THE STABILITY OF SUPERMASSIVE STARS*†

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ABSTRACT

The studies presented in this paper constitute an extension of previous work on relaxation oscillations in supermassive stars. It is shown that a relatively small amount of rotation is sufficient to remove the general-relativistic instability which arises in such stars when rotation is absent. The post-Newtonian equations for the binding energy and for the frequency of the fundamental mode of radial oscillation are derived and the close connection between these two quantities is exhibited. The equatorial instability associated with contraction under rotation is investigated and the results used to estimate the limiting mass in which hydrogen burning can be effective as a source of energy during relaxation oscillations. This limit is found to be at least $10^8\ M_\odot$ and perhaps as high as $10^9\ M_\odot$ whereas, without rotation, the limit arising from general relativistic considerations is $10^6\ M_\odot$.

INTRODUCTION

In an attempt to understand the source of the energy emitted by the extended radio sources associated with elliptical galaxies, Hoyle and Fowler (1963*a*) suggested the possibility that a mass of the order of $10^8\ M_\odot$ has condensed in the galactic nucleus into a supermassive[1] star in which nuclear-energy generation takes place. In a subsequent paper (Hoyle and Fowler 1963*b*) the release of gravitational energy during general-relativistic collapse after the exhaustion of nuclear fuels was discussed. In a third paper (Hoyle and Fowler 1965) processes occurring during the formation of the massive star were discussed. The alternate possibility that fragmentation into smaller stars occurs was considered, and subsequent collisions between fragments was given some attention. In the present paper it is assumed that a single massive star has formed, either directly or through an intermediate stage involving fragmentation into smaller stars, which are ultimately destroyed by collisions between them. From this basic assumption the paper proceeds to a discussion which is limited to the problem of the dynamic stability of the massive star during the period in which nuclear-energy generation takes place.

In the first reference mentioned above it was shown that the radiative luminosity of a *stable* supermassive star ($M > 10^3\ M_\odot$) is proportional to the mass according to the approximate relation

$$L \approx 2 \times 10^{38}(M/M_\odot) \text{ ergs sec}^{-1}\,, \tag{1}$$

where M is the mass of the star and $M_\odot$ is the mass of the Sun. On the assumption that one-half of the hydrogen in the star is processed to helium, the available energy is

$$Q \approx \tfrac{1}{2} \times 7 \times 10^{-3} Mc^2 \approx 6 \times 10^{51}(M/M_\odot) \text{ ergs}\,, \tag{2}$$

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[1] The designation "supermassive" applies throughout this paper to stars with mass $M > 10^3\ M_\odot$. The prefix "super" will frequently be omitted, but the stars under discussion in this paper are not to be confused with stars with masses between $30\ M_\odot$ and $100\ M_\odot$ which are frequently called massive stars.

so that the lifetime for the main-sequence stage of a supermassive star is

$$\tau \approx Q/L \approx 3 \times 10^{13} \text{ sec} \approx 10^6 \text{ yr} \tag{3}$$

independent of mass. It was also found that hydrogen burning through the CNO bi-cycle takes place at a central temperature near 8×10^7 ° K and that the effective surface temperature during hydrogen burning is approximately 7×10^4 ° K indicating strong emission in the ultraviolet. A major unsolved problem concerned the mechanism by which the optical energy output is transformed into the high-energy particles and magnetic field necessary to produce the radio emission on the basis of current synchrotron theory.

The discovery (Schmidt 1963; Oke 1963; Greenstein and Matthews 1963) and subsequent investigation (Greenstein and Schmidt 1964; Oke 1965) of the quasi-stellar radio sources (QSS's) shows that starlike objects associated with certain radio sources do indeed have very large luminosities in the optical range. The observed luminosities are claimed to be of the order of 10^{46} ergs sec^{-1} which is expected from equation (1) for $M \sim 10^8 M_\odot$. Lifetimes (Greenstein and Schmidt 1964) of the QSS's fall in the range 10^3–10^6 years on various models. Thus it is tempting to associate the source of energy in the QSS's with nuclear burning in massive stars. Subsequent gravitational energy release and possible connections with the *extended* radio sources are left aside for the time being. In fact, the association with QSS's and radio galaxies is not the only motivation for this paper. The stability of massive stars is a problem of interest and significance per se.

Support for the massive-star model is given by the observed variability (Smith and Hoffleit 1963; Matthews and Sandage 1963; Sharov and Efremov 1963; Sandage 1964; Geyer 1964) of the optical radiation from the QSS's. In addition to luminous flashes with durations of the order of days or weeks, there is evidence for cyclic variations with periods of the order of 10 years. It is generally agreed that the occurrence of the cyclic variations is crucial to the question whether the primary radiating object is a single coherent massive star (Hoyle and Fowler 1963*a*, *b*) or a system of smaller stars as discussed by numerous authors (Burbidge 1961; Woltjer 1964; Ulam and Walden 1964; Field 1964; Hoyle and Fowler 1965; Gold, Axford, and Ray 1965). It is difficult on the basis of random collisions or supernova outbursts in a system of many stellar objects to explain variations which exhibit a regular periodicity. Thus, without prejudice to the problem of the reality of the cyclic variations since only additional and more precise observations will settle this matter, the possibility is investigated in this paper that such variations can arise from pulsations in a single massive star. The general-relativistic instability (Chandrasekhar 1964*a*, *b;* Fowler 1964*a*) which occurs in non-rotating stars is discussed along with the relaxation oscillations (Fowler 1965) which may thereby be produced. In the major conclusion of the paper it is shown that the general-relativistic instability is completely removed during nuclear burning by a relatively small amount of rotation especially if differential rotation is taken into account.

An elegant treatment of the stability of supermassive stars using the exact equations of general relativity has been given by Chandrasekhar (1964*a*, *b;* 1965), and applications to polytropic gas spheres have been made by Tooper (1964). An analysis of the binding energy has been given by Iben (1963), and a general discussion of the binding energy and the various modes of oscillation has been given by Bardeen (1965). In the interest of simplicity and some gain in physical insight the following discussion will be restricted to the post-Newtonian approximation (Fowler 1964*a*) to the relativistic equations.

This restriction can be justified on the grounds that only the Newtonian and post-Newtonian terms in the Schwarzschild line element have been verified in the three so-called crucial tests of general relativity. There is even some question concerning the correspondence between observation and theory in the advance of the perihelion of Mercury which constitutes a test of the coefficient of the post-Newtonian term in the line element.

In determining the post-Newtonian terms a further approximation is made in that these terms are evaluated using the equilibrium configurations given by the Newtonian approximation. It must be emphasized that this cannot be justified without recourse to the detailed analysis of the exact formal solutions and the post-Newtonian approximation as given by Chandrasekhar. Only by such a detailed analysis can the conditions be determined under which this procedure gives a fair approximation to the correct results.

Even though it has important effects, rotation can be taken to be small and need only be treated in the Newtonian approximation and only for the case where distortion from spherical symmetry can be neglected. The two starting points will be (1) the equation for the binding energy of a star in hydrostatic equilibrium and (2) the radial equation for dynamic equilibrium throughout the star. The object is to derive useful relations for the binding energy and for the frequency of the fundamental mode of radial oscillation and to exhibit the connection between these two quantities. Because of the order of approximation to which the derivations are restricted, the results are applicable only to supermassive stars ($M > 10^3 M_\odot$) in which the ratio β of gas pressure to gas plus radiation pressure is small ($\beta < 0.1$) and can be approximated by equation (A21) in the Appendix.

BINDING ENERGY OF A MASSIVE STAR IN HYDROSTATIC EQUILIBRIUM

Let us neglect rotation for the moment. The binding energy E_b of a star is equal but opposite in sign to the total energy E exclusive of the rest mass energy when infinitely dispersed at zero temperature and is given by (Fowler 1964*a*)

$$-E_b = E = (M - M_0)c^2 . \tag{4}$$

The gravitational or inertial mass measured by an external observer is

$$M = \int dM_r = \int \rho dV , \tag{5}$$

where dV is the "coordinate" volume, $\rho = \rho_0 + u/c^2$ is the total mass-energy density in mass units per unit coordinate volume, ρ_0 is the rest mass density of nuclei and ionization electrons, and u is the internal energy density of gas and radiation, and includes the rest mass energy of particles created in the medium at elevated temperatures such as electron-positron pairs (see the Appendix for further discussion). If spherical symmetry is assumed, $dM_r = \rho dV = 4\pi r^2 \rho dr$, where r is the coordinate radial variable and M_r is the mass energy internal to r. The integration is taken from zero to R, the coordinate radius of the star.

If the rest mass is to be computed at any stage of contraction the rest mass density must be integrated over the "proper" volume elements according to

$$M_0 = \int \rho_0 \left(1 - \frac{2GM_r}{r c^2}\right)^{-1/2} dV , \tag{6}$$

where G is the gravitational constant and the square-root term converts coordinate volume to proper volume. Because of atomic and nuclear processes there is always some ambiguity in choosing ρ_0 and thus in calculating M_0. The point is that the nuclear rest masses per nucleon may change in dispersing the star from radius R to infinity. Unlike the creation and annihilation of pairs these nuclear changes may be irreversible. We arbitrarily choose to define the binding energy relative to the rest mass energy of the nuclei and ionization electrons existing at a given stage of contraction as specified by the parameter R. Our choice is not necessarily unique or single-valued, but this is a problem in detail and not in principle.

Since M_r is related to ρ and not to ρ_0, it is convenient to retain ρ and u in expressing E so that

$$E=\int u\left(1-\frac{2GM_r}{rc^2}\right)^{-1/2}dV+\int \rho c^2\left[1-\left(1-\frac{2GM_r}{rc^2}\right)^{-1/2}\right]dV, \quad (7)$$

$$E=H-\Omega . \quad (8)$$

The first term in equation (7) is the proper internal energy of the star, which we designate by H in equation (8). The second term in equation (7) is the mass energy of the star minus the proper mass energy. If the sign is reversed this is just the gravitational binding energy (taken positive), which we designate by Ω in equation (8).

It is now appropriate to expand H and Ω, to retain only the Newtonian (subscript 0) and post-Newtonian (subscript 1) terms, and to introduce the Newtonian term for the rotational energy which we designate by Ψ_0. The result is

$$E\approx H_0-\Omega_0+\Psi_0+H_1-\Omega_1 , \quad (9)$$

$$E\approx\int u dV-\int\frac{GM_r}{r}\rho dV+\tfrac{1}{2}\int r^2\omega^2\sin^2\theta\,\rho dV+\int\frac{GM_r}{rc^2}u dV-\tfrac{3}{2}\int\frac{G^2M^2}{r^2c^2}\rho dV . \quad (10)$$

The definition of the various terms in equation (9) will be obvious from the order of the terms in equation (10). In equation (10) ω is the angular velocity and θ is the polar angle measured from the axis of rotation. It will develop that $H_0 - \Omega_0$ is proportional to β and is thus small and comparable to $H_1 - \Omega_1$. We discuss only cases where Ψ_0 is comparable within a factor of 10 to these two differences in the internal and gravitational energy terms.

EQUATION OF DYNAMIC EQUILIBRIUM

Again let us neglect rotation for the moment. The exact general-relativistic equation for dynamic equilibrium in the spherically symmetric case has been written by Misner and Sharp (1964) and others as

$$y^2\ddot{r}+y\dot{r}^2\frac{dy}{dr}=-\frac{1}{\rho}\frac{dp}{dr}\left(\frac{1+y^2\dot{r}^2/c^2-2GM_r/rc^2}{1+p/\rho c^2}\right)-\frac{GM_r}{r^2}-\frac{4\pi Gpr}{c^2}, \quad (11)$$

where

$$y=\frac{\rho+p/c^2}{\rho_0}=1+\frac{u}{\rho_0c^2}+\frac{p}{\rho_0c^2}. \quad (12)$$

It will be noted that the left-hand side of equation (11) can be written in the more compact form $yd(y\dot{r})/dt$.

We now proceed to write equation (11) in the post-Newtonian approximation and to apply it to small perturbations (δ) about hydrostatic equilibrium. Conditions at equilibrium will be designated by the subscript "eq." It will be clear that the two terms containing $\dot{r}^2$ can be neglected since $\dot{r}=\dot{r}_{eq}=0$ at hydrostatic equilibrium and $\delta\dot{r}^2 = 2\dot{r}_{eq}\delta\dot{r}_{eq}=0$ to first order. This leaves $y^2\ddot{r}$ on the left-hand side of equation (11) where the Newtonian term in y is unity and the post-Newtonian terms are much smaller than unity in all applications made in this paper. After the manipulations on equation (11) which follow, it will develop that the Newtonian term on the right-hand side is small and comparable to the post-Newtonian term. Thus it is unnecessary to retain second-order terms in the factor y^2 and in equation (11) we replace y^2 in $y^2\ddot{r}$ by unity.

Since the left-hand side of equation (11) has now been reduced to the classical Newtonian acceleration, $\ddot{r}$, with no ambiguities in space-time measurements, it will be clear that small rotational effects can be introduced in the approximation of the Newtonian centrifugal acceleration, $r\omega^2 \sin^2\theta$. Thus the post-Newtonian equivalent of equation (11) for small rotation in supermassive stars is

$$\ddot{r} \approx r\omega^2 \sin^2\theta - \frac{1}{\rho}\frac{dp}{dr}\left(1 - \frac{p}{\rho c^2} - \frac{2GM_r}{r c^2} + \ldots\right) - \frac{GM_r}{r^2} - \frac{4\pi G p r}{c^2}. \quad (13)$$

Since $dp/dr = \rho GM_r/r^2$ in Newtonian hydrostatic equilibrium with no rotation, equation (13) can be written, to the order of the approximations being made in this discussion, as

$$\ddot{r} \approx r\omega^2 \sin^2\theta - \frac{1}{\rho}\frac{dp}{dr} - \frac{GM_r}{r^2}\left(1 + \frac{p}{\rho c^2} + \frac{2GM_r}{r c^2} + \ldots\right) - \frac{4\pi G p r}{c^2}. \quad (14)$$

Multiply equation (14) by $r\rho dV$ and integrate over the entire star. The result is

$$\begin{aligned}\int r\,\ddot{r}\rho dV \approx & -\int 4\pi r^3 dp - \int \frac{GM_r}{r}\rho dV + \int r^2\omega^2 \sin^2\theta\,\rho dV \\ & -\int \frac{GM_r}{r c^2} p dV - 2\int \frac{G^2 M_r^2}{r^2 c^2}\rho dV - 4\pi\int \frac{G p r^2}{c^2}\rho dV. \end{aligned} \quad (15)$$

The first and last terms on the right-hand side can be integrated by parts from $r = 0$, where $M_r = 0$, to $r = R$, where $p = 0$, to yield

$$\int r\,\ddot{r}\rho dV \approx 3\int p dV - \Omega_0 + 2\Psi_0 + \int \frac{GM_r}{r c^2} p dV - 3\int \frac{G^2 M_r^2}{r^2 c^2}\rho dV. \quad (16)$$

From the discussion in the Appendix, $p = (\Gamma_4 - 1)u \approx \frac{1}{3}(1 + \beta/2)u$ so that $p \approx \frac{1}{3}u$ when β is small and it is natural to define a mean value of Γ_4 such that $\int p dV = \langle\Gamma_4 - 1\rangle \int u dV$. To the approximation of interest we can use this $\langle\Gamma_4\rangle$ in the fourth term on the right-hand side of equation (16). The result is the virial equation

$$\int r\ddot{r}\rho dV \approx 3\langle\Gamma_4 - 1\rangle H_0 - \Omega_0 + 2\Psi_0 + \langle\Gamma_4 - 1\rangle H_1 - 2\Omega_1. \quad (17)$$

Under conditions of hydrostatic equilibrium, $\ddot{r} = 0$ everywhere and a simple virial relation is obtained between H_0, Ω_0, etc. For numerical calculations of the binding energy it is most convenient to eliminate H_0 in substituting into equation (9) and the result is

$$E_{\rm eq} \approx -\frac{\langle 3\Gamma_4 - 4\rangle}{3\langle\Gamma_4 - 1\rangle}\Omega_0 - \frac{\langle 5 - 3\Gamma_4\rangle}{3\langle\Gamma_4 - 1\rangle}\Psi_0 + \tfrac{2}{3}H_1 + \frac{\langle 5 - 3\Gamma_4\rangle}{3\langle\Gamma_4 - 1\rangle}\Omega_1. \quad (18)$$

Equation (A19) then yields

$$E_{\rm eq} \approx -\frac{\bar{\beta}}{2}\Omega_0 - (1 - \bar{\beta})\Psi_0 + \tfrac{2}{3}H_1 + (1 - \bar{\beta})\Omega_1. \quad (19)$$

For small $\bar{\beta}$ in massive stars

$$E_{\rm eq} \approx -\frac{\bar{\beta}}{2}\Omega_0 - \Psi_0 + \tfrac{2}{3}H_1 + \Omega_1, \quad (20)$$

where, in recapitulation

$$\Omega_0 = \int \frac{GM_r}{r}\rho dV = 4\pi G\int \rho r M_r dr, \quad (21)$$

$$\Psi_0 = \tfrac{1}{2}\int r^2\omega^2 \sin^2\theta\,\rho dV = \pi\int \rho r^4\omega^2 \sin^3\theta\, dr\, d\theta, \quad (22)$$

$$H_1=\int \frac{GM_r}{rc^2}\,udV=\frac{4\pi G}{c^2}\int urM_r dr \approx \frac{12\pi G}{c^2}\int prM_r dr\,, \tag{23}$$

$$\Omega_1=\tfrac{3}{2}\int \frac{G^2M_r^2}{r^2c^2}\,\rho dV=\frac{6\pi G^2}{c^2}\int \rho M_r^2 dr\,. \tag{24}$$

In the last approximation in equation (23) we have used $p = (\Gamma_4 - 1)u \approx u/3$ for massive stars. In equation (20) it will be noted that all terms are *small* when this equation is applied to *slowly rotating, massive stars*. This circumstance arises from the fact that $H_0 - \Omega_0$ in equation (9) becomes proportional to $\bar{\beta}$ through equations (17) and (A19).

ADIABATIC RADIAL PULSATION

In order to determine the angular frequency, σ_R, of the fundamental mode of radial oscillation, equation (17) is applied to a perturbation of the form

$$\frac{\delta r}{r}=\frac{\delta R}{R}\exp(-i\sigma_R t)\,. \tag{25}$$

The result is

$$-\sigma_R^2 I\,\frac{\delta R}{R}\approx 3\langle\Gamma_1-1\rangle\delta H_0-\delta\Omega_0+2\delta\Psi_0+\langle\Gamma_1-1\rangle\delta H_1-2\delta\Omega_1\,, \tag{26}$$

where

$$I=\int r^2\rho dV \tag{27}$$

is the moment of inertia of the star about the origin of coordinates. I is equal to $\frac{3}{2}$ the usual moment of inertia about the axis of rotation if the distortion from spherical symmetry is ignored. In deriving equation (26) we used equation (A27) in the Appendix. Again we overlook the fact that the average Γ_1 in the coefficient of δH_1 is not quite the same as that in the coefficient of δH_0. If the oscillation is adiabatic, the energy equation becomes

$$\delta E=\delta H_0-\delta\Omega_0+\delta\Psi_0+\delta H_1-\delta\Omega_1=0\,. \tag{28}$$

If equation (28) is employed to eliminate δH_0 in equation (26), the result is

$$-\sigma_R^2 I\,\frac{\delta R}{R}=\langle 3\Gamma_1-4\rangle\delta\Omega_0+\langle 5-3\Gamma_1\rangle\delta\Psi_0-2\langle\Gamma_1-1\rangle\delta H_1-\langle 5-3\Gamma_1\rangle\delta\Omega_1\,. \tag{29}$$

APPLICATIONS TO POLYTROPIC MODELS

Within the approximations which have been carefully specified, equations (18) and (29) are quite general. Further elucidation requires that Ω_0, etc., be specified as functions of the stellar radius R and mass M and that $\delta\Omega_0$, etc., be related to δR through these quantities. This can only be done for specific stellar models. For our purposes polytropic models specified by the index n in the relation $pV^{1+1/n}$ = const. or p = const. $\rho_0^{1+1/n}$ are of sufficient diversity and accuracy.

Considerable simplification arises from the fact that our interest is concentrated on *slowly rotating*, *massive stars* in which the Newtonian terms in equations (18) and (29) are small and of the same order of magnitude as the post-Newtonian terms. This means that the integrals for Ω_0, Ψ_0, H_1, and Ω_1 can be evaluated using the run of the variables throughout the star given by the solution of the classical Lane-Emden polytropic equations without rotation. In particular it is not necessary to distinguish between ρ_0 and ρ or between M_0 and M in keeping with the general presumption that $M_0 - M$ is small compared to either one of them. Only one new physical concept must be introduced—namely, that for an *isolated* star, angular momentum must be conserved through all stages of contraction or of oscillation.

The Newtonian gravitational binding energy in units of Mc^2 can be expressed in terms of the convenient dimensionless parameter $2GM/Rc^2$ as

$$\frac{\Omega_0}{Mc^2}=\frac{3}{2(5-n)}\left(\frac{2GM}{Rc^2}\right), \tag{30}$$

so

$$\frac{\delta\Omega_0}{\Omega_0}=-\frac{\delta R}{R}. \tag{31}$$

Rotational terms in Ω_0 result in terms of order $\beta\omega^2$ in E or $\sigma_R{}^2$ and can be neglected when both β and ω^2 are small. The Newtonian rotational energy is given in terms of the *conserved* angular momentum, Φ, by

$$\frac{\Psi_0}{Mc^2}=\frac{\Phi^2}{2(ckMR)^2}, \tag{32}$$

where for *uniform* rotation $k=(2I/3MR^2)^{1/2}$ is the radius of gyration in units of R and $\Phi=k^2MR^2\omega=$ const. Differential rotation will be discussed in what follows. Once established under the conservation of angular momentum for all mass elements in a star, differential rotation requires $\Psi_0 \propto R^{-2}$ just as for uniform rotation, so that in any case

$$\frac{\delta\Psi_0}{\Psi_0}=-2\frac{\delta R}{R}. \tag{33}$$

It has been shown (Fowler 1964*a*, *b*) that the integrals for H_1 and Ω_1 in units of Mc^2 involve the dimensionless parameter $(2GM/Rc^2)$ to the second power as might be expected on general grounds. Numerical coefficients can be derived analytically for some polytropes and can be evaluated numerically for others. For the quantities of greatest interest, the result can be expressed as

$$\frac{H_1}{Mc^2}=\zeta_n'\left(\frac{2GM}{Rc^2}\right)^2, \qquad \frac{\Omega_1}{Mc^2}=\zeta_n''\left(\frac{2GM}{Rc^2}\right)^2, \tag{34}$$

and

$$\frac{2}{3}\frac{H_1}{Mc^2}+\frac{\Omega_1}{Mc^2}=\zeta_n\left(\frac{2GM}{Rc^2}\right)^2, \tag{35}$$

where, for example, $\zeta_0'=0.064$, $\zeta_0''=0.161$, $\zeta_0=0.204$, $\zeta_1'=0.116$, $\zeta_1''=0.241$, $\zeta_1=0.318$, $\zeta_2'=0.219$, $\zeta_2''=0.417$, $\zeta_2=0.563$, $\zeta_3'=0.513$, $\zeta_3''=0.923$, $\zeta_3=1.265$, $\zeta_4'=2.12$, $\zeta_4''=3.66$, and $\zeta_4=5.07$. J. M. Bardeen and S. P. S. Anand have called the author's attention to the fact that $\zeta_n \approx 5.07/(5-n)^2$.

In any case

$$\frac{\delta H_1}{H_1}=\frac{\delta\Omega_1}{\Omega_1}=-2\frac{\delta R}{R}. \tag{36}$$

Thus equation (29) becomes

$$\sigma_R{}^2 I \approx \langle 3\Gamma_1-4\rangle\Omega_0+2\langle 5\text{–}3\Gamma_1\rangle\Psi_0-4\langle\Gamma_1-1\rangle H_1-2\langle 5\text{–}3\Gamma_1\rangle\Omega_1\,. \tag{37}$$

The Newtonian terms in this equation are identical to those given by Chandrasekhar and Lebovitz (1962) in their equation (111). For $\langle\Gamma_1\rangle \approx \frac{4}{3}+\bar{\beta}/6$ and $\bar{\beta}$ small as in supermassive stars

$$\sigma_R{}^2 I \approx \frac{\bar{\beta}}{2}\Omega_0+2\Psi_0-\tfrac{4}{3}H_1-2\Omega_1\,. \tag{38}$$

In those cases where $\langle\Gamma_1\rangle$ and $\langle\Gamma_4\rangle$ can be taken to be equal, as, e.g., when $\bar{\beta}$ is small and $\langle\Gamma_1\rangle \approx \langle\Gamma_4\rangle \approx \frac{4}{3} + \bar{\beta}/6$, then it will be clear from equations (18), (30), (32), (34), and (37) that

$$\sigma_R{}^2 \approx 3\langle\Gamma_1 - 1\rangle \frac{R}{I}\frac{dE_{\text{eq}}}{dR} \qquad (\langle\Gamma_1\rangle \approx \langle\Gamma_4\rangle), \quad (39)$$

$$\sigma_R{}^2 \approx \frac{R}{I}\frac{dE_{\text{eq}}}{dR} \qquad (\beta \ll 1). \quad (40)$$

This important relation has been previously (Fowler 1964*a*) used in the case of non-rotating massive stars and will be discussed further in what follows. A circumstance under which equations (39) and (40) do not hold will be noted near the conclusion of this paper.

In order to make the analysis which follows as transparent as possible it will prove expedient to specify a particular polytropic index. For massive stars it is well known that the case $n = 3$, for which β = constant, yields a fairly accurate representation for the internal structure. For $n = 3$, equations (20) and (38) become

$$\frac{E_{\text{eq}}}{Mc^2} \approx -\tfrac{3}{8}\beta\left(\frac{2GM}{Rc^2}\right) + 1.265\left(\frac{2GM}{Rc^2}\right)^2 - \tfrac{1}{2}\left(\frac{\Phi}{ckMr}\right)^2 \quad (n = 3), \quad (41)$$

$$\sigma_R{}^2 \approx \frac{Mc^2}{I}\left[\tfrac{3}{8}\beta\left(\frac{2GM}{Rc^2}\right) - 2.53\left(\frac{2GM}{Rc^2}\right)^2 + \left(\frac{\Phi}{ckMR}\right)^2\right] \quad (n = 3). \quad (42)$$

These equations display the Newtonian gravitational term in $1/R$, the Newtonian rotational term in $1/R^2$, and the general-relativistic post-Newtonian term in $1/R^2$. The dependence on powers of $1/R$ can be replaced by dependence on powers of the central temperature, T_c, by use of the relation (Fowler 1964*a*)

$$T_c = \frac{5.83 \times 10^{18}}{R}\left(\frac{M}{M_\odot}\right)^{1/2} \qquad (n = 3). \quad (43)$$

ROTATIONAL STABILITY VERSUS GENERAL-RELATIVISTIC INSTABILITY

The fundamental mode of radial oscillation becomes dynamically unstable when $\sigma_R{}^2 < 0$ or σ_R becomes imaginary in equation (25). In the case of no rotation, $\Phi = 0$, it has been noted (Chandrasekhar 1964*a*, *b;* Fowler 1964*a*) that instability sets in for contraction below a critical radius given for $\sigma_R{}^2 \leq 0$ in equation (42) by

$$R_{\text{cr}} = \frac{6.74}{\beta} R_g = 3.4 \times 10^5 \left(\frac{M}{M_\odot}\right)^{3/2} \text{cm} \quad (\Phi = 0,\ n = 3), \quad (44)$$

where $R_g = 2GM/c^2$ is the limiting gravitational radius or Schwarzschild coordinate radius and β has been evaluated using $\mu = 0.73$ for a representative mixture (Fowler 1965) of 50 per cent hydrogen, 47 per cent helium, and 3 per cent heavy elements by mass. From equations (43) and (44) the critical central temperature, above which instability sets in, is

$$T_{\text{cr}} = 1.7 \times 10^{13}\,(M_\odot/M)\,{}^\circ\text{K} \qquad (\Phi = 0,\ n = 3). \quad (45)$$

At the critical radius and central temperature, E_{eq} reaches a minimum value and the binding energy reaches a maximum value as indicated by equation (40). This is illustrated in Figure 1 for $M = 10^6\,M_\odot$ where $E_{\text{eq}}/M_\odot c^2$, σ_R, and the period $\Pi = 2\pi/\sigma_R$ are shown as functions of R and T_c. In the calculations $I = 0.113\,MR^2$ for a polytrope of index $n = 3$ has been used. The situation can be understood physically in the following way. To the left of the minimum in E_{eq} in the Newtonian range, an adiabatic perturbation (constant E) toward smaller radii leads to more energy than that required for equilibrium and thus, for any physically reasonable equation of state, to more pressure than that necessary for hydrostatic equilibrium. Thus the contraction is opposed. An adiabatic perturbation toward larger radii leads to less energy and less pressure than that required for hydrostatic equilibrium and thus expansion is opposed. The same argument

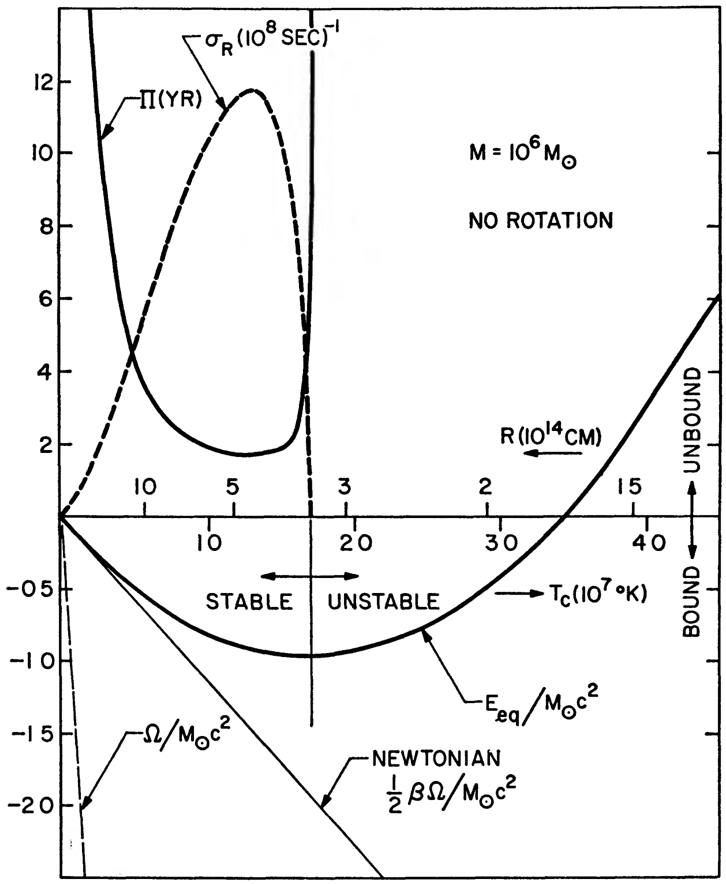

FIG. 1.—The binding energy and the frequency and period of the fundamental mode of radial oscillation in a non-rotating star with mass equal to $10^6 M_\odot$.

used to the right of the minimum (Fowler 1964*a*) indicates that a contraction leads to less pressure than that needed for hydrostatic equilibrium while an expansion leads to more so that the system is dynamically unstable to adiabatic perturbations. It will be noted that the minimum E_{eq} given by $-9\bar{\beta}^2Mc^2/64\zeta_n(5-n)^2$ has magnitude $\sim M_\odot c^2$, which is independent (Fowler 1965) of the mass M, and that the minimum period during stable contraction is of order of 1 year. In general the minimum period is given by

$$\Pi_{min} = 1.7 \times 10^{-12}(M/M_\odot)^2 \text{ yr} \qquad (\Phi = 0, n = 3)\,. \tag{46}$$

The critical temperature is only 1.7×10^7 ° K for $M = 10^6 M_\odot$, and this is considerably below the temperature of 8×10^7 ° K which, as has been previously noted (Hoyle and Fowler 1963*a*), is necessary for hydrogen burning through the CNO bi-cycle. This means that there is no source of the energy required for hydrostatic equilibrium above

1.7×10^7 ° K or for contraction below 3.5×10^{14} cm so that the instability results in gravitational collapse until the onset of nuclear burning. The resulting relaxation oscillations for $M \leq 10^6 M_\odot$ have been discussed by Fowler (1965). For $M \geq 10^6 M_\odot$, the onset of hydrogen burning is not sufficient to prevent continued gravitational collapse in a non-rotating star. This has placed a serious limitation on the energy available in that model (Fowler 1965) which depicts QSS's as non-rotating massive stars undergoing relaxation oscillations, as hydrogen burning in a star with $M = 10^6 M_\odot$ yields only $\sim 10^{58}$ ergs and the required energies are in some cases of the order of 100 times this figure.

In equation (42) the general-relativistic term which leads to instability varies as R^{-2} and is negative. For constant angular momentum, Φ, the rotational term also varies as R^{-2} but is positive. *Thus for large enough* Φ, *the general-relativistic instability discussed above is removed by rotation.* In physical terms the rotation prevents the gravitational collapse which would otherwise result from the general-relativistic instability. Relative to the magnitude of the angular momentum common to astronomical systems, the required

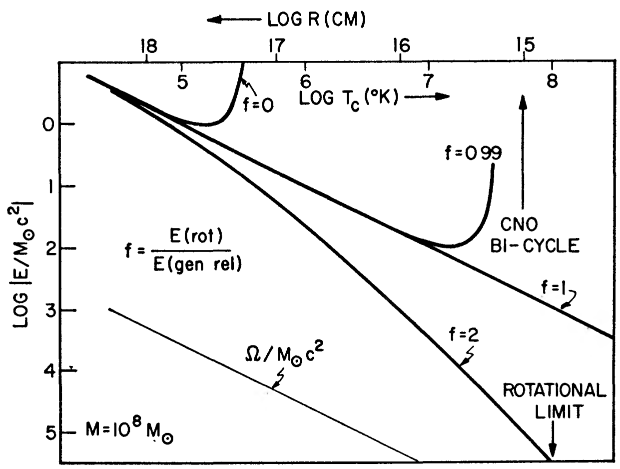

FIG. 2.—The binding energy of a rotating star with mass equal to $10^8 M_\odot$

Φ is quite small. For the rotational and general-relativistic terms in equation (42) to cancel, the critical angular momentum for stability is given by

$$\Phi_{cr} = (2k^2 \zeta_n)^{1/2} \frac{2GM^2}{c}, \quad (47)$$

where we have generalized to any n. Since the angular momentum is conserved, it is simplest to calculate Φ_{cr} at the stage where the stellar mass is dispersed uniformly as a gaseous cloud. In this case $n = 0$, $k^2 = \frac{2}{5}$, and $\zeta_0 = 0.204$ so that

$$\frac{\Phi_{cr}}{M} = 3.6 \times 10^{15} \left(\frac{M}{M_\odot}\right) \text{cm}^2 \text{ sec}^{-1}. \quad (48)$$

Even for $M = 10^8 M_\odot$ this angular momentum per unit mass is very small compared to the typical value, 10^{30} cm^2 sec^{-1}, which applies to the rotation of the solar system in the Galaxy.

The rotational effects are illustrated for a star with mass $M = 10^8 M_\odot$ in Figures 2 and 3. Figure 2 exhibits the dependence of $E_{eq}/M_\odot c^2$ on R and T_c while Figure 3 shows the dependence of the period Π on these same quantities. The curves have been calculated for $f = 0$, 0.99, 1, and 2, where f is the ratio of the rotational energy to the "general-relativistic" energy represented by the post-Newtonian terms in equations (20) and

(35). For a given angular momentum f remains constant during homologous contraction. The calculations have been made for polytropic index $n = 3$.

It will be noted that dynamic stability at the temperature required for hydrogen burning through the CNO bi-cycle sets a lower limit on f very close to unity. For reasons to be discussed in the next section, there probably exists an upper limit on f of the order of 10. The period of the fundamental radial oscillations at hydrogen burning varies rapidly with f being of the order of 1 year for $f = 1$ and 10 days for $f = 2$. If is extremely doubtful, however, that small amplitude, linear oscillations characterized by exactly these periods will occur. From the work of Ledoux (1941) and of Schwarzschild and Härm (1959) it is more probable that large-amplitude, non-linear pulsations will be set up at

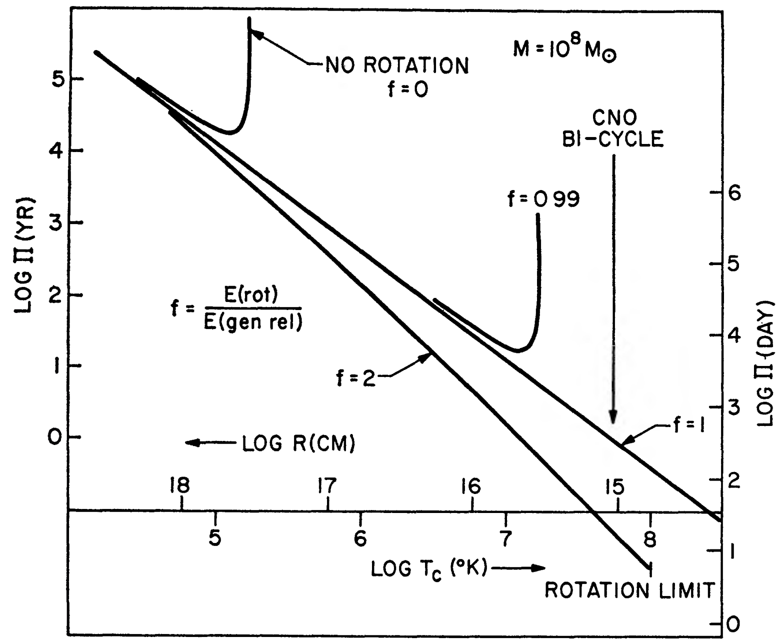

FIG. 3.—The period of the fundamental mode of radial oscillation of a rotating star with mass equal to $10^8\ M_\odot$.

the onset of nuclear burning. The energy generation will indeed take place in a relatively short period followed by a longer period of expansion to large radius and then recontraction during which the energy is transmitted to the surface of the star and radiated away. Relaxation oscillations of this nature in supermassive stars have been previously discussed (Fowler 1965), and the possible connections with the periodicity and exotic forms of energy emission in quasi-stellar objects have been pointed out. Only one point need be added to that discussion: variations in the magnetic field which accompany the oscillations will accelerate electrons to relativistic energies through the betatron mechanism.

Here we emphasize that rotation extends the mass range in which stable relaxation oscillations triggered by hydrogen burning can occur up to masses of the order of $10^8\ M_\odot$ or somewhat more. This extends the available nuclear energy in such objects to at least 10^{60} ergs. These limits were $10^6\ M_\odot$ and 10^{58} ergs without rotation. With rotation as the stabilizing agent, a star of mass $10^8\ M_\odot$ can serve as the energy source in a QSS with total luminosity equal to 2×10^{46} ergs sec^{-1} for a period as long as 10^6 years as noted in the Introduction.

For the record we note the period in supermassive stars with $f = 1$, $n = 3$, $k^2 = 0.075$, $\mu = 0.73$:

$$\Pi = \left(\frac{8\pi^2 k^2 R^3}{\beta G M}\right)^{1/2} = \left(\frac{6\pi k^2}{\beta G \bar{\rho}}\right)^{1/2} = 2.8 \times 10^{-21} R^{3/2} \left(\frac{M_\odot}{M}\right)^{1/4} \text{yr}$$
$$= 3.9 \times 10^7 T_c^{-3/2} \left(\frac{M}{M_\odot}\right)^{1/2} \text{yr}. \qquad (49)$$

In equation (49), $\bar{\rho}$ is the mean density of the stellar matter.

THE LIMIT OF ROTATIONAL STABILITY

Even though the rotation required to remove the general-relativistic instability is quite clearly available under typical astronomical circumstances as discussed in the previous section, the question arises whether the required angular momentum will lead to equatorial instability before sufficient contraction and high enough central temperature for hydrogen burning is reached.

It is first necessary to prescribe somewhat more precisely the central temperature required for hydrogen turning through the CNO bi-cycle. Fowler (1965) has given the average energy generation per gram per second, $\bar{\varepsilon}$, throughout the star and, when multiplied by the mass, this yields the nuclear-energy generation rate as

$$M\bar{\varepsilon} \approx 8.8 \times 10^{-44} \left(\frac{M}{M_\odot}\right)^{1/2} T_c^{11} \text{ ergs sec}^{-1}. \qquad (50)$$

When $M\bar{\varepsilon}$ from equation (50) is equated to L from equation (1), it is found that the central temperature, T_{cn}, required for *nuclear*-energy generation through the CNO bi-cycle is

$$T_{cn} \approx 2.5 \times 10^7 \left(\frac{M}{M_\odot}\right)^{1/22} {}^\circ\text{K} \qquad (\text{CNO bi-cycle}), \qquad (51)$$

so that $T_{cn} \approx 6 \times 10^7$ ° K for $M = 10^8 M_\odot$. This is lower than the estimate, $T_{cn} \sim 8 \times 10^7$ ° K, found originally by Hoyle and Fowler (1963*a*), but is somewhat more precise. It will be noted that the critical central temperature, equation (45), for general-relativistic instability is *less* than that required for hydrogen burning, equation (51), for all masses $M \gtrsim 4 \times 10^5 M_\odot$. This emphasizes the stability limitation on non-rotating models for supermassive stars (Fowler 1965). It may well be that explosions such as that observed in M82 by Lynds and Sandage (1963), which involved $5.6 \times 10^6 M_\odot$, can be attributed to the onset of general-relativistic instability during hydrogen burning without rotation.

With the required temperature in hand it is now necessary to ascertain the limiting central temperature at which rotation, governed by the conservation of angular momentum, leads to the equatorial instability characterized by loss of mass at the equator. It is probably true that a star survives this instability and that nuclear-energy generation at the center is not terminated by the loss of mass at the surface, but nonetheless this limitation is well worth investigating in some detail. The analysis which has been made to the present point in this paper has been limited to spherical symmetry in the post-Newtonian approximation. Thus the conclusions to follow require that the angular momenta considered be much less than the critical angular momentum at which distortion from spherical symmetry is large.

The problem is best discussed in terms of angular velocity rather than angular momentum since the critical limiting angular velocity is given quite simply by equating the equatorial centrifugal force to the gravitational attraction at the equator with the result that

$$\omega_{cr} = \left(\frac{GM}{R^3}\right)^{1/2} . \tag{52}$$

In terms of the angular velocity the rotational energy can be written as

$$\Psi_0 = \tfrac{1}{2} K^2 M R^2 \omega_R^2 . \tag{53}$$

Equation (53) has been written to include the case of differential rotation; $\omega_R = \omega(R)$ is the angular velocity at the equatorial radius and K is a constant which can be determined when $\omega = \omega(r)$ is specified as a function of the radius. For uniform rotation $\omega = \omega_R$ and $K = k$, the radius of gyration in units of R. Equations (52) and (53) can be combined to yield

$$\frac{\Psi_0}{Mc^2} = \tfrac{1}{4} K^2 \left(\frac{2GM}{Rc^2}\right) \left(\frac{\omega_R}{\omega_{cr}}\right)^2 , \tag{54}$$

in which the ratio of the equatorial angular velocity to its critical limiting value appears explicitly.

It should be noted that equation (54) should not be taken to imply that the factor 2 does not appear on the right-hand side of equation (33). Equation (54) applies to the relatively slow changes between equilibrium states. During the faster changes which occur during small radial oscillations, it would seem reasonable to assume that angular momentum is conserved. Then equation (33) can be employed as written with Φ_0 evaluated from equation (54) with $\omega_R = \omega_{cr}$ as the limiting case. Under these circumstances it will be clear that equation (40) no longer holds and that dynamical instability ($\sigma_R^2 = 0$) no longer sets in at the minimum in the equilibrium energy-curve. In fact if $\bar{\beta}\Omega_0/2$ in equation (38) is neglected it is found that $\sigma_R^2 = 0$ when $E_{eq} = 0$ rather than at the minimum value for E_{eq}.

The ratio of the rotational energy to the "general-relativistic" energy can be evaluated from equations (35) and (54) as

$$f = \frac{K^2}{4\zeta_n} \left(\frac{Rc^2}{2GM}\right) \left(\frac{\omega_R}{\omega_{cr}}\right)^2 . \tag{55}$$

This equation can be solved for the minimum radius for rotational stability at the equator and combined with equation (43) to yield the maximum central temperature. The results are

$$R_{min} \sim 1.5 \times 10^6 \frac{f}{K^2} \left(\frac{M}{M_\odot}\right) \left(\frac{\omega_{cr}}{\omega_R}\right)^2 \text{ cm} \qquad (n = 3) \tag{56}$$

and

$$T_{max} \sim 3.9 \times 10^{12} \frac{K^2}{f} \left(\frac{M_\odot}{M}\right)^{1/2} \left(\frac{\omega_R}{\omega_{cr}}\right)^2 {}^\circ\text{K} \qquad (n = 3) . \tag{57}$$

These equations are clearly only rough approximations in view of the fact that distortion from spherical symmetry has been neglected.

If angular momentum loss keeps ω_{cr}/ω_R fixed so that Ψ_0 is proportional to R^{-1} from equation (54), then R and T for maximum binding energy are given by equations (56) and (57) with $f = 2$. The maximum binding energy relative to the rest mass energy is then given by $(K^4/64\zeta_n)(\omega_R/\omega_{cr})^2$. This ratio can become of the order of several per cent in the case of differential rotation. For zero binding and the onset of dynamical instability, R and T are given by equations (56) and (57) with $f = 1$. These conclusions require that $\bar{\beta}\Omega_0/2$ be neglected in equation (38).

For uniform rotation $K^2 = k^2 = 0.075$ for a polytrope of index 3, and thus $T_{\max}$ is only 3×10^7 ° K for a star with $M = 10^8 M_\odot$ even when the minimum $f = 1$ and maximum $\omega_R/\omega_{cr} = 1$ are used in equation (57). This is not sufficient for hydrogen burning. The limiting mass which can be stabilized by uniform rotation during hydrogen burning is approximately $10^7 M_\odot$.

Differential rotation with an increase in angular velocity toward the center of the star results in a marked increase in K^2 and thus in $T_{\max}$. Two models with differential rotation have been considered. In the first model the massive star is assumed to contract from a cloud with polytropic index $n = 0$ to a structure with index n in such a way that each *spherical* shell retains its angular momentum. This model is not self-consistent in that the Newtonian equation for hydrostatic equilibrium cannot be satisfied by a polytropic relation between p and ρ when the centrifugal forces are not neglected. The second model is that of Stoeckly (1965) in which the star contracts in such a way that the angular momentum is conserved in each cylindrical shell (but not each ring) parallel to the axis of rotation. In this model the polytropic relation may be employed when centrifugal forces are included in the equation for hydrostatic equilibrium. The results for the two models are fortunately very similar as will be noted in the following tabulation:

	n				
	0	1	2	3	4
K^2 (spherical model)... . .	0 400	0 629	1 14	2 61	10 8
K^2 (cylindrical model)	0 400	0 624	1 10	2 47	9 8

If the value for K^2 for $n = 3$ on the cylindrical contraction model is substituted into equation (57), it is found that

$$T_{\max} \sim \frac{10^{13}}{f} \left(\frac{M_\odot}{M}\right)^{1/2} \left(\frac{\omega_R}{\omega_{cr}}\right)^2 {}^\circ\mathrm{K} \qquad (n = 3). \quad (58)$$

Now for $f = 1$ and $(\omega_R/\omega_{cr})^2 = \frac{1}{3}$ it is seen that $T_{\max} \sim 10^8$ ° K even for $M = 10^9 M_\odot$, and this is greater than the $T_{cn} \approx 7 \times 10^7$ ° K required for hydrogen burning in a star of this mass. On a more conservative evaluation with $f = 2$ and $(\omega_R/\omega_{cr})^2 = \frac{1}{5}$, the temperature is $T_{\max} \sim 10^8$ °K for $M = 10^8 M_\odot$. This is the "rotational limit" indicated on Figures 2 and 3. Because of the approximation involved in the assumption of spherical symmetry it is difficult to assess with any accuracy the limiting mass for which rotation is large enough to remove the general-relativistic instability but not so large that mass loss occurs at the equator. It can be noted that the central temperature will be higher for a given equatorial radius when distortion is taken into account and this fact tends to increase the limiting mass. We conclude that $10^8 M_\odot$ is a conservative upper limit but cannot completely exclude masses as high as $10^9 M_\odot$.

For the record we note the rotational period, P_R, at the periphery in supermassive stars:

$$P_R = \frac{\pi K R^2 c}{GM(2 f \zeta_n)^{1/2}}. \quad (59)$$

For $n = 3, f = 1, \zeta_3 = 1.264$, and $K^2 = 2.47$ this becomes

$$P_R = 2.2 \times 10^{-23} R^2 \left(\frac{M_\odot}{M}\right) \mathrm{yr} = 7.5 \times 10^{14} T_c^{-2} \mathrm{yr}. \quad (60)$$

Thus the peripheral rotational periods are the order of 0.1 year for $T_c \sim 10^8$. The mean rotational period is about $\frac{1}{5}$ this value.

Some consideration must be given to the situation in which the initial angular momentum is large so that $f \gg 1$ and the maximum temperature for rotational stability falls well below that necessary for hydrogen burning. This problem arises in all attempts to understand star formation. Some mechanism for transfer of the angular momentum to the material spun off at the equator must be invoked as has been done by Hoyle (1960) and others. We need only argue that such mechanisms can be effective for supermassive stars as well as has clearly been the case for the Sun and other ordinary stars. Angular momentum transfer to ejected material eventually reduces the angular momentum to the point where f falls in the reasonable range from 1 to 10.

Figure 2 illustrates one point which should be emphasized. For quite reasonable values of f the binding energy $E_b = |E_{eq}|$ becomes quite large and, when $f = 2$, almost one order of magnitude larger than the nuclear-energy resources of the star. Since this energy must be lost during contraction, it is another source of the observed energy emissions. It will be noted that this arises because the coefficient of Ψ_0 in equation (19) is $1 - \bar{\beta}$ and $\bar{\beta}$ is small in massive stars. In ordinary stars $\bar{\beta} = 1$ and this contribution to the binding energy vanishes.

In conclusion it can be pointed out that any physical phenomenon which leads to a positive term proportional to $1/R^2$ in the frequency equation (42) will, if large enough, remove the general-relativistic instability in supermassive stars. Thus turbulent kinetic energy associated with convection or internal magnetic disturbances scales as $1/R^2$ and will be effective in this regard.

The author is indebted for assistance by Dr. J. Bardeen at many stages in the analysis presented in this paper, and by Dr. Robert Stoeckly in the treatment of differential rotation. He is indebted to Barbara Zimmerman for performing all numberical computations and integrations. This investigation arose out of discussions with Dr. Ian W. Roxburgh at the Second Texas Symposium on Relativistic Astrophysics held December 15–18, 1964, at the University of Texas, Austin, and has benefited from subsequent discussion with him on the problems of rotation in massive stars. The author is indebted to Dr. Roxburgh for a preprint of the manuscript of his independent work (Roxburgh 1965) in which he has come to much the same conclusions as presented in this paper.

APPENDIX

In this Appendix the relations used in the main text between the internal energy density u, and pressure p, both in ergs cm^{-3}, are discussed and the use of various expressions for the "effective ratio of specific heats" is clarified. It is sufficiently general for our purposes to consider the medium to be approximately non-degenerate and to be made up of nuclei, ionization electrons, electron-positron pairs, and radiation as treated in detail by Fowler and Hoyle (1964). Then from equations (B62) and (B43) of this reference one has

$$u = xnkT + 2n_+ m_e c^2 + aT^4 \qquad \text{and} \qquad p = qnkT + \tfrac{1}{3}aT^4 , \tag{A1}$$

where $n = n_0 + n_N + 2n_+$ is the number density of all particles—ionization electrons, nuclei, and electron-positron pairs; x is the mean kinetic energy per particle in units of kT; and q is a factor, close to unity in value, which incorporates the deviations from Boyle's law in the gas. The internal energy density includes everything except the rest mass-energy density, $\rho_0 c^2$, of the nuclei and the associated ionization electrons.

The number density of paired electrons and positrons is

$$2n_+ \approx (n_0^2 + 4n_1^2)^{1/2} - n_0 , \tag{A2}$$

where

$$n_0 = Zn_N = \frac{\rho_0 Z}{M_u A} = 6.02 \times 10^{23} \rho_0 \frac{Z}{A} \tag{A3}$$

is the original number of ionization electrons, n_N is the number density of nuclei, ρ_0 is the rest mass density, M_u is the atomic mass unit, Z is the mean number of free electrons per nucleus, A is the mean nuclear mass plus that of associated electrons in atomic mass units, and

$$n_1 = \frac{1}{\pi^2 z} \left(\frac{m_e c}{\hbar} \right)^3 K_2(z). \tag{A4}$$

In equation (A5), $z = m_e c^2/kT$ and $K_2(z)$ is the modified Bessel function of second order. Numerically one has

$$\begin{aligned} n_1 &\approx 1.521 \times 10^{29} T_9^{3/2} \exp(-5.93/T_9) \text{ cm}^{-3} \qquad (T_9 < 3) \\ &\approx 1.688 \times 10^{28} T_9^3 \text{ cm}^{-3} \qquad (T_9 > 3) . \end{aligned} \tag{A5}$$

Because of the low density in massive stars for a given temperature, the numer of positrons and paired electrons becomes comparable to the number of ionization electrons at relatively low temperatures, e.g., at 5×10^8 ° K in a star with $M = 10^8 M_\odot$. This is above the temperature for hydrogen-to-helium conversion however.

The factor x in equation (A1) is equal to $\frac{3}{2}$ for non-relativistic, non-degenerate electrons and nuclei and has been tabulated for relativistic, non-degenerate electrons by Chandrasekhar (1939) as U/PV in his Table 24 (p. 397). The entries in this table also apply to the pair positrons under relativistic non-degenerate conditions. Although the entries in the table range from $x = \frac{3}{2}$ up to maximum value, $x = 3$, there are circumstances (Fowler and Hoyle 1964) under which x can be as high as 3.15, in which case $q = 1.05$. At low temperatures pairs can be neglected, the electrons and nuclei may recombine into atoms and molecules, and in any case x can be found in terms of the specific heat at constant volume c_V or the ratio of specific heats γ from

$$\frac{d(xT)}{dT} = c_V = \frac{1}{\gamma - 1}. \tag{A6}$$

When x is constant, one has

$$x \approx c_V = \frac{1}{\gamma - 1}. \tag{A7}$$

Under the circumstances of major interest in this paper, the nuclei are ionized, the electrons are non-relativistic and non-degenerate, and pairs can be neglected. Then $\gamma = \frac{5}{3}$ and $x = c_V = \frac{3}{2}$.

If $\beta = qnkT/p$ is introduced as the ratio of gas pressure to total pressure and $1 - \beta = aT^4/3p$ as the ratio of radiation pressure to total pressure, then from equations (A1) the dimensionless ratio of internal energy density to pressure is given by

$$\frac{u}{p} = 3 - (\beta/q)[3q - x - z(2n_+/n)]. \tag{A8}$$

As is required relativistically this ratio approaches 3 at very high temperatures independent of β, as then $kT \gg m_e c^2$, $z \to 0$, $x/q \to 3$, and $2n_+ \to n$. When pairs are first copiously produced, this ratio can exceed 3 under certain circumstances. The relativistic behavior for β is discussed in detail by Fowler and Hoyle (1964); it passes through a minimum near zero in massive stars but increases to a limiting value, $\beta = \frac{7}{11}$, at high temperatures when pairs become copious.

The customary non-relativistic expression for u/p is found by setting $q = 1$, $x = (\gamma - 1)^{-1}$, and $n_+ = 0$ so that

$$\frac{u}{p} \approx \frac{\beta}{\gamma - 1} + 3(1 - \beta) = \frac{3(\gamma - 1) - \beta(3\gamma - 4)}{\gamma - 1}. \qquad \text{(NR)} \tag{A9}$$

It is convenient at this point to introduce a quantity which is very similar to the adiabatic coefficients Γ_1, Γ_2, and Γ_3 defined by Chandrasekhar (1939, pp. 57, 58). We denote this quantity by Γ_4 and define it by

$$\Gamma_4 - 1 \equiv p/u = \frac{1}{3-(\beta/q)[3q-x-z(2n_+/n)]} \tag{A10}$$

$$\approx \frac{\gamma-1}{3(\gamma-1)-\beta(3\gamma-4)} = \frac{\gamma-1}{1+(1-\beta)(3\gamma-4)}. \quad \text{(NR)} \tag{A11}$$

Thus

$$\Gamma_4 = \frac{4-(\beta/q)[3q-x-z(2n_+/n)]}{3-(\beta/q)[3q-x-z(2n_+/n)]} = \tfrac{4}{3} + \frac{(\beta/q)[3q-x-z(2n_+/n)]}{9-3(\beta/q)[3q-x-z(2n_+/n)]}, \tag{A12}$$

$$\Gamma_4 \approx \frac{4(\gamma-1)-\beta(3\gamma-4)}{3(\gamma-1)-\beta(3\gamma-4)}, \quad \text{(NR)} \tag{A13}$$

$$\Gamma_4 \approx \tfrac{4}{3} + \frac{\beta(3\gamma-4)}{9(\gamma-1)-3\beta(3\gamma-4)}. \quad \text{(NR)} \tag{A14}$$

It will be clear from the definition of Γ_4 that averaging over the entire volume of the star yields

$$\int p dV = \int (\Gamma_4 - 1) u dV = \langle \Gamma_4 - 1 \rangle \int u dV . \tag{A15}$$

The appropriate mean value for Γ_4 is that obtained by averaging over each element of internal energy, udV.

Extreme relativistic conditions arise when $x = 3q$ and $z = 0$ in equations (A12) in which case $\Gamma_4 = \frac{4}{3}$ as expected. Under intermediate circumstances Γ_4 can be found by using the first of equations (A12). However, under the circumstances of major interest in this paper, equation (A13) with $\gamma = \frac{5}{3}$ is applicable and

$$\Gamma_4 \approx \frac{8-3\beta}{6-3\beta}, \qquad \gamma = \tfrac{5}{3}, \quad \text{(NR)} \tag{A16}$$

$$\approx \tfrac{4}{3} + \frac{\beta}{6}, \qquad \beta \ll 1, \quad \text{(NR)} \tag{A17}$$

where the second approximation holds for small β. This is the same approximation that holds for the first of Chandrasekhar's adiabatic coefficients, $\Gamma_1 \equiv -d \ln p/d \ln V$ when β is small. As a matter of fact, in massive stars during hydrogen burning β is quite small, being given by (Fowler and Hoyle 1964)

$$\beta \approx \frac{4.28}{\mu}\left(\frac{M_\odot}{M}\right)^{1/2} \qquad M > 10^3 M_\odot \qquad (n=3), \quad \text{(NR)} \tag{A18}$$

where $\mu = A/(Z+1)$ is the mean "molecular" weight. Note that $\beta \sim 10^{-3}$ for $M = 10^8 M_\odot$ and $\mu = \frac{1}{2}$ (hydrogen). As discussed in the main text it is the smallness of β and the closeness of Γ_1 and Γ_4 to $\frac{4}{3}$ which makes the Newtonian terms in the binding energy and pulsation frequency correspondingly small and thus brings the rotational and general-relativistic terms into prominence in these quantities. It will be noted that Γ_1 and Γ_4 are effective ratios of specific heats under appropriate circumstances.

In the above analysis the ratio p/u at a given time and position in a star has been equated to $\Gamma_4 - 1$. In addition it is necessary to establish relationships between δp and δu and between

$\delta\int p dV$ and $\delta\int u dV$ when the star is subject to an *adiabatic* perturbation at all points. The general-relativistic adiabatic relation is

$$\delta Q = \delta(\rho c^2 V) + p\delta V = 0\,, \tag{A19}$$

where it is required that the volume V apply to a fixed number of baryons throughout the adiabatic change. This requirement follows from the generally accepted physical law of the *conservation of baryons*. Under the conditions of interest in this paper the only baryons involved are protons and neutrons, free or incorporated in nucleons as nuclei.

In order to proceed, it is necessary to recall once again the relation

$$\frac{\delta p}{p} \equiv -\Gamma_1 \frac{\delta V}{V} \tag{A20}$$

by which Chandrasekhar's first adiabatic coefficient is defined. If equations (A19) and (A20) are appropriately manipulated it is found that

$$\frac{\delta p}{p} = \Gamma_1 \frac{\delta \rho}{\rho + p/c^2} \quad \text{and} \quad \frac{\delta(pV)}{\delta(\rho c^2 V)} = \Gamma_1 - 1\,. \tag{A21}$$

Under some circumstances it is of interest to consider adiabatic changes during which no nuclear or atomic processes occur so that the rest mass associated with a given number of baryons (or nucleons) does not change. Under these circumstances $\rho_0 V$ is an invariant and $\delta(\rho c^2 V) = \delta(\rho_0 c^2 V) + \delta(uV)$ becomes just equal to $\delta(uV)$. Then

$$\frac{\delta p}{p} = \Gamma_1 \frac{\delta \rho_0}{\rho_0} \qquad (\rho_0 V = \text{const.})\,, \tag{A22}$$

$$\frac{\delta(pV)}{\delta(uV)} = \frac{\delta(p/\rho_0)}{\delta(u/\rho_0)} = \Gamma_1 - 1 \qquad (\rho_0 V = \text{const.})\,, \tag{A23}$$

and

$$\frac{\delta p}{\delta u} = \frac{\Gamma_1}{\Gamma_4}\frac{p}{u} = \frac{\Gamma_1}{\Gamma_4}(\Gamma_4 - 1) \qquad (\rho_0 V = \text{const.})\,. \tag{A24}$$

Now consider the variations $\delta\int p dV$ and $\delta\int u dV$ corresponding to adiabatic changes made throughout the entire star. These can be written, respectively, as

$$\delta\int p dV = \delta\int (pV)\frac{dV}{V} = \int \delta(pV)\frac{dV}{V} = \int (\Gamma_1 - 1)\,\delta(uV)\frac{dV}{V} \tag{A25}$$

and

$$\delta\int u dV = \delta\int (uV)\frac{dV}{V} = \int \delta(uV)\frac{dV}{V} \qquad (\rho_0 V = \text{const.})\,. \tag{A26}$$

The second equality in each of these equations derives from the fact that dV and V must each apply to a fixed number of baryons during any perturbation. Thus dV/V is replaceable by dN_B/N_B, where N_B is the total number of baryons in the star and is therefore clearly invariant to any perturbation under consideration. In the last equality in (A25), equation (A23) has been used. Then, since Γ_1 and its average $\langle\Gamma_1\rangle$ over udV are constant to first order during any perturbation, it ultimately follows that

$$\delta\int p dV = \delta\int(\Gamma_4 - 1)u dV = \langle\Gamma_1 - 1\rangle\delta\int u dV \qquad (\rho_0 V = \text{const.})\,. \tag{A27}$$

It will be clear that Γ_4 is not constant during an adiabatic perturbation, and, in fact, it can be shown that

$$\frac{\delta\Gamma_4}{\Gamma_4 - 1} = (\Gamma_4 - \Gamma_1)\frac{\delta V}{V}\,. \tag{A28}$$

Comparison of equation (A27) with (A15) indicates that $\langle\Gamma_1\rangle$ replaces $\langle\Gamma_4\rangle$ in relations involving adiabatic perturbations. To first order in small β, Γ_1 and Γ_4 and hence $\langle\Gamma_1\rangle$ and $\langle\Gamma_4\rangle$ are equal. This can be seen from Chandrasekhar's non-relativistic expression for Γ_1 which corresponds to equation (A13). This expression is

$$\Gamma_1 \approx \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta+12(\gamma-1)(1-\beta)} \qquad \text{(NR)} \tag{A29}$$

$$\approx \tfrac{4}{3} + \frac{(4\beta-3\beta^2)(3\gamma-4)}{36(\gamma-1)-3\beta(12\gamma-13)} \qquad \text{(NR)} \tag{A30}$$

$$\approx \frac{32-24\beta-3\beta^2}{24-21\beta}, \qquad \gamma = 5/3 \qquad \text{(NR)} \tag{A31}$$

$$\approx \tfrac{4}{3} + \frac{\beta}{6} \approx \Gamma_4, \qquad \beta \ll 1 \qquad \text{(NR)}. \tag{A32}$$

Fowler and Hoyle (1964, p. 289), give the relativistic expression for Γ_1 when pairs are included. Actually Γ_1 does not differ greatly from Γ_4 over the range $0 \leq \beta \leq 1$ as illustrated in Table A1.

TABLE A1

VALUES OF Γ_1 AND Γ_4 FOR $\gamma = \frac{5}{3}$

β	Γ_1	Γ_4	β	Γ_1	Γ_4
0 0.	1 333	1 333	0 5	1 426	1 444
01 . .	1 335002	1 335008	0 6	1 449	1 476
.1. .	1 350	1 351	0 7 .	1 476	1 512
.2	1 368	1 370	0 8 .	1 511	1 556
.3	1 386	1 392	0 9	1 563	1 606
0 4... .	1.405	1 417	1 0	1 667	1 667

The two are equal at the two extremes of this range with $\Gamma_1 = \Gamma_4 = \frac{4}{3}$ at $\beta = 0$ and $\Gamma_1 = \Gamma_4 = \frac{5}{3}$ at $\beta = 1$ for $\gamma = \frac{5}{3}$. In addition for small β, Γ_1 and Γ_4 are equal for any γ since

$$\Gamma_1 \approx \Gamma_4 \approx \tfrac{4}{3} + \frac{\beta}{3}\,\frac{\gamma-4/3}{\gamma-1}, \qquad \beta \ll 1 \qquad \text{(NR)}. \tag{A33}$$

For convective stability it is necessary that

$$\frac{d\ln p}{dr} > \Gamma_1 \frac{d\ln\rho}{dr} \qquad \text{(conv. stab.)}. \tag{A34}$$

This is a necessary and sufficient condition in general relativity except in very special physical conditions involving only small regions in a star where the effect of general-relativistic modifications is not of crucial importance. Since p and ρ usually decrease with r it is often convenient to use expression (A34) rewritten as

$$\left|\frac{d\ln p}{dr}\right| < \Gamma_1 \left|\frac{d\ln\rho}{dr}\right| \qquad \text{(conv. stab.)}. \tag{A35}$$

For a polytrope at index n with $p \propto \rho^{1+1/n}$ this yields

$$\left(1+\frac{1}{n}\right) < \Gamma_1,$$

or

$$n > \frac{1}{\Gamma_1 - 1} \qquad \text{(conv. stab.)} \qquad \textbf{(A36)}$$

$$> 3(1 - \beta/2) , \qquad T_9 < 1 .$$

Thus the polytrope $n = 3$ which has been used extensively throughout this paper is convectively stable except when electron-positron pair formation reduces Γ_1 below $\frac{4}{3}$ in the range $1 < T_9 < 3$. Formation of other particles will reduce Γ_1 below $\frac{4}{3}$ in additional ranges at still higher temperatures.

An important quantity in the considerations discussed in this paper is $\bar{\beta}$, the ratio of gas pressure to total pressure averaged over the internal energy distribution in the star. See, for

TABLE A2

n	R_n	M_n	$\rho_c/\bar{\rho}$	$\langle\mu\beta\rangle/(\mu\beta)_c$	$(\mu\beta)_c(M/M_\odot)^{1/2}$	$\langle\mu\beta\rangle(M/M_\odot)^{1/2}$
0 .	2 4494	4 8988	1 0000	1 8729	2 3569	4 4142
0 5	2 7528	3 7871	1 8361	1 5525	2 8088	4 3607
1 0	3 1416	3 1416	3 2899	1 3634	3 1743	4 3278
1 5	3 6538	2 7141	5 9907	1 2343	3 4879	4 3051
2 0	4 3529	2 4111	11 4025	1 1383	3 7691	4 2900
2 5	5 3553	2 1872	23 4065	1 0625	4 0299	4 2817
3 0	6 8969	2 0182	54 1825	1 0000	4 2788	4 2788
3 5	9 5358	1 8906	152 884	0 9465	4 5237	4 2817
4 0 .	14 9716	1 7972	622 408	0 8992	4 7734	4 2922
4 5	31 8365	1 7378	6189 47	0 8558	5 0416	4 3146
5 0 .	∞	1 7321	∞	0 8136	5 3727	4 3712

example, equations (19) and (20). It can be shown from the analysis of Fowler (1964*a*) and Fowler and Hoyle (1964) that, for massive stars,

$$\frac{\langle \mu\beta \rangle}{(\mu\beta)_c} = \frac{\int \theta^{(5n+1)/4} \xi^2 d\xi}{\int \theta^{n+1} \xi^2 d\xi} , \qquad \textbf{(A37)}$$

where θ and ξ are the customary dimensionless variables in the Lane-Emden equation for the polytrope of index n and at the center (c) of the star

$$(\mu\beta)_c = \left[\frac{3}{4\pi}(n+1)^3 \frac{\mathfrak{R}^4 M_n^2}{a G^3 M_\odot^2}\right]^{1/4} \left(\frac{M_\odot}{M}\right)^{1/2} . \qquad \textbf{(A38)}$$

Numerical values for $(\mu\beta)_c(M/M_\odot)^{1/2}$ and $\langle\mu\beta\rangle\,(M/M_\odot)^{1/2}$ are tabulated in Table A2. Note that the latter quantity is approximately independent of n. In Table A2 R_n and M_n are the constants of integration corresponding to radius and mass, respectively, for the Lane-Emden equation. For μ = constant, equation (A37) yields $\bar{\beta}/\beta_c$.

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