent lower. The core mass for model A is sensibly larger than that reported in the previous investigation (Schwarzschild and Härm 1962), 0.549 $M_\odot$. The increase, amounting to some 9 per cent, seems rather large. It may simply come from the improved mathematical technique and the complete models obtained in the later work.

For the lower value of X_e which we have adopted, the information is even more scanty. Hayashi *et al.* (1962) suggested for $X_e = 0.61$, $Z = 0.02$, a core mass of 0.42 $M_\odot$, while

[1] *Note added in proof:* This remark may be too strong in view of a recent discussion (L. Searle and A. W. Rodgers, *Ap. J.*, **143**, 809, 1966) of NGC 6397, one member of which appears to be low in helium by a factor in the range 5–10.

their estimate for $X_e = 0.90$, $Z = 10^{-3}$, was 0.53 $M_\odot$. By comparison with Härm and Schwarzschild's work, it seems reasonable that for $X_e = 0.65$ and $Z_{\rm CNO} \lesssim 10^{-2}$ a core mass $\sim$0.45 $M_\odot$ may be relevant.

In view of the uncertainties, models were computed for two core mass values for each choice of X_e. It is hoped that the values chosen will cover a reasonable range of possibilities. For $X_e = 0.90$ core masses of 0.6 and 0.5 $M_\odot$ were used, while for $X_e = 0.65$ values of 0.5 and 0.4 $M_\odot$ were used.

The final sequences of models computed were therefore the following:

(i) $X_e = 0.90$; $Z_{\rm CNO} = 0.5\ Z = 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}$; $M_c = 0.6\ M_\odot, 0.5\ M_\odot$;
$M_s \sim 1.2\ M_c$ to $1.25\ M_\odot$.

(ii) $X_e = 0.65$; $Z_{\rm CNO} = 0.5\ Z = 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}$; $M_c = 0.5\ M_\odot, 0.4\ M_\odot$;
$M_s \sim 1.2\ M_c$ to $1.25\ M_\odot$.

The results of observational importance are presented graphically in Figures 1 and 2. Some mathematical details of the models of 1.25 $M_\odot$ are given in Tables 1 and 2 (num-

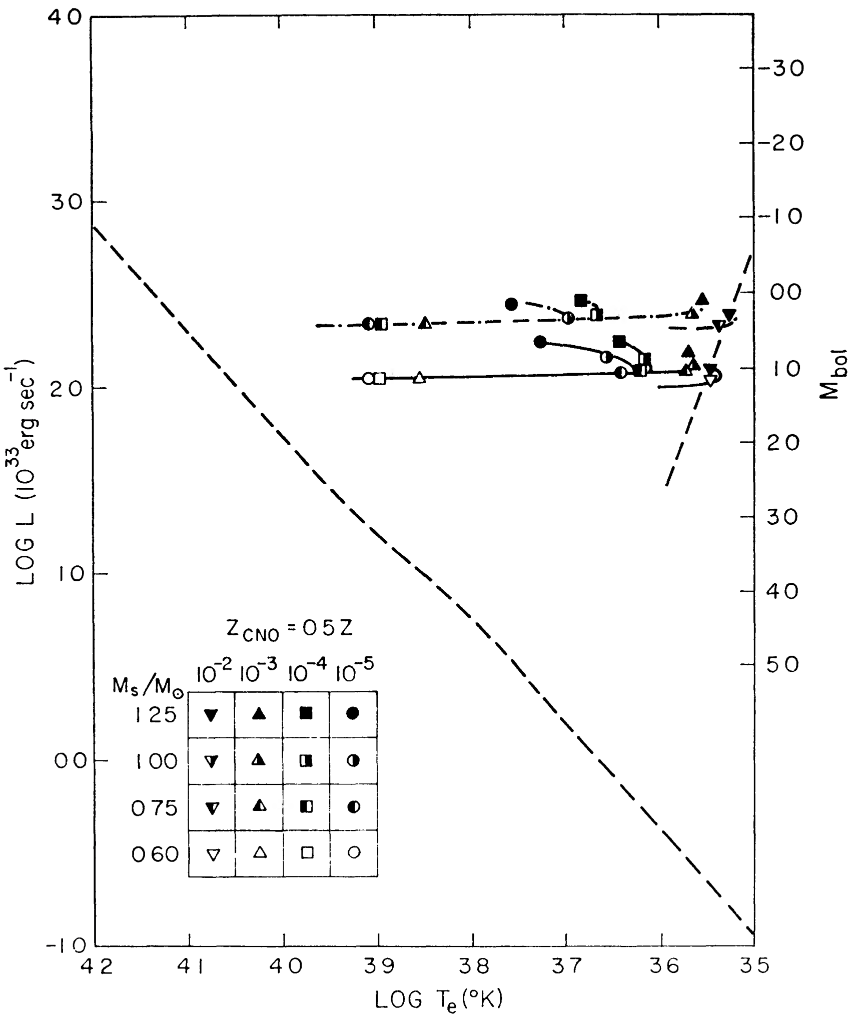

FIG 1.—Models for $X_e = 0.90$. The symbols, which give the total mass, are joined by lines denoting the value of the core mass, 0.6 $M_\odot$ (*dash-dot line*) and 0 5 $M_\odot$ (*solid line*). The main sequence and giant branch are taken from Faulkner (1964).

bered to correspond with the relevant figure), while normal points for all the curves are given in Table 3. For purposes of orientation, main-sequence and giant-branch models are also plotted, for $X = 0.65$, $Z_{\rm CNO} = 10^{-2}$. These were taken from the author's thesis (Faulkner 1964). Main-sequence models computed by the present program acted as a check on the formulae deduced by Faulkner *et al.* (1965*a*) for the surface boundary condition. The earlier main-sequence models used an explicit surface program for obtaining conditions at $T = 10^5$ ° K. The difference between the two main sequences was hardly discernible, amounting to 0.002 at most in log T_e.

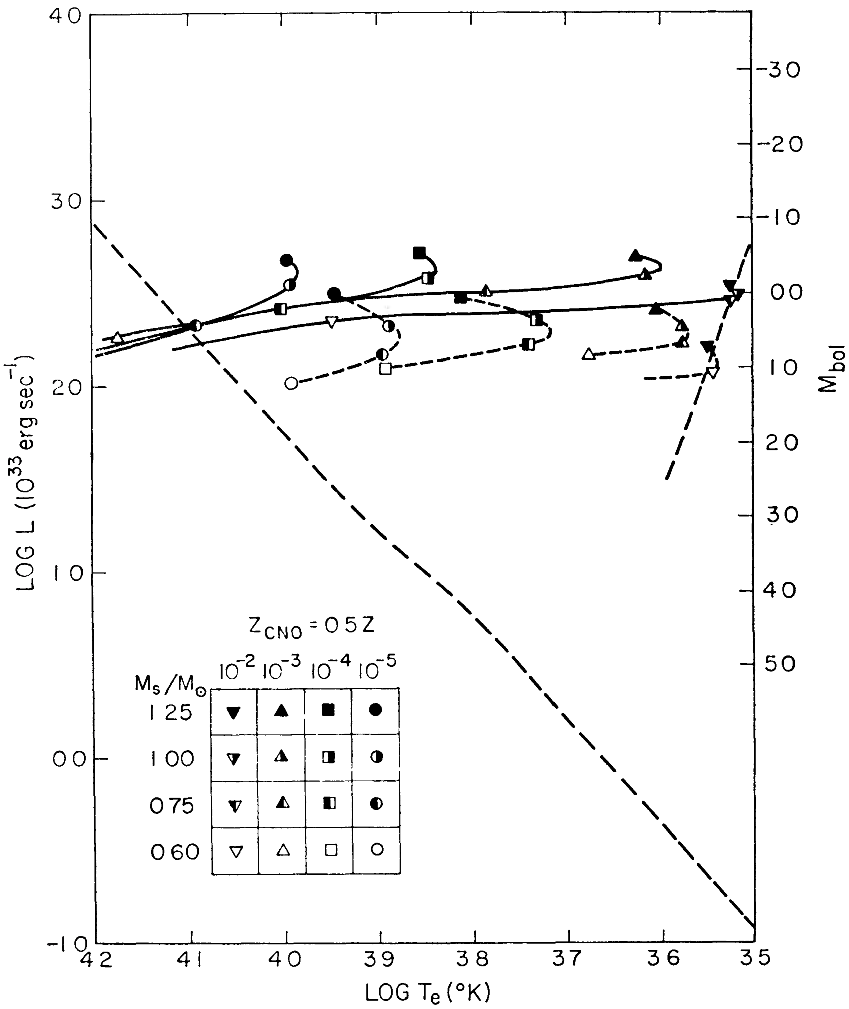

FIG. 2.—Models for $X_e = 0\ 65$. The symbols, which give the total mass, are joined by lines denoting the value of the core mass, 0.5 $M_\odot$ (*solid line*) and 0.4 $M_\odot$ (*dashed line*). The main sequence and giant branch are taken from Faulkner (1964).

The outstanding feature of the present results is the existence of a critical turning point as the total mass is varied. The critical point marks the lowest value of T_e which may be attained by any models with a given composition and core mass. The critical mass at which the turning point occurs is approximately twice the core mass.

Fixing attention on one value of X_e, it is seen that the precise position of the turning point is an extremely sensitive function of the metal composition. Thus, taking the upper series of curves in Figure 2, which have $M_c = 0.5\ M_\odot$, the critical point is at ~10000°, ~7000°, ~4000°, and ~3300° K for $Z_{\rm CNO} = 10^{-5}$, 10^{-4}, 10^{-3}, and 10^{-2}, respectively.

The critical point for $Z_{\text{CNO}} = 10^{-2}$ would be even further to the right but for the surface boundary condition which establishes a forbidden region (Hayashi and Hōshi 1961; Faulkner *et al.* 1965*a*, *b*); the giant branch runs very close to the position where the whole stellar envelope would be convecting, which gives a physical limit to the mean polytropic index. This forces all the models with $Z_{\text{CNO}} = 10^{-2}$ and $0.70\ M_{\odot} \lesssim M_s \lesssim 1.25\ M_{\odot}$ to occupy an extremely small region of the H-R diagram. The effect of Z_{CNO} on the critical point is also present in Figure 1, but not nearly to the same extent.

TABLE 1*

DETAILS OF MODELS WITH $M_s = 1.25\ M_{\odot}$, $X_e = 0.90$

	$Z_{\text{CNO}}=0\ 5Z$							
	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
	$M_c/M_{\odot}=0\ 6$				$M_c/M_{\odot}=0\ 5$			
log P_c ..	19 9305	19 9358	19 9435	19 9474	20 0337	20 0410	20 0492	20 0532
log T_c .	8 0975	8 0968	8 0959	8 0958	8 0848	8 0839	8 0829	8 0827
log L_{core}	2 3895	2 3756	2 3545	2 3329	2 1165	2 0969	2 0736	2 0470
log T_j ..	7 4827	7 4675	7 4317	7 3618	7 4829	7 4612	7 4186	7 3459
log ρ_{je}	2 0850	2 0231	1 9082	1 7324	2 2164	2 1368	2 0201	1 8441
log R_j	9 7120	9 7182	9 7297	9 7476	9 6458	9 6538	9 6655	9 6840
log L_s	2 4511	2 4662	2 4636	2 3681	2 2495	2 2529	2 2165	2 0892
log T_e .	3 7554	3 6820	3 5543	3 5242	3 7260	3 6412	3 5697	3 5423

* The subscript c denotes the center, j the jump in composition (je being the external side) Units are c g s except for L_{core} and L_s, where the unit is 10^{33} ergs sec^{-1} The observational quantities log L_s and log T_e for these and other models with X_e = 0 90 are plotted in Fig 1.

TABLE 2*

DETAILS OF MODELS WITH $M_s = 1.25\ M_{\odot}$, $X_e = 0\ 65$

	$Z_{\text{CNO}}=0\ 5Z$							
	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
	$M_c/M_{\odot}=0\ 5$				$M_c/M_{\odot}=0\ 4$			
log P_c .	20 0344	20 0489	20 0567	20 0535	20 1720	20 1840	20 1895	20 1850
log T_c	8 0845	8 0828	8 0820	8 0827	8 0679	8 0665	8 0659	8 0665
log L_{core}	2 1105	2 0733	2 0517	2 0463	1 7358	1 7040	1 6874	1 6798
log T_j ..	7 5725	7 5324	7 4865	7.4242	7 5628	7 5205	7 4703	7 4030
log ρ_{je} ..	2 2778	2 1125	1 9694	1 8948	2 3533	2 1992	2 0791	2 0277
log R_j ..	9 6511	9 6688	9 6829	9 6884	9 5800	9 5962	9 6070	9 6099
log L_s	2 6867	2 7153	2 7043	2 5403	2 4983	2 4974	2 4382	2 2124
log T_e .	3 9920	3 8524	3 6223	3 5263	3 9440	3 8139	3 6048	3 5498

* Subscripts as in Table 1 The observational quantities log L_s and log T_e for these and other models with X_e = 0 65 are plotted in Fig 2

TABLE 3

NORMAL POINTS FOR FIGURES 1 AND 2

LOG L (10^{33} ergs sec^{-1})	LOG T_e — Z_{CNO} 10^{-5}	10^{-4}	10^{-3}	10^{-2}
	$X_e=0\ 90,\ M_c/M_\odot=0\ 60$			
2 33	...	...	...	3 588
2 35	3 920	3 920	3 920	3 518
2 37	3 750	3 750	3 750	3 525
2 40	3 693	3 664	3 590	
2 45	3 755	3 673	3 554	
	$X_e=0\ 90,\ M_c/M_\odot=0\ 50$			
2 04	...	...	...	3 553
2 05	>3 900	>3 900	>3 900	3 538
2 09	3 650	3 650	3 630	3 542
2 10	3 631	3 628	3 586	
2 15	3 630	3 612	3 562	
2 20	3 667	3 620	3 567	
2 25	3 726	3 638		
	$X_e=0\ 65,\ M_c/M_\odot=0\ 50$			
2 35	4 075	4 075	4 090	3 940
2 40	4 050	4 025	4 030	3 800
2 45	4 024	3 970	3 955	3 560
2 50	4 003	3 920	3 845	3 521
2 55	3 991	3 873	3 728	3 528
2 60	3 984	3 842	3 628	
2 65	3 985	3 837	3 596	
2 70	3 999	3 847	3 621	
	$X_e=0\ 65,\ M_c/M_\odot=0\ 40$			
2 02	3 990	..		3 720
2 05	3 965	...		3 580
2 10	3 930	3 893		3 539
2 15	3 904	3 829	...	3 541
2 20	3 885	3 770	3 625	3 548
2 25	3 876	3 726	3 575	
2 30	3 877	3 714	3 571	
2 35	3 888	3 723	3 577	
2 40	3 904	3 746	3 589	
2 45	3 923	3 781	3 614	
2 50	3 944	3 820		..

The result obtained in Figure 2 is already extremely suggestive. According to our models, non-mixed post–helium-flash stars with a given envelope hydrogen concentration must move to a position in the H-R diagram with a surface temperature greater than a certain critical value. The critical temperature is a function of metal content only (the appropriate core mass being also a function of the metal content). The implications of this result will be discussed in § V. We merely note here that, whatever the total mass, a post–helium-flash gap in the horizontal branch is predicted by our models, a gap of length correlated with metal deficiency. The results obtained in Figure 2 may be fortuitous, yet there is a remarkable agreement with the observations for M2 and M92. These clusters are metal-deficient by a factor $\sim$200 (i.e., $Z_{\text{CNO}} \sim 5 \times 10^{-5}$ to $\sim 10^{-4}$) and their gaps extend to $B - V \sim 0.3$–0.4 mag., i.e., $T_e \sim 6000°$–$7000°$ K. This is taken as a preliminary confirmation of our theoretical results.

The effect of metal content is so striking that it seems worthwhile making an excursion in order to understand how it comes about. The next section is therefore devoted to an examination of the reasons for this remarkable effect, and its relation to what was previously known.

IV. UNDERSTANDING THE RESULTS

A. THE INFLUENCE OF METAL CONTENT ON THE STRUCTURE OF A STAR HAVING A SHELL ENERGY SOURCE

Metal content generally affects two physical properties of stellar material, the opacity and the energy generation in the CNO bi-cycle. In the present context, the former effect is unimportant for $Z \lesssim 10^{-3}$. However, considerable control is exerted over the structure of the models via the shell source of energy. Once the nature of this control is appreciated, the results we have obtained are easily understood. We begin with some general remarks on stellar structure.

Consider first homogeneous stars in which energy generation is naturally concentrated toward the center. The virial theorem demands a relation between the internal density and temperature, without actually fixing either. A choice for T_c implies a value for ρ_c and thus, by integration, the energy flux in order of magnitude, provided that a reasonable fraction of the star is in radiative equilibrium. The requirement that the source of the flux passing through the radiative region be nuclear energy determines which choice of T_c is correct. The radius scale is set to a large extent by the associated value of ρ_c.

This apparently excellent physical situation leads somewhat paradoxically to mathematical divergencies from the point of view of the model calculator. Suppose an inward integration commences with slightly incorrect surface values. The march of variables inward will be slightly erroneous, but serious consequences arise only as the mass tends to zero, where we require both luminosity and radius to vanish simultaneously. Unless the temperature and density have precisely the required balance, the energy-generation rate with its usually high-power dependence leads to the luminosity greatly overshooting or undershooting the required zero. Similar remarks hold to a lesser extent for the radius. For small values of the mass, these two variables are extremely sensitive functions of the surface boundary values. On the other hand, pressure and temperature, which do not have to satisfy such stringent conditions, are much less sensitive.

Conversely, the divergent tendencies are reversed for outward integrations. Pressure and temperature drop to small fractions of their central values, and the required total mass may be overshot, or may never be reached, depending on small changes in the central conditions. Physically, it is clearly immaterial to the center whether $\rho_{\text{surface}}/\rho_{\text{center}} = 10^{-4}$ or 10^{-10}, while to the photosphere it is vital.

The above considerations can now be applied to the present models. The composition jump (denoted by j) is sufficiently removed from the center that P_j and T_j are already sensitive functions of P_c and T_c, while L_j and R_j are closely determined. The structure of the helium core is thus almost independent of the hydrogen-rich exterior. However, the possibility now arises that the inward divergence in the equations can occur at the composition jump, the possible site of a hydrogen-burning shell. If (as is the case for our present models) radiative equilibrium in the envelope demands more flux than the core can provide, the shell source is in a position to dominate the structure.

We can imagine reducing the values of Z_{CNO} and Z separately. The reduction of Z_{CNO} decreases the shell contribution to the flux. As the envelope opacity is unaltered, the necessary extra flux is supplied by raising the shell temperature and density. This can be done without affecting the center significantly in any way; the sensitivity of these shell variables to the central conditions means that they can themselves be adjusted to meet the energy requirements of the radiative envelope. Thus while two models with differing Z_{CNO} have essentially the same core, the density is higher in the envelope of that with the lower value of Z_{CNO}. A reduction in Z (but not Z_{CNO}) where the envelope opacity is affected has similar consequences. This time we may expect an actual increase in the total luminosity, which again demands increased temperatures and densities in the shell and subsequently in the envelope. The qualitative argument is fully confirmed by explicit models in which Z and Z_{CNO} have been independently reduced.

To sum up, the core and the envelope are largely independent of one another. Reductions in Z_{CNO} or Z lead to higher densities in the envelope. The remaining mass may be fitted into the model in a smaller radius. (Though the situations are not strictly comparable, it is tempting to ascribe some of the difference between the main-sequence–subgiant transitions of Population I and II to the same effects.) Thus a shell source of energy can exert a strong controlling effect on the structure of a star through its ability to regulate the physical conditions at the interface.

The effects described above are clearly exhibited in Figures 1 and 2. To enable a more detailed side-by-side comparison to be made, we have tabulated some of the relevant quantities for all our models of mass 1.25 $M_\odot$ in Tables 1 ($X_e = 0.90$) and 2 ($X_e = 0.65$).

It is worthwhile at this point to compare the Härm and Schwarzschild model A in the appropriate evolutionary stage (No. 9) and our model which is closest to this, $M_s = 1.25\ M_\odot$, $X_e = 0.90$, and $Z_{\text{CNO}} = 10^{-3}$. The differences, particularly in the total luminosity, are greater than can be accounted for by the slight mass difference or the small differences in metal content, and appear to be due to major differences in two of the physical laws used in model A. Assuming our equation (2) to be the correct expression for the triple-alpha reaction, we find the power-law approximation in model A to be too low by a factor $\sim$20 at the temperatures of interest. Ordinarily this would have very little effect on the total luminosity of a star with only a central energy source. Higher central temperature and density would contrive to keep the flux level approximately the same. This indeed happens in the core of model A—log T_c is incremented by $\sim$0.036, log P_c by $\sim$0.137, i.e., $\Delta \log \rho_c / \Delta \log T_c \sim 3$, as required by the virial theorem. The core luminosity is within 1 per cent or so of ours (incidentally confirming the argument that the structure of the helium core is determined by its own outer radiative equilibrium and the energy-generating law). Model A is thus prepared at the center to have a higher run of temperature than our model. It might still be possible to avoid a higher shell source temperature because of the latter's sensitivity to the precise central conditions. Here the second major difference plays its part. The opacity formula used in model A severely underestimates the lower temperature opacities. Our formulae, though not highly accurate, nevertheless represent the lower temperature opacities of Cox and Stewart (1964) more closely. The opacities which we obtain have been checked against the values for the Kippenhahn II mixture. For the densities we have at $T_6 = 10.0$, 1.0, and 0.1, our opacities are too low by $\sim$1.06, $\sim$1.13, and $\sim$6. For the same densities and temperatures, the Härm and Schwarzschild opacities are too low by $\sim$1.3, $\sim$2.9, and $\sim$80, respectively. Thus over a large range of temperature, the envelope opacities in model A are low compared to both the Cox-Stewart opacities and to ours. A much larger flux contribution from the shell source is required, and the shell source temperature is $T_6 \sim 30$ rather than ~ 27 as in our model. The considerable difference in total luminosity is thus explained.

B. MOLECULAR WEIGHT DISCONTINUITIES AND THE CRITICAL POINT PHENOMENON

The left-right displacements of the sequences in Figures 1 and 2 are explained by the argument above. The shapes of the sequences and, in particular, why there are critical turning points must be otherwise explained. The essential point is that as we move along a particular sequence in the direction of increasing total mass, an increasingly larger outer fraction of the model has a lower molecular weight than that in the core. The problem of replacing a variable part of a model of fixed total mass by material of different molecular weight has already been discussed in the literature.

Discontinuities in molecular weight as a way of obtaining increased radii were originally proposed by Öpik (1938), who obtained modest extensions by changes in the deep interior. A discontinuous decrease of μ in the outer layers was used by Hoyle and Lyttleton (1942, 1949) to explain the large radii of red giants. The triple-alpha reaction had yet to be discovered, and the models consisted of helium-enriched hydrogen-burning cores, surrounded by inert hydrogen envelopes. An interesting physical argument was produced for the extended radii—material outside the discontinuity is less tightly bound, as thermal energy per unit mass is inversely proportional to mean molecular weight. The maximum extension occurred when the replaced mass was approximately one-half of the total. This result was also discussed in a systematic way by the Bondis (Bondi 1950; Bondi and Bondi 1950). The result which we have obtained for any fixed metal content is a simple consequence of combining the above effect with models of the pure helium main sequence (Cox and Giuli 1961).

An approximate method for replacing a small outer part of the mass by a hydrogen-rich envelope was produced by Cox and Salpeter (1961). However, their approximations break down before reaching the region that is of interest to us here—in particular the assumption that there is negligible energy production in the shell.

It is of interest that the present models can only be placed at the giant end of the horizontal branch because of two things—higher opacity in the envelope and the nuclear-reaction rate of the CNO bi-cycle. Models with $Z_{\text{CNO}} = 0$ rely on a pp shell source, and for small enough Z the critical point barely reaches the main sequence. For masses not far removed from the critical mass, the models are on, or to the left of, the main sequence. As masses increase above the present range the temperature in the shell rises to an extent that a new mechanism, the NeNa cycle, can be brought into operation. Such models could arise if the CNO elements in the vicinity of the shell had been scoured out, as proposed by Fowler, Burbidge, Burbidge, and Hoyle (1965). Is it merely a coincidence that such models occupy a region of the H-R diagram close to that of the main-sequence A stars?

V. OBSERVATIONAL COMPARISON AND IMPLICATIONS

We have already noted the suggestive agreement between the critical points for the metal-weak models of Figure 2 and the blue sides of the horizontal-branch gap in metal-weak clusters. Whether this property is maintained during subsequent evolution will be crucial to the proposed identification. Should evolution carry the models steadily to the red, the property will be destroyed. Evolution to the blue would, however, maintain it through newly arriving stars, while at the same time producing the blue ends of the horizontal branch.

It might be thought that the favored value of X_e, 0.65, is too low for Population II stars. If $X_e = 0.90$ is to be preferred and our explanation for the gap retained, one or both of two statements must hold. The envelope opacity must be too high, or the CNO bi-cycle rate must be too large. As our opacities are, if anything, low in the important regions by $\sim$5–15 per cent compared to the Cox-Stewart values, the former seems unlikely. The few uncertainties remaining in the CNO bi-cycle do not seem capable of altering the energy-generation rate by very large factors (Caughlan and Fowler 1962). Indeed, the argument could be turned around to say that the present rate is certainly preferable to that used by Haselgrove and Hoyle (1958), which relied upon a resonance in $N^{14}(p,\gamma)O^{15}$. The much higher energy-generation rate meant that their model with $Z = 10^{-3}$ was as close to the forbidden region as one of our models would be with $Z \sim 10^{-1}$. Even for the low X_e value, it would have required Z_{CNO} lower by $\sim 10^2$ than the observed estimates in order for the gaps to exist. Thus an opportunity for astrophysical prediction of the character of nuclear energy levels may well have been missed.

The implications of a "cosmic helium abundance" are profound. It would indeed be most intriguing were the globular clusters telling us that matter has locally, and possibly universally, been processed to temperatures $\gtrsim 2.5 \times 10^{10}$ ° K (Hoyle and Tayler 1964). One further consequence of a cosmic helium abundance (whatever its value) would be a correlation of horizontal-branch luminosity with metal content in the sense noted by Sandage and Wallerstein (1960). Figures 1 and 2 already demonstrate this correlation. The effect of metal content on the core masses, as previously mentioned, would further increase the separation in the sense observed. A separation in log L of 0.2–0.3 would not appear unreasonable, i.e., $\Delta M_{bol} \sim 0.5$–0.75 mag. It could arise as an immediate consequence of one obvious way of populating the horizontal branches to the blue of the critical point. Appreciable mass loss in the giant stage would enable the stars of one cluster to fall anywhere along the sequences plotted in our figures. To call on mass loss may appear to be an arbitrary step. However, Oke (1965) has observational evidence for lower masses on the horizontal branch, while Christy (1965) believes from his theoretical studies of RR Lyrae stars that their masses are lower than 1 $M_\odot$—and probably as low as 0.5–0.6 $M_\odot$. Christy's models also suggest a helium content $\sim$0.30 if a satisfactory agreement with the observed pulsations is to be obtained, an interesting and quite independent argument for the higher value.

As far as we are aware, the evolution of stars with high helium but extremely low metal content has not been extensively investigated. This is possibly because of prejudices that helium and metal contents should vary together, or because of the line-blanketing results for subdwarfs. The closest approach to the type of composition considered here is that of Hoyle's Type IIb model (Hoyle 1959), with $X = 0.75$ and $Z = 10^{-3}$. Sandage (1962) used an argument based on homology to deduce that this composition led to an evolved sequence having the wrong slope with respect to the main sequence. The adopted technique also included an arbitrary initial translation in log T_e to an assumed zero-age main sequence, justified by appealing to the known uncertainties in model radii. This weakens the argument for discarding the models because subsequent radii are incorrect, as does the additional fact that it is not very safe to use homology to scale the Type IIb model either up or (as here required) down the main sequence. The initial model is situated at the hump in the main sequence which occurs precisely because several physical factors are changing in importance, rendering homology inapplicable. Homology can be used for the Type IIa model preferred by Sandage because it is considerably below the hump for its own composition, and only models below the hump are relevant. However, the energy generation in the *pp*-chain was inadvertently overestimated in the early evolutionary stages, so that the consequent differential changes were themselves in error.

The matter seems sufficiently open to doubt to justify further investigation. Should explicit calculations for models with $X = 0.65$ and $Z_{CNO} = 10^{-4}$ confirm the tendency to a wrong slope, the "big bang" theory would be presented with a serious problem. On the other hand, the shapes of Population II sequences above the turnoff are very poorly matched, as Sandage shows. While it is possible that a correct evaluation of the *pp*-rate (or the correct exponential laws for both *pp* and CNO rates) might bring better agreement, we have already suggested a general argument for lower hydrogen values as an aid in matching the sequence shapes. It may well be that higher helium content with very low metals can resolve this difficulty of the theory.

Returning to our models, the luminosity levels themselves do not rule out one or other of our values for X_e. Indeed, if our assignment of probable core masses is correct, the luminosity levels are very closely the same. Bearing in mind that our envelope opacities are slightly low, and also the errors in the assignment of core masses, it is still possible to say that the horizontal branches should be at $M_{\text{bol}} \sim -0.2$ to $+0.5$ mag. With allowances for the bolometric correction, this is the range in which the observations mainly lie. Theory and observation appear equally capable of deciding what the true level of the horizontal branches must be. In this connection it is interesting to compare our models with the incipient horizontal branch of M67 (Eggen and Sandage, 1964). At $M_v \sim 1.5$, the luminosities may well agree with the lowest envelope in Figure 2—the undoubtedly high values of both Z and Y probably requiring $M_c \sim 0.4\ M_\odot$. If mass loss is the explanation for the horizontal branch, all the stars with $V_0 \lesssim 11.5$ and $(B - V)_0 \lesssim 0.8$ in Eggen and Sandage's Figure 3 have masses $\lesssim 0.6\ M_\odot$. Stars which lose no mass and start with $\gtrsim 0.7\ M_\odot$ would move to a position against the subgiant branch with $M_{\text{bol}} \sim 0.6$ mag. in our Figure 2. It is amusing to note that, in their Figure 3, Eggen and Sandage have a cluster of five stars at precisely this position if their modulus and a bolometric correction ~ 0.4 mag. are correct.

In spite of the evidence that can be advanced for mass loss, we do not feel that it need be an essential requirement of a horizontal-branch theory. In this regard it should be noted that if the initial hydrogen content is as low as 65 per cent, the masses of stars with the present turnoff colors of the globular clusters are in the range ~ 0.65–$0.75\ M_\odot$ (Faulkner 1966). Models in this mass range are most sensitive to the mass in the core.

Evolution of the present models is being studied in collaboration with Dr. Icko Iben, Jr., of the Massachusetts Institute of Technology, and will be reported on when completed. However, it is quite instructive to make rough estimates of time scales, etc., from the static models. Following Deinzer and Salpeter (1964) and Divine (1964), we assume helium burning goes through to oxygen when sufficient carbon has been produced. For all our cores approximately one quarter is convecting. Assuming that helium exhaustion sets the time scale, we find that $t \sim M_c^{-2.5}$ (as $L_c \sim M_c^{3.5}$), giving less than 5×10^7 yr for $M_c = 0.6\ M_\odot$, but as much as 1.4×10^8 yr for $M_c = 0.4\ M_\odot$. Woolf's (1964) figure of $\sim 2.3 \times 10^8$ yr seems definitely at variance with the former value. However, Woolf's calculation itself depended upon the assumption of high X_e. A lower value of X_e would reduce the time scale in Woolf's calculation and, as we have seen, imply the lower range of values for M_c. In addition, the time scales and the rates of shell burning are such that the models with $X_e = 0.90$ will retain essentially the same core mass throughout core helium burning. For $X_e = 0.65$ the core mass will increase by the order of 10 per cent, aiding movement to the blue. Our conclusion is that the present evidence definitely indicates a preference for the lower value of the envelope hydrogen content for the globular clusters.

To sum up the evolution of metal-weak models: it is expected that a steady, slowly brightening evolution to the blue will occur. After a time scale set by central helium exhaustion, evolution back to the giant branch will occur, as normally happens with exhaustion of a central fuel. This is expected to be a relatively fast phase. The scheme fits in well with observation. It suggests that the horizontal-branch gap need not necessarily be empty. A few stars may be expected there in the rapid stages of evolution in a sufficiently dense cluster, going both to and from the main horizontal-branch region. Subsequently the models will climb back up the giant branch. Conditions deep down are different from those in the earlier giant period, so the two branches need not be exactly coincident. This may be responsible for the observed doubling of the giant branch mentioned by Arp (1955).

For stars with almost normal metal abundance, the "forbidden region" may still play an important role. Our argument above has essentially asserted that during evolution the critical points for a given external composition and core mass move to the left in the H-R diagram. For the richer metal compositions it is quite possible that the critical point is moved little, if at all, from the forbidden region. In this case the models will be expected to brighten slightly, while remaining closely concentrated to the giant branch, exactly as observed in the metal-rich clusters.

We come finally to those clusters which, like M13 or M22 (Arp and Melbourne 1959), appear to violate the general correlation of metal abundance with horizontal-branch gap. Throughout this paper we have assumed that globular cluster stars have ages which demand masses little removed from the critical mass for their composition and core mass—essentially the range ~ 0.7–$\sim 1.25\ M_{\odot}$. If this is in general correct, then, a cluster which is significantly younger (as suggested in the case of M13 by Baum, Hiltner, Johnson, and Sandage 1959) or older will have a wider gap, as the mass will be larger or smaller than the critical value. In addition, even if the suggestion that there is a basic cosmic helium abundance is correct, there still exists the possibility of local enhancement. As can be seen from Figures 1 and 2, changes in the helium content can simulate the effects of changes in the metal content. We cannot pretend that these arguments represent a truly satisfactory explanation for the clusters like M13. However, they do demonstrate that the metal content (or at least, the supposed value) need not be the only determining factor.

VI. CONCLUSION

Models have been derived for stars which have passed through the helium flash. When identified with the horizontal branches of globular clusters, the general correlation of the observed horizontal-branch gap with metal content receives a satisfactory explanation. The identifications and further speculations on the subsequent developments seem to require the following conclusions: (i) There is no mixing between core and envelope during the helium flash; (ii) there is a fairly uniform helium concentration in the envelopes of globular cluster stars at a value ~ 0.35.

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