A MODEL OF FERMI ACCELERATION AT SHOCK FRONTS WITH AN APPLICATION TO THE EARTH'S BOW SHOCK*

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ABSTRACT

A model of first-order Fermi acceleration at shock fronts is developed. A hydromagnetic shock is assumed to be propagating toward an isolated magnetic mirror in an otherwise uniform magnetic field which is not parallel to the shock front. The behavior of an ensemble of particles trapped between the mirror and the shock is studied, and a description of the balance of particle injection into the trapping region, loss through the shock or mirror, and energy gain is obtained. Injection criteria are discussed, and an attractive source of particles eligible for acceleration is shown to be the quasi-thermalized plasma behind the shock, a small fraction of which may leak out in front of the shock and be accelerated.

Application of the model to the Earth's bow shock leads to an attractive model for the energetic electron pulses which have been observed outside of the magnetosphere. If a small fraction of the $\sim$1-keV electrons in the quasi-thermalized plasma behind the shock escape to the region in front of the bow shock, many quantitative features of the observed pulses are explained by a model in which each pulse is due to the acceleration of these electrons by a mirror approaching the bow shock. In particular, the observed cutoff in $>$30-keV pulses a few Earth radii beyond the shock is readily explained. The model predicts, further, that the cutoffs for more energetic particles should be at smaller distances from the shock. The observed energy spectra are in excellent agreement with the model. Some critical observations are suggested.

I. INTRODUCTION

The first part of this paper develops a simple theory of charged-particle acceleration by hydromagnetic shock fronts. The discussion is a development of some ideas recently suggested by Jokipii and Davis (1964), who proposed Fermi acceleration at the Earth's bow shock as a promising explanation of energetic electron pulses observed beyond the magnetopause (Fan, Gloeckler, and Simpson 1964; Frank and Van Allen 1964; Anderson, Harris, and Paoli 1965; Montgomery, Singer, Conner, and Stogsdill 1965). The present discussion is split into two parts. First, a general approach to the behavior of particles trapped between a shock and a magnetic mirror is developed. A differential equation describing the average behavior of an ensemble of trapped particles in energy and time is derived and a general solution presented. Useful limiting forms of the solution are discussed. Finally, conditions required for the acceleration of particles are discussed and possible sources of eligible particles considered. The second part of the paper is devoted to an application of the theory to the interpretation of energetic electron pulses observed beyond the magnetosphere.

Central to much of contemporary astrophysics is the problem of electromagnetic acceleration of charged particles. Two basic acceleration mechanisms involving large-scale magnetic fields have been applied in many models. They are betatron acceleration in an increasing magnetic field as proposed by Swann (1933), and the Fermi mechanism involving interaction with moving magnetic irregularities (Fermi 1949, 1954). Most attention has been directed toward statistical aspects of the Fermi mechanism and the question of how rapidly particles gain energy over a sequence of many interactions. Shock fronts have been introduced mainly as a mechanism for scattering particles in pitch angle

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(Parker 1958). Wentzel (1963) and Parker (1958) have also considered statistical aspects of acceleration between series of converging shocks. Axford and Reid (1963) have suggested that compression of energetic protons between the Earth's bow shock and an approaching interplanetary shock front may explain certain polar-cap absorption events, although a detailed model was not worked out. In this paper we are interested in the general problem of first-order Fermi acceleration of particles trapped between a single-plane shock front and an isolated magnetic irregularity (which may or may not be another shock) moving toward it.

Because much of the following discussion depends on the existence of a shock front in an essentially collisionless plasma, a brief digression on the nature of the assumed shock is in order. Observations point to the existence of irreversible changes in fluid parameters, analogous to fluid-dynamical shocks, in collisionless magnetized plasmas such as the solar wind (Ness, Scearce, and Seek 1964; Bridge, Egidi, Lazarus, Lyon, and Jacobson 1964; Sonett, Colburn, Davis, Smith, and Coleman 1964). Although it is not yet clear to what extent the fluid-dynamical analogy is correct for such a shock, fluid-dynamical terms will be used below to describe it. None of the results derived, however, depends on a definite model of a shock—all that is required is the observed relatively thin propagating front, behind which is a turbulent magnetic field and a quasi-thermalized plasma.

II. ACCELERATION OF CHARGED PARTICLES

Consider a situation in which a shock propagates into a magnetic field $\boldsymbol{B}_0$ which is not parallel to the (locally) plane shock front. There then exists a coordinate system in which the shock front is stationary and the plasma flow velocity is either parallel or antiparallel to the average magnetic field on either side of the shock (Bazer and Ericson 1959). We will work entirely in this coordinate system. $\boldsymbol{B}_0$, the magnetic field on the side in which the plasma flows toward the shock, is assumed uniform except for a small region of irregular field ($B_M > B_0$) which acts as a magnetic mirror and which is a distance $L(t) = L_0 - \beta_0 c(t - t_0)$ along $\boldsymbol{B}_0$ from the shock. The mirror may be stationary with respect to the incoming plasma, in which case $\beta_0 c$ is simply the plasma flow velocity; or the mirror may be moving relative to the plasma so that $\beta_0 c$ may differ from the plasma velocity. Figure 1 illustrates the assumed configuration. We initially consider the shock to be analogous to a "fast" hydromagnetic shock (Bazer and Ericson 1959), for which the average post-shock magnetic-field intensity B_1 is greater than B_0. An energetic particle incident from B_0 onto either B_1 or B_M may be reflected from the region of stronger field and be trapped for a while between the shock and the mirror. Suppose for the present that the mirror conserves the magnetic moment of the particle. The interaction of such a particle with a moving magnetic mirror is summarized in Appendix I. It is found that in the limit that β_0^2 can be neglected compared to unity and the particle velocity $v_p \gg c\beta_0$ each reflection from the mirror increases the particle's parallel momentum $P_{||}$ and changes the particle's total energy W by $2\beta_0 c P_{||}$. The reflection from the shock front is more complex, and a general treatment would require detailed knowledge of the shock structure. In the reference system chosen, the average electric fields at the shock are curl-free, and we assume the particle gains, on the average, no energy upon reflection at the shock front. Under the assumption that a particle with pitch angle $\theta_p = \cos^{-1}|P_{||}/P| = \cos^{-1}\mu$ makes $\mu v_p/2L(t)$ round trips per second, where $v_p = c\sqrt{(1 - m_0^2c^4/W^2)}$ is the particle velocity, the particle increases its energy at the rate

$$\frac{dW}{at} = \beta_0 c \mu^2 \frac{W(1 - 1/W^2)}{L(t)}, \tag{1}$$

where W has been expressed in units of m_0c^2. In general $L(t)$ will depend upon μ and the structure of the mirror field. Below we will be considering many particles, assuming their average behavior at a given energy W to be given by equation (1) averaged over pitch angle. We therefore refer to an average $L(t)$. As $L(t) = L_0 - \beta_0 c(t - t_0)$ approaches the particle gyroradius, equation (1) begins to break down and the rate of energy gain is no longer simply related to $L(t)$.

In addition, we will neglect collisions with the ambient plasma. In § III below, the restrictions this places on the ambient plasma and magnetic-field configuration are considered, and possible sources of particles eligible for acceleration are discussed.

In Appendix I the conditions under which a moving magnetic mirror reflects particles

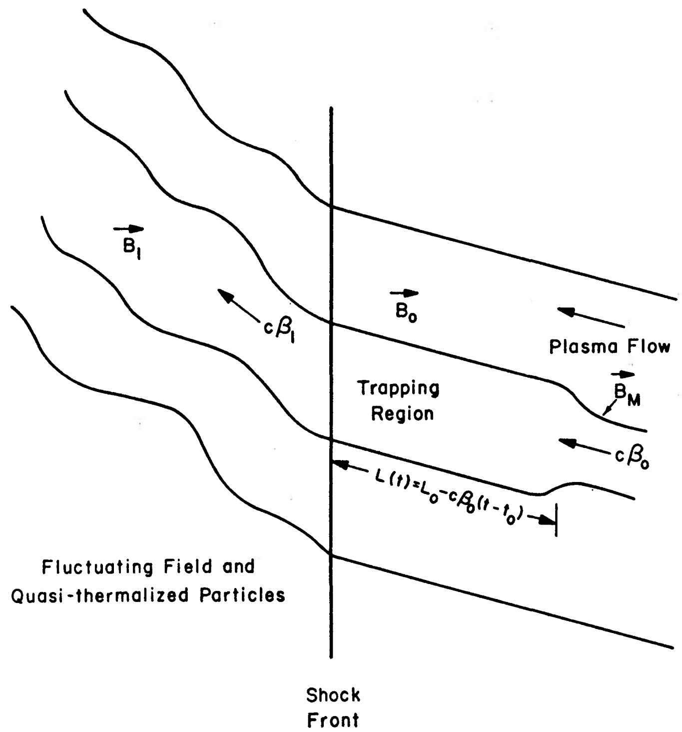

FIG. 1.—Schematic diagram illustrating the assumed magnetic field configuration and relevant parameters.

are briefly discussed. If β_0^2 can be neglected compared to unity, the mirror in general reflects those particles for which, in B_0, $\mu < \mu_c$ where μ_c is given by

$$\frac{1-\mu_c^2}{[\mu_c + \beta_0(1 + m_0^2 c^2/P^2)^{1/2}]^2} = \frac{B_0/B_M}{1 - B_0/B_M}, \tag{2}$$

and allows all others to be transmitted. Note that, as the particle velocity v_p becomes less than or of the order of $c\beta_0$, this is a very stringent condition on μ and few particles are reflected. We are chiefly interested in the other limit, $v_p \gg c\beta_0$, for which equation (2) reduces to $\mu_c \sim \sqrt{(1 - B_0/B_M)}$. If the shock front also conserved the magnetic moment, a particle would gain energy at the rate given by equation (1), but the attendant increase of $\mu = |P_{||}/P|$ would lead to loss of the particle at a relatively low energy. If, however, the shock were thin compared with r_g, the gyroradius, or if the fields at the shock fluctuate rapidly compared with the gyrofrequency, the particle would be scattered in pitch angle. Since it may be scattered to larger pitch angle (smaller μ), it then is possible to accelerate some particles to very high energies. Similarly, if the mirror violated the magnetic moment of the particle, there would be scattering in pitch angle at the mirror. In the latter case, equation (1) may not correct for each particle. It appears reasonable, however, that the average of equation (1) over a large number of trapped particles may be a reasonable approximation to the average rate of energy gain. The validity of equation (1) for $L(t) \gtrsim r_g$ will be assumed in what follows.

We are therefore led to consider an ensemble of trapped particles. Let $n(W,t)dW$ be the total number of particles with energy in the range W to $W + dW$ which are found in the trapping region at time t, per cm^2 normal to $\boldsymbol{B}_0$. Then $\rho_t(W,t) = n(W,t)/L(t)$ gives the number density of trapped particles. The average energy gain per particle of energy W is assumed to be given by equation (1) if it is averaged over pitch angle. We neglect the spread in energy caused by the spread in μ over the ensemble. This requires that particles be stirred sufficiently in pitch angle that, after many reflections, it is unlikely that a particle has gained appreciably more or less than the average energy, $\langle \Delta W \rangle$. Suppose that scattering is quite efficient and the particle gains $\delta W = \langle \delta W \rangle \pm F$ per cycle, where F is of the order of $\langle \delta W \rangle$ and is assumed to be completely uncorrelated from cycle to cycle. If it is first assumed that F and $\langle \delta W \rangle$ are constant, a straightforward application of a random-walk analysis yields the familiar result that, if ΔW is the energy change in N cycles, $\langle \Delta W^2 \rangle - \langle \Delta W \rangle^2 \simeq NF^2 \simeq N\langle \delta W \rangle^2$. Since $\langle \Delta W \rangle = N\langle \delta W \rangle$, the rms fluctuation $\pm \langle \delta W \rangle \sqrt{N}$ is relatively unimportant for large N. A similar analysis may be carried out if $\langle \delta W \rangle$ is a slowly varying function of energy. We are therefore apparently justified in neglecting the fluctuations about $\langle \Delta W \rangle$.

To account for the loss of particles, introduce a loss factor X, defined as the probability per cycle that a particle is scattered into the loss cone of the mirror or otherwise lost from the trapping region. If the scattering were sufficient to keep the pitch-angle distribution nearly isotropic, the loss at the mirror would be X_i. In the limit that the particle velocity $v_p \gg c\beta_0$, it is shown in Appendix I that $X_i \simeq 1 - \sqrt{(1 - B_0/B_M)}$. Since it is unlikely that particles are preferentially scattered into the loss cone of the mirror, we may conclude that X_i represents an upper bound on X. For X_i assumes that on each cycle the loss cone is uniformly populated; but if scattering or the change in pitch angle due to acceleration are insufficient to "refill" the loss cone on each succeeding cycle, the fraction of particles in the loss cone may be substantially less than in the isotropic case. It is therefore probable that $X \lesssim X_i$.

If particles in the energy range W to $W + dW$ are introduced into the trapping region at a rate $S(W,t)dW$ cm^{-2} sec^{-1}, the conservation of particles may be expressed, in terms of the above definitions, as

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial W}\left[\frac{aW}{L(t)}\left(1 - \frac{1}{W^2}\right)n\right] - \frac{bn}{L(t)}\left(1 - \frac{1}{W^2}\right)^{1/2} + S(W,t), \tag{3}$$

where for convenience W is again expressed in units of m_0c^2 and we have defined $a = \beta_0 c\langle \mu^2 \rangle$, $b = cX\langle \mu \rangle/2$ which will be assumed constants, independent of W and t. For an isotropic pitch-angle distribution, $\langle \mu^2 \rangle = \frac{1}{3}$ and $\langle \mu \rangle = \frac{1}{2}$. Equation (3), representing the behavior averaged over pitch angle, should give a reasonable approximation to the behavior of the particle as a function of energy. It is, of course, possible to attempt to obtain the dependence on pitch angle through assumptions on the scattering and use of a Fokker-Planck equation. Such an analysis depends on the detailed form of the scattering in pitch angle and is probably not useful at this stage. The general solution to equation (3) is readily obtained in terms of an integral of Green's function n_g over the source function, as is discussed in Appendix II. The boundary conditions are chosen to correspond to the physical situation in which there are no trapped particles at $t = t_0$. At this time, either S is turned on or S is non-zero but the magnetic mirror becomes connected to the shock. Also, we assume no trapped particles of zero kinetic energy. Thus, equation (3) is to be solved for $t > t_0$, $W > 1$ with $n(W,t_0) = n(1,t) = 0$. The general solution is derived in Appendix II and can be expressed formally as

$$n(W,t) = \frac{WL(t)[W+\sqrt{(W^2-1)}]^{-b/a}}{(W^2-1)^{(1-c\beta_0/2a)}}\int_1^\infty dW_1 \frac{[W_1+\sqrt{(W_1^2-1)}]^{b/a}S(W_1,t')}{a(W_1^2-1)^{c\beta_0/2a}} \times \theta(W-W_1)\,\theta\Big[\Big(\frac{L(t)}{L_0}\Big)^{-2a/c\beta_0} - \frac{W^2-1}{W_1^2-1}\Big], \tag{4a}$$

where

$$t' = \frac{L_0}{\beta_0 c} + \Big(\frac{W^2-1}{W_1^2-1}\Big)^{c\beta_0/2a}\Big(t - t_0 - \frac{L_0}{\beta_0 c}\Big) + t_0 \tag{4b}$$

and the step function $\theta(x) = 1$ if $x > 0$; 0 if $x < 0$. Obviously $n(W,t)$ may be quite complex, but it is in principle obtainable from equation (4) if $S(W,t)$ is known. In spite of the complexity of equation (4), some very useful results can be obtained if reasonable, heuristic assumptions are made concerning $S(W,t)$. These are considered in order of decreasing generality.

If, as may usually be expected, $S(W, t)$ has a cutoff energy W_2, above which S is essentially zero, it then follows from equation (4) that there is a corresponding cutoff in $n(W,t)$ at $W_{\max}(t)$, where

$$\frac{W^2_{\max}(t)-1}{W_2^2-1} = \Big[\frac{L(t)}{L_0}\Big]^{-2a/c\beta_0}. \tag{5}$$

This cutoff energy, which increases as the mirror approaches the shock, is due to the requirement that a particle makes many round trips of duration $2L(t)/c\mu\sqrt{(1-1/W^2)}$ to increase its energy. This may also be derived by a direct application of equation (1).

It is also quite reasonable that $S(W,t)$ has addition a lower cutoff at $W_1 > 1$, below which it is also zero. If $S(W,t)$ may also be regarded as constant over a time required to accelerate particles from W_1 to the energies of interest, one finds the very useful result that over the energy range $W_2 < W < W_3(t)$, the shape of the energy is independent of time and of the detailed form of S. For W_a and W_b in this range,

$$\frac{n(W_a)}{n(W_b)} = \frac{W_a}{W_b}\Big(\frac{W_b^2-1}{W_a^2-1}\Big)^{(1-c\beta_0/2a)}\Big[\frac{W_a+\sqrt{(W_a^2-1)}}{W_b+\sqrt{(W_b^2-1)}}\Big]^{-b/a} \tag{6}$$

and $W_3(t)$ is given by

$$\frac{W_3^2-1}{W_1^2-1} = \Big[\frac{L(t)}{L_0}\Big]^{-2a/c\beta_0}. \tag{7}$$

There are, of course, no particles with energies below W_1. To facilitate comparison with observed spectra, we note that under the assumption that the pitch-angle distribution is independent of energy the flux is proportional to $v_p n = n(W)c\sqrt{(1-1/W^2)}$. Thus, if $F(W)$ is the flux at energy W, we have

$$\frac{F(W_a)}{F(W_b)} = \Big(\frac{W_b^2-1}{W_a^2-1}\Big)^{(1-c\beta_0/a)/2}\Big[\frac{W_a+\sqrt{(W_a^2-1)}}{W_b+\sqrt{(W_b^2-1)}}\Big]^{-b/a}. \tag{6'}$$

If E is the kinetic energy, $W = 1 + E$, and the flux in the non-relativistic limit is readily shown to be

$$\frac{F(E_a)}{F(E_b)} = \Big(\frac{E_b}{E_a}\Big)^{(1-c\beta_0/a)/2}\exp\Big\{-\frac{b}{a}[\sqrt{(2E_a)} - \sqrt{(2E_b)}]\Big\}. \tag{6''}$$

Between $W_3(t)$ and $W_{\max}(t)$, $n(W,t)$ decreases to zero in a manner dependent on S. Above $W_{\max}(t)$, of course, $n(W,t)$ is zero. As $W_3(t)$ and $W_{\max}(t)$ increase with time (or decreasing $L(t)$) we have a constantly lengthening region in energy for which the spectrum is given by equation (6). This is a particularly interesting result if an essentially time-independent $S(W,t)$ is sharply peaked in energy at $W \simeq W_2 \simeq W_1$. In this case equation (6) gives the (time-independent) spectrum except for the constantly increasing cutoff at $W_{\max}(t) \simeq W_3(t)$. This last assumption, although stringent, may be used to give a zeroth-order approximation to the spectrum if $S(W,t)$ is not well known. In principle, observations at the cutoff energies would give the energy dependence of S, if it were time-independent. In § III it is shown that the quasi-thermalized plasma behind the shock may provide an attractive source with the desired properties of time independence and a sharply peaked spectrum.

The assumptions introduced make the derived spectrum only approximate. The cutoff at $W_{\max}(t)$ will be sharp only to the extent that it is improbable that a particle gain appreciably more or less than the average energy. It has been shown that the rms spread in energy is of the order of $W/\sqrt{N}$ and may therefore be negligible if the number of cycles N is large. This is facilitated by a small fractional gain in energy per cycle and therefore may be a better approximation for electrons than protons at low energies.

Also, as stated above, the analysis breaks down when $L(t)$ approaches r_g. The situation for $L(t) \lesssim r_g$ is difficult to analyze and we will confine attention to cases where $L_0 \gg r_g$ and $L(t)$ only approaches r_g a negligible time before $L(t)$ reaches zero. The main effect, then, is a limitation in the maximum energy attainable in a given event. This maximum energy $W_{L=0}$ can be roughly approximated by substituting $L(t) \simeq r_g = Pc/eB_0$ in equation (5):

$$(W^2{}_{L=0} - 1)^{1+a/c\beta_0} \simeq (W_2{}^2 - 1)\left[\frac{L_0 eB_0}{m_0 c^2}\right]^{2a/c\beta_0}. \quad (8)$$

This, although crude, should give a reasonable estimate of the allowed energies.

III. SOURCES OF PARTICLES TO BE ACCELERATED

Until now the existence of particles that can be trapped and accelerated has been rather cavalierly assumed. It remains to determine the conditions under which a given particle may be accelerated and then to consider possible sources of eligible particles. The conditions that a particle must meet are that it have a reasonable probability of being reflected from the shock or the mirror and that the energy gains according to equation (1) offset the losses.

Consider first the reflection process. As discussed in Appendix I, losses at the mirror begin to be large for particles whose velocities are not somewhat larger than $c\beta_0$. This effectively rules out the acceleration of thermal ions since $c\beta_0$ is of the order of the shock velocity relative to the plasma and is therefore likely to be greater than the mean thermal ion velocity. The acceleration of the bulk (or thermal) ions and electrons is also ruled out because they cannot be reflected at the shock front. For reflection of an appreciable fraction of the plasma would not be consistent with the existence of a shock front. In the absence of detailed knowledge of the shock structure it is not possible to be more explicit and it will be assumed that particles (either ions or electrons) must satisfy both of the above-mentioned general constraints to be effectively accelerated. They must be both suprathermal and have velocities appreciably greater than $c\beta_0$. One may conclude that injection of electrons is favored over protons or other ions because at a given suprathermal energy the electrons will have a far greater velocity.

Collisional energy losses can be treated more explicitly. We consider only the situation in which the ambient plasma is fully ionized. First, note that since we are interested in the injection of relatively low-energy particles, it is reasonable to neglect synchrotron radiation and bremsstrahlung. The effects of these on relativistic particles are considered in the next paragraph. Since collisional losses in a fully ionized plasma decrease with increasing energy, it is sufficient to require that at $t = t_0$.

$$-\left(\frac{dW}{dt}\right)_{\text{collisions}} < \left(\frac{dW}{dt}\right)_{\text{acceleration at } t=t_0} . \tag{9}$$

Spitzer (1956) gives the characteristic energy-loss time for non-relativistic test particles as

$$t_E \simeq \frac{1}{G[\sqrt{(m_1 v_p^2/2kT)}]} \frac{m_p^2 v_p^3}{32\pi e^4 Z_1^2 Z_p^2 n_1 \ln \Lambda}, \tag{10}$$

where

$$G(x) = \left[\int_0^x \exp(-y^2)dy - x\exp(-x^2)\right] / x^2\sqrt{\pi}$$

is tabulated by Spitzer (1956), ln Λ is the usual Coulomb logarithm and n_1, m_1, and T refer to the field particles through which the test particle, m_p, v_p is moving. Spitzer also notes that for $v_p \gg \sqrt{(2kT/m_1)}$, $G \ln \Lambda$ must be replaced by $0.5\ (1 + m_1/m_p)^{-2}$. Setting

$$\left(\frac{dW}{dt}\right)_{\text{collision}} \simeq -\frac{E}{t_E},$$

where $E = W - m_0c^2 \ll m_0c^2$ is the particle kinetic energy and using the non-relativistic limit of equation (1), equation (9) becomes

$$\frac{\beta_0 c \langle \mu^2 \rangle m_p^2 v_p^3}{n_1 L_0} > 16\pi e^4 Z_1^2 Z_p^2 G[\sqrt{(m_1 v_p^2/2kT)}] \ln \Lambda . \tag{11}$$

If inequality (11) is satisfied for a given group of test particles in the ambient plasma, and if they also satisfy the general conditions for reflection obtained in the preceding paragraph, they are then subject to acceleration as discussed in § II.

As an example of the magnitudes involved, consider a typical shock traveling at 1000 km sec^{-1} through an ambient plasma at a temperature of 10^{5} °–10^{6} ° K. It will accelerate 1-keV electrons if $n_1 L_0 \lesssim 10^{18}$ cm^{-2} and 1-keV protons if $n_1 L_0 \lesssim 10^{19}$ cm^{-2}. For nominal L_0 values of 10^{10}–10^{12} cm, this is clearly not a serious restriction until densities approach coronal values of 10^7 cm^{-3}.

For completeness we briefly consider bremsstrahlung and synchrotron radiation losses for relativistic electrons. If $W \gg m_0c^2$, the losses for a particle in a proton-electron gas of density n_1 cm^{-3} and magnetic-field intensity B can be written

$$-\frac{dW}{dt} = 1.37 \times 10^{-16} n_1 W \left(\ln \frac{W}{m_0 c^2} + 0.36\right) \quad \text{(bremsstrahlung)},$$

$$-\frac{dW}{dt} = 1.57 \times 10^{-15} B^2 \left(\frac{W}{m_0 c^2}\right)^2 \quad \text{(synchrotron)},$$

where cgs units are used throughout (see, e.g., Ginzburg and Syrovatsky 1964). These are to be compared with the relativistic form of equation (1),

$$\frac{dW}{dt} = \beta_0 c \frac{\mu^2}{L(t)} W .$$

Putting in a nominal value of 1000 km sec^{-1} for the shock velocity $c\beta_0$, one finds that bremsstrahlung and synchrotron radiation are unimportant unless, respectively, $n_1 L \gtrsim$

10^{22} cm^{-2} for bremsstrahlung and $LB^2W/m_0c^2 \gtrsim 10^{16}$ cm gauss^2 for synchrotron radiation. For most applications these limits will be well satisfied.

It is clear that ambient energetic particles and a few suprathermal plasma particles may satisfy the requirements and be accelerated, although they are not likely to be numerous except under special circumstances. A much more attractive source is the thermalized plasma behind the shock, some particles of which may escape to the front of the shock and be accelerated. If a plasma of mean particle mass $\langle m \rangle$ flows into the shock at velocity v_0 and leaves at v_1, energy conservation indicates that the mean increase in thermal energy per particle across the shock is of order $\Delta E \simeq f\langle m \rangle (v_0^2 - v_1^2)/2$, where f is a factor of order unity depending on the thermal degrees of freedom behind the shock and the form of equipartition. This expectation is supported by observations of keV electrons behind the Earth's bow shock (Freeman, Van Allen, and Cahill 1963). For strong shocks in a relatively cold plasma, this represents an appreciable increase, particularly for electrons. This source is possibly not very efficient for the ions in the plasma since $\Delta E \simeq f\langle m \rangle (v_0^2 - v_1^2)/2$ may not be sufficient to make the ion velocity much greater than $c\beta_0 \sim v_0$, as required for efficient reflection at the mirror. It is reasonable to conjecture that a fraction of these particles may escape along the magnetic field to the trapping region ahead of the shock, providing an $S(W,t)$ that is very nearly time-independent and which may be assumed to be peaked at a kinetic energy of the order of $\langle m \rangle (v_0^2 - v_1^2)/2$ or a few keV for a shock moving at 1000 km sec^{-1}. The energy spectrum of energetic particles may therefore be well represented by equations (6) and (7). It will be found in the latter part of this paper that this source for electrons is strongly indicated for the Earth's bow shock.

IV. PARTICLES BEHIND THE SHOCK

Having determined the spectrum in the trapping region, we proceed to consider the behavior of particles behind the shock. Since we are considering an essentially collisionless plasma, the details of the shock structure are not known. It is expected that the magnetic field in the region immediately behind the shock front is rapidly fluctuating, and this expectation is confirmed by observations behind the Earth's bow shock (Sonett and Abrams 1963; Ness, Scearce, and Seek 1964). In such a turbulent field it is convenient to treat the average particle motion in the diffusion, or random-walk, approximation with mean free path λ determined by the magnetic-field structure. It will be assumed that λ is somewhat greater than the particle gyroradius, in which case the motion is mainly along the average postshock magnetic field B_1, with a diffusion coefficient $K \simeq \lambda v_p$. It should be noted that the qualitative nature of the conclusions will not be changed if the diffusion is three-dimensional. Attention is confined to that region behind the shock in which λ is constant. The field fluctuations may damp out a long distance behind the shock, but this will not be considered. Then the particle number density $\rho(t,x,W)dW$ cm^{-3}, as a function of time, the distance x along B_1 from the shock front ($x = 0$), and energy, is obtained from the one-dimensional diffusion equation

$$\frac{\partial \rho}{\partial t} = +K \frac{\partial^2 \rho}{\partial x^2} - c\beta_1 \frac{\partial \rho}{\partial x}, \tag{12}$$

where $c\beta_1$ is, in analogy with $c\beta_0$, the average velocity of the magnetic-field irregularities behind the shock. The second term on the right side of equation (12) thus gives the effect of the convective motion of the scattering centers. The relative motion of the irregularities, which could give rise to further Fermi acceleration behind the shock, will be neglected in this treatment. Equation (12) is to be solved for $t > t_0$, $x > 0$, subject to the boundary conditions that $\rho(t_0,x,W) = 0$ and $\rho(t,0,W) = \rho_t(W,t)$, the density in the trapping region as determined from equation (4). The last boundary conditions would have to be modified slightly if the flux of trapped particles through the shock front into the diffusion region were less than the flux at $x = 0$ predicted by equation

(12). This case will not be treated because it implies that the effective reflection coefficient of the shock is low, of order $r_s = (1 - B_0/B_1)$, and acceleration inefficient. The general solution to equation (12) is easily found by standard mathematical techniques, as discussed briefly in Appendix III. The present discussion is confined to a consideration of the general nature of the solutions.

As discussed in § II, conditions are usually such that there is a maximum energy at a given time which is given by equation (5). For any given W above the injection energy, therefore, $\rho(t,0,W) = 0$ for $t < t_W$ where $W_{\max}(t_W) = W$. Further, if the source function for injected particles is essentially constant and is peaked at $W = W_2$, then $\rho(t,0,W)$ rises sharply at t_W to a constant value and remains there until the mirror is engulfed by the shock. The solution for this case is discussed in Appendix III. The particle diffusion velocity at $x = 0$ is shown to be

$$(v_D)_{x=0} = \left[\frac{K}{\pi(t-t_W)}\right]^{1/2} \exp\left[-\frac{c^2\beta_1^2}{4K}(t-t_W)\right] - \frac{c\beta_1}{2}\, erfc\left[\frac{c\beta_1}{2}\left(\frac{t-t_W}{K}\right)^{1/2}\right] + c\beta_1 . \tag{13}$$

Consider first $\beta_1 > 0$, so that the irregularities essentially move with the plasma, and assume K small enough so that the time for $(v_D)_{x=0}$ to approach $c\beta_1$ is small. The flux is then mostly due to convection of particles with magnetic irregularities. Clearly this result is also true for any $\rho(t,0,W)$ that does not change appreciably over the time it takes $(v_D)_{x=0}$ to become essentially equal to $c\beta_1$. The energy spectrum of the accelerated particles close behind the shock should, therefore, in the absence of rapid changes in $\rho(t,0,W) = \rho_t(W,t)$, be essentially the same as in the trapping region. In addition, if $c\beta_1 \ll v_p(W)$, the trapped-particle velocity, most of the particles incident on the shock front are reflected. The effective reflection coefficient of the shock can then be written for $\beta_1 > 0$ as

$$r_s \simeq 1 - \frac{c\beta_1}{v_p(W)\langle\mu^3\rangle}, \tag{14}$$

where $\langle\mu^3\rangle = \frac{1}{4}$ for an isotropic distribution. We see that particles can be efficiently accelerated only if $c\beta_1 \ll v_p(W)$. Since ions are much slower than electrons at a given nonrelativistic energy, we again expect electrons to be accelerated more efficiently than protons or other ions. It is also clear that it is not necessary to restrict consideration to "fast" shocks, where the average field is larger behind the shock. Also if $\beta_1 < 0$ the flux through the shock would be reduced still further. However, application to the Earth's bow shock in the latter part of this paper suggests that $\beta_1 > 0$.

At time $t_c = t_0 + L_0/\beta_0 c$, $L(t) = 0$ and the magnetic mirror is convected through the shock front. It accelerates no more particles and $\rho_t \simeq 0$ for $t > t_c$. This will, of course, affect the structure of the shock, but the effect may not be important. If the effects on the shock are neglected, we can treat the behavior of the particles as above except that the boundary condition at $x = 0$ becomes $\rho(t,0,W) \simeq 0$ for $t > t_c$. This is again a well-defined mathematical problem and can be treated by the techniques of Appendix III. The particles stay on the same line of force but now escape slowly from the end at $x=0$. Since the diffusion coefficient is $K \simeq \lambda v_p$, it is expected that the decay of the intensity is faster for more energetic particles so that the energy spectrum softens (steepens) with distance from the shock.

The density of particles along the line of force at a given energy is schematically depicted in Figure 2 at two different times, $t_W < t < t_c$ and $t_c < t$, where it should be remembered that the intensity is zero everywhere for $t < t_W$.

It has been demonstrated that the irregular magnetic fields behind the shock may dramatically reduce the flux of particles through the shock, causing the shock front to have effectively unit reflectivity. However, that fraction of reflected particles which has penetrated to the diffusion region may gain more or less energy than that which is reflected at the shock proper. There are two reasons for this. The particles will take longer to complete a cycle; and the irregularities behind the shock may be moving. The first of these effects is probably small since the mean depth of penetration into the region behind the shock is of the order of a few times λ and will be assumed to be much less than $L(t)$.

The second effect is more important and may make the model relatively inefficient for weak shocks. The importance of the effect may be estimated as follows. The fraction of particles that are reflected at the shock front proper is about $r_s \simeq 1 - B_0/B_1$, since for

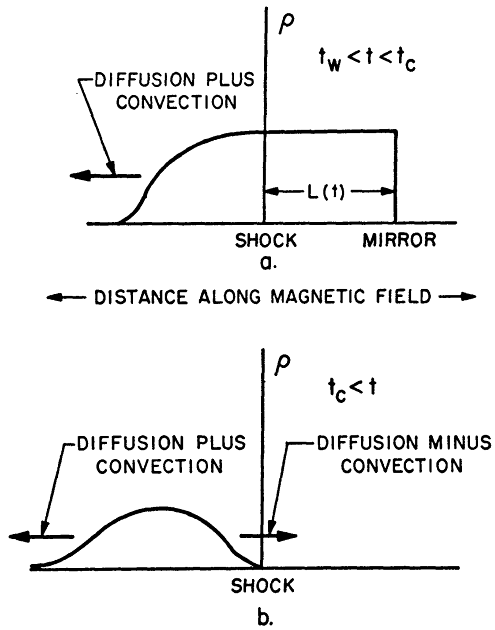

FIG. 2.—A schematic sketch of the variation of ρ along a given field line for energy W at two different times. As discussed in the text, $\rho = 0$ for $t < t_W$. (*a*) indicates the behavior of ρ for a time t such that $t > t_W$, but the mirror is still some distance from the shock. As indicated by the heavy arrow, particles have begun to move into the region behind the shock by a combination of diffusion and convection. (*b*) shows the situation for a time $t > t_c$ so that the mirror has been absorbed by the shock. Particles now diffuse in both directions in addition to being convected by the magnetic irregularities behind the shock front.

an isotropic pitch-angle distribution this is independent of the conservation of magnetic moments (Wentzel 1964). For these particles, the energy gain is given by equation (1). It is readily determined that the remainder of the particles gain more or less by the factor $(\beta_0 - \beta_1)/\beta_0$. A measure of the efficiency of the shock is then

$$q = 1 + \frac{\beta_1}{\beta_0}(r_s - 1), \tag{15}$$

which may be of order unity for a strong shock and be nearly zero for a weak shock. We will restrict our attention to strong shocks where the efficiency may be taken to be unity.

V. LIMITATIONS ON PARTICLE DENSITY

One might inquire about collective effects of the accelerated particles. In the development of the model, single-particle behavior has been assumed and this clearly limits the density of accelerated particles. A reasonable criterion for the validity of the model is that the trapped-particle kinetic-energy density be substantially less than that of the magnetic field. That is,

$$\int_1^\infty \rho_t(W)(W-1)\,dW < \frac{B_0^2}{8\pi m_0 c^2}, \tag{16}$$

where again W is expressed in units of m_0c^2. If equation (16) is not satisfied, the collective behavior of the particles becomes important and the model is not applicable. The process of acceleration would then itself modify the ambient magnetic-field configuration and perhaps the shock structure.

This depends on the value of the parameters for each case. Clearly if X were nearly zero, the spectrum would be quite flat and equation (16) may not be satisfied. Under most conditions, however, the spectrum will be sufficiently steep so that equation (16) is satisfied.

VI. APPLICATIONS

a) Interpretation of Electron Pulses Observed beyond the Magnetosphere

The theory of Fermi acceleration described above will now be applied to electrons at the Earth's bow shock. Recent observations of energetic electron pulses beyond the magnetosphere are first reviewed. It is shown that the most probable interpretation of the observations is in terms of the transient presence of electrons in an extended region of the transition region between the bow shock and the magnetosphere. It is then noted that the presence of electrons beyond the bow shock and their apparent confinement to a narrow region beyond suggest the presence of magnetic irregularities beyond the shock. We are thus naturally led to an application of the ideas of the model discussed above. It is found that the model leads to a compelling quantitative understanding of most features of the phenomenon. A detailed explanation of the energetic electron events observed in the vicinity of the geomagnetic tail will not be attempted here.

The observations under consideration consist of pulses of $\gtrsim$30-keV electrons which are observed beyond the magnetosphere and, at times, beyond the Earth's bow shock. They consist typically of fluxes of about 10^5 cm^{-2} sec^{-1} ($>$30 keV), or about 10 times background, which last only a few minutes at the satellite and are extremely variable in magnitude and location. The short duration of the pulses indicates that they are of local origin and are probably accelerated in or near the magnetosphere of the Earth. The pulses are seen in the data of Van Allen and Frank (1959). Observations by Fan *et al.* (1964) and Anderson *et al.* (1965) using IMP-I have done much to reveal the relationship of the pulses to the magnetosphere and to the bow shock. Comparison with the IMP-I magnetometer and plasma data shows that the electron pulses occur beyond the bow shock, although they tend strongly not to be found more than a few Earth radii beyond it. In addition, between the bow shock and the magnetosphere, the pulses were observed much more often when far from the subsolar point than when near it. Anderson *et al.* (1965) report further that the pulses tend to be more intense and more frequent near the magnetosphere. Frank and Van Allen (1964) have recently reported similar observations on Explorer XIV. They find the occurrence of the pulses to be positively correlated with k_p and report measurable fluxes of 1.6-MeV electrons (of the order of 10 cm^{-2} sec^{-1} $>$ 1.6 MeV).

More recently, detailed measurements of integral spectra of electrons in the energy range 50 keV $< E <$ 325 keV, together with their spatial and temporal variations at $R \simeq 17\ R_E$, have been reported by Montgomery *et al.* (1965). Their observations near the bow shock and in the transition region are apparently consistent with the earlier observations. They report a striking asymmetry in the occurrence of events between the dawn and evening sides of the magnetosphere, the pulses being appreciably more numerous on the dawn side. They also report events on the night side of the magnetosphere, in the vicinity of the magnetospheric tail, which apparently last for several hours.

How are the elctron pulses to be interpreted? Observations alone cannot answer this conclusively since one is faced with the usual ambiguity between spatial and temporal variations inherent in observations from a single, moving satellite. The extreme variability of the observed pulses argues against their being due to permanent belts of energetic electrons. There are, furthermore, compelling theoretical arguments against interpreting the pulses as due to the presence of electrons in localized regions of space, through which the satellite passes in a few minutes (Jokipii and Davis 1964). Any energetic electrons must drift through the magnetic field with the bulk velocity of the solar wind, at about 400 km/sec. They also tend to spiral rapidly, about 10^{10} cm/sec,

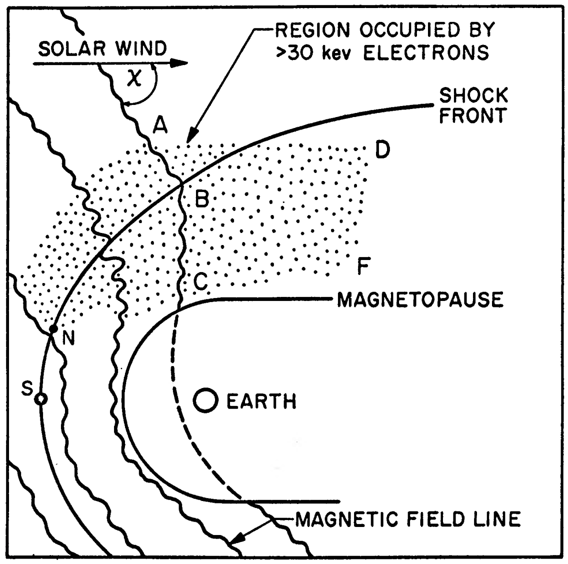

FIG. 3.—Schematic representation. This indicates the suggested general interpretation of a pulse. Acceleration occurs somewhere in the region *NABCN* or perhaps in the magnetosphere. Boundaries *NAD* and *NCF* may not be sharp due to motion of particles along field lines. *N* can occur at various distances from *S*, the subsolar point. The solar wind blows accelerated particles into the region *ABCFD* and beyond. When trapping or acceleration ceases, the whole region drifts downwind.

along the lines of force. Thus, if a localized source acts for more than a few tens of seconds, the flux must be appreciable in an extended region much larger than the 2000 km that a satellite can traverse in a few minutes. It is suggested, therefore, that the pulses are due to the transient presence of electrons in an extended region, as in Figure 3. The duration of their presence is that of the observed pulses and the region typically encompasses much of the transition region between the bow shock and the magnetopause. The motion of the electrons in the transition region will be, as discussed in § IV of this paper with $\beta_1 > 0$, essentially convection with the plasma together with a relatively slow diffusion along the field. Since more pulses were observed in the transition region when far from the subsolar point than when near it, it is reasonable that the region of electrons may on different occasions have its upwind edge at various distances from the subsolar point. The solar wind extends these regions downwind into the transition region. More regions would therefore be likely to sweep past the satellite when it is far from the subsolar point than when near it, explaining the frequency of pulses away from the subsolar point. Similarly, the tendency for pulses to be more intense and numerous near the magnetopause may be a result of the flow around the magnetosphere and the associated deflection and compression of the plasma and magnetic-field lines near the magnetopause.

b) The Acceleration of Electrons at the Bow Shock

Within the above general interpretation of the pulses, it remains to determine the source of the electrons. Several possibilities have been suggested, although detailed models are lacking. It has been proposed that the electrons are of magnetospheric origin (Anderson *et al.* 1965), or that they are heated by plasma instabilities in the transition region (Fredricks, Scarf, and Bernstein 1965). First note that, whatever their source, the energetic electrons must be able to penetrate to the region in front of the shock. The observed cutoff in the pulses at a few times 10^9 cm in front of the shock indicates the presence of magnetic irregularities which reflect the electrons back toward the shock. Since these irregularities probably move relative to the shock (they are probably carried with the solar wind, which has a velocity several times the Alfvén velocity), there is a possibility that the particles are accelerated by a Fermi mechanism, and perhaps most of the acceleration is accomplished in this manner.

We are thus naturally led to the hypothesis that each pulse of electrons is caused by one or more magnetic irregularities approaching the bow shock. The motion of particles behind the shock has already been discussed, and it is clear that the motion is consistent with the hypothesis. The remainder of this paper is devoted to demonstrating that the whole phenomenon can be qualitatively explained by this model. Note that the low ambient density in the solar wind makes it unnecessary to worry about collisional energy losses.

The bow shock is expected to heat particles to energies of the order of $\langle m\rangle(v_0^2 - v_1^2)/2$, or about 1 keV for a solar-wind velocity of 500 km/sec. The existence behind the bow shock of electrons in this energy range is supported by observations of Freeman *et al.* (1963). Consider now the following sequence of events for the general configuration of Figure 3. A field line, moving with the solar wind, has a magnetic irregularity B_M upwind from the bow shock. At time t_0 this field line connects with the shock and a small fraction of the $\sim$1-keV electrons found behind the shock begins escaping along the field toward B_M and is subsequently accelerated as discussed previously. For simplicity, we make the approximations that the bow shock is essentially plane and that the injected electrons are sharply peaked at about 1 keV. The spectrum is then given by equations (6) and (7), with a $W_1 \sim W_2 \sim 1$ keV and b/a determined by the solar-wind conditions. The bow shock is a strong shock, so q in equation (15) will be taken to be essentially unity.

Putting in characteristic dimensions and noting that the interplanetary magnetic field is usually inclined at about 45° to the wind velocity (Ness *et al.* 1964), one finds that a typical L_0 should be of the order of a few times 10^{10} cm, or a characteristic dimension of the shock surface. Setting $W_1 \sim W_2 \sim 1$ keV and using equation (7), one finds that $W_{\max}(t) \simeq 30\text{ keV} + m_0c^2$ (i.e., 30-keV electrons are first produced) corresponds to $L(t)$ of the order of 10^9 cm, or a few R_E. Thus, pulses of 30-keV or higher electrons would tend not to be seen more than a few Earth radii beyond the bow shock, in excellent agreement with the observed cutoff in $>$30-keV pulses. Similarly, the duration of $>$30-keV pulses should be of the order of the time required for the mirror, traveling at the solar-wind velocity of 500 km/sec, to traverse this distance of a few R_E, or the observed few minutes. Note that it follows from equation (8) that for such an event and a 5-γ (1-$\gamma = 10^{-5}$ gauss) magnetic field, the maximum energy $W_{L=0}$ is of the order of 1 MeV. Also in gratifying agreement with observation is the expectation that, since the interplanetary magnetic field is preferentially inclined toward the dawn side of the magnetosphere as in Figure 3, the pulses should occur preferentially on the dawn side of the magnetosphere.

The average statistical features of the observed pulses are thus shown to be in good agreement with the Fermi acceleration hypothesis. That aspect of a single pulse most accessible to measurement is its energy spectrum. The agreement between theory and observations is found below to extend to the details of the energy spectra.

According to the discussion in § IV, the predicted energy spectra during the main phase of an event should be almost the same in the transition region, near the bow shock, as in the trapping region. Before a detailed comparison with the data can be made, two parameters—the flux at some energy and the ratio b/a—must be determined. Either or both vary considerably and depend on as yet unknown parameters of the solar wind and interplanetary magnetic field. In view of this, it is most feasible to make a two-parameter fit to the spectrum in each case and then determine whether the required

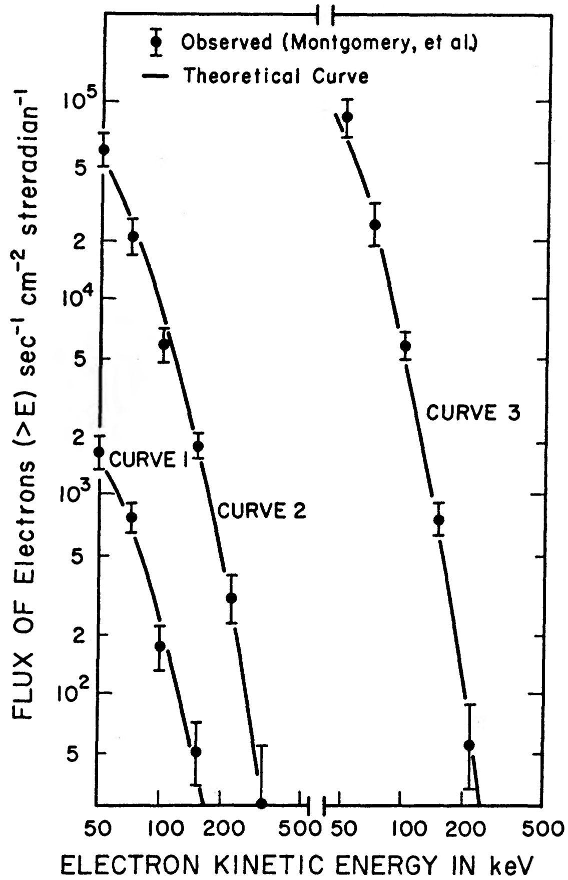

FIG. 4.—The dots with error bars indicate three integral spectra reported by Montgomery *et al.* (1965) at a distance of 17 R_E. The spectra were obtained at three different times during the relatively constant phase of an event located at ecliptic latitude $-5°$, ecliptic longitude 249°, and magnetic latitude 11°. This is probably in the transition region on the morning side of the magnetosphere. The solid lines are two-parameter theoretical fits to the data. The values of b/a required to fit curves are: for curve *1*, $b/a = 18$; for curve *2*, $b/a = 16.5$; and for curve *3*, $b/a = 22.3$.

parameters are reasonable. This was done for three integral electron spectra reported by Montgomery *et al.* (1965) taken at different times during the relatively constant phase of an event. Although the authors did not state whether the observations were made in the transition region or the magnetospheric tail, the reported location of the observations, ecliptic latitude $-5°$, longitude 249°, magnetic latitude 11°, at a distance of 17 R_E indicates that they were probably made in the transition region. The present model should therefore be applicable. Figure 4 displays the results. The dots with error bars are the observed points and the solid lines are two-parameter theoretical fits to the observations, assuming the spectra to be the same as in the trapping region. The predicted integral spectra were obtained from equation (6′) by a three-point Gaussian integration. It was assumed that the spectrum (6′) extended to $W = \infty$ in the integrations. The steepness of the spectrum makes this a reasonable approximation. Because the function is very smooth and well behaved, it is believed that errors in this integration procedure are negligible. One may also utilize equation (6″). This procedure yields very much the same integral spectrum as numerical integration of equation (6′). In the energy range of interest, the spectrum is approximately an exponential rigidity spectrum. The curves are rather insensitive to variations in $\langle\mu^2\rangle$ and the isotropic value of $\frac{1}{3}$ was used.

The agreement is clearly good and is better than a straight line, or power-law, fit. That the shapes of the spectra are in so close agreement indicates that the agreement is meainingful in spite of there being two adjustable parameters.

In addition, Frank and Van Allen (1965) have reported a partial spectrum for one pulse. They find integral fluxes $\simeq 10^6$ cm^{-2} sec^{-1} at $E > 40$ keV; $\simeq 30$ cm^{-2} sec^{-1} at $E > 1.6$ MeV and an upper bound of about 10^4 cm^{-2} sec^{-1} at $E > 230$ keV. These are consistent with a value of about 12 for b/a. The spectrum was therefore somewhat harder (less steep) in this event. Since a harder spectrum corresponds to a larger wind velocity or a larger mirror ratio, it is reasonable that this pulse was unusually intense and occurred on a day with a high k_p index.

Are the required parameters reasonable? The required flux of injected electrons $\gtrsim 1$ keV is of the order of 10^6 cm^{-2} sec^{-1} in each case, so that only about 1 per cent of the approximately 3×10^8 electrons cm^{-2} sec^{-1} incident on the shock need escape to be accelerated. Also the energy density of the accelerated particles >1 keV is of the order of 10^{-11} erg cm^{-3}, or much less than that of a magnetic field of 5 γ. The required values of b/a are not very different as would be expected for the same event. The differences may be due to the mirror ratio being different on different magnetic lines of force. To show that values for b/a in the range 15–20 are consistent with our knowledge of the solar wind, more analysis is necessary.

Note first that the solar wind is usually inclined at an angle to the interplanetary field, as indicated schematically in Figure 3. In order to obtain an estimate for $c\beta_0$, we must transform to a frame in which the shock is stationary and the solar-wind velocity is either parallel or antiparallel to the interplanetary magnetic field. In the limit that β_0^2 can be neglected compared to unity this transformation does not change the magnetic field and so is simple to carry out. As shown in Appendix IV, one obtains, in terms of $\boldsymbol{v}_w$ the wind velocity, the angle χ between $\boldsymbol{B}_0$ and $\boldsymbol{v}_w$ ($\cos\chi = -(\boldsymbol{B}_0\cdot\boldsymbol{v}_w)(\boldsymbol{B}_0\cdot\boldsymbol{n})/B_0 v_w|\boldsymbol{B}_0\cdot\boldsymbol{v}_w|$), and the angle α between $\boldsymbol{n}$, the shock normal and $\boldsymbol{v}_w$ ($\cos\alpha = \pm\boldsymbol{v}_w\cdot\boldsymbol{n}/v_w$ the sign of α being the sign of $|\boldsymbol{B}_0\cdot\boldsymbol{n}| - |\boldsymbol{B}_0\cdot\boldsymbol{v}_w/v_w|$),

$$c\beta_0 = \frac{v_w}{\cos\chi}\left[\frac{1}{1+\sin\alpha\tan\chi}\right]. \tag{17}$$

Assuming a nominal wind velocity of 500 km sec^{-1} and setting $\chi = 45°$, the average spiral angle of the interplanetary magnetic field, we find $\beta_0 \simeq 2.5 \times 10^{-3}$ for $\alpha \simeq 0$ which corresponds to the subsolar point. Away from the subsolar point α increases slowly and β_0 decreases slightly as the mirror approaches the shock. For an isotropic pitch-angle distribution $\langle\mu\rangle \simeq \frac{1}{2}$, $\langle\mu^2\rangle \simeq \frac{1}{3}$, and $b/a \simeq 0.3 \times 10^3 X$. Therefore, values of b/a in the range 15–20 require that X be about 0.05–0.10. Setting $X_i = 0.05$–0.10 suggests that B_M/B_0 must be about 7–10, but, since $X \leq X_i$, this is probably only an upper limit on the required mirror ratio. Because Frank and Van Allen find the pulses to be correlated with k_p and therefore perhaps with v_w (Snyder, Neugebauer, and Rao 1963), these estimates based on quiet-day solar-wind parameters may be conservative.

We find, therefore, that the proposed model of Fermi acceleration is in excellent quantitative agreement with the electron pulses observed near the Earth's bow shock. It is expected that protons should also be accelerated, although because of their larger gyroradii and lower velocities they should be accelerated less efficiently. Evidence for proton events has been reported (Montgomery *et al.* 1965).

Further observations are desirable to further test predictions of this model. Analysis of simultaneous energetic-particle and magnetic-field measurements, both in the transition region and beyond the bow shock would help to determine the magnetic-field configurations associated with the particles. Detailed energy spectra are, of course, essential. A very useful test of the Fermi acceleration hypothesis would be an analysis of the variation of the cutoff distance from the bow shock as a function of energy. Confirmation of the predicted effect would be strong support for this model. The success of this model in explaining efficient acceleration at the Earth's bow shock would appear to make it an attractive mechanism in other situations where shocks may be expected.

I would like to express gratitude to Professor Leverett Davis, Jr., for continued interest and for many valuable suggestions concerning this work. The financial support of the National Science Foundation in the form of a graduate fellowship is gratefully acknowledged. This work was also supported in part by NASA grant NsG 426.

Note added in proof January 24, 1966.—Recent observations (Fan, C. Y., Gloeckler, G., and Simpson, J. A., to be published, *J. Geophys. Res.*, 1966) indicate that the spectra of the electron pulses behind the bow shock become somewhat steeper (soften) with increasing distance from the subsolar point. Thus, exact correspondence between the spectrum calculated for the trapping region (eq. [6]) and pulses observed behind the shock should not be expected. The general conclusions concerning the agreement between the observed and calculated spectra, however, remain valid.

APPENDIX I

REFLECTION OF A PARTICLE BY A MOVING MAGNETIC MIRROR

Consider a particle moving in a magnetic field $\boldsymbol{B}_0$ toward a moving magnetic mirror of intensity $B_M = sB_0$, where $\boldsymbol{B}_M$ is initially assumed parallel to $\boldsymbol{B}_0$. The mirror moves with velocity $c\beta_0$ along $\boldsymbol{B}_0$ and the restriction to $\boldsymbol{B}_0$ and $\boldsymbol{B}_M$ parallel is unnecessary if $\beta_0^2 \ll 1$.

The assumption that the particle's magnetic moment $P_\perp^2/B$ is conserved yields a criterion for reflection of the particle at the mirror, where $P_\perp$ and $P_\|$ are, respectively, the particle's momenta perpendicular and parallel to $\boldsymbol{B}$. Noting that the particle's energy is also constant in the rest frame of the mirror, one readily obtains the general result that any particle with $\mu_0 = |P|/P|_{B=B_0} < \mu_c$ is reflected, where μ_c satisfies

$$\frac{(1-\beta_0^2)(1-\mu_c^2)}{[\mu_c + \beta_0(1+m_0^2c^2/P^2)^{1/2}]^2} = \frac{1}{s-1}. \tag{I.1}$$

Once μ_c is determined, the transmission factor X of the mirror for a known distribution of pitch angles $\theta_p = \cos^{-1}\mu_0$ can be calculated:

$$X = \frac{\int_0^{\cos^{-1}\mu_c} n(\theta_p)\,d\theta_p}{\int_0^{\pi/2} n(\theta_p)\,d\theta_p}. \tag{I.2}$$

If the distribution is isotropic, $n(\theta_p)d\theta_p = A \sin\theta_p d\theta_p$ and,

$$X_i = 1 - \mu_c.$$

The energy gain of a reflected particle is also readily calculated. Note first that in the rest frame of the mirror the energy and perpendicular momentum of the particle are unchanged, but that the parallel momentum is reversed. A particle having an initial energy W_i and parallel momentum $(P_\|)_i$ will have $W^* = \gamma_0[W_i + \beta_0 c(P_\|)_i]$ and $P_\|^* = \gamma_0[(P_\|)_i + \beta_0 W_i/c]$ in the rest frame of the mirror. Reversing $P_\|^*$ and transforming back to the original coordinate system, one readily obtains

$$W_f - W_i = \Delta W \simeq 2\gamma_0^2\beta_0[c(P_\|)_i + \beta_0 W_i]. \tag{I.3}$$

If the particle makes $\mu v_p/2L(t)$ reflections per second, where $2L(t)$ is the distance traveled between reflections, the rate of energy gain can be written

$$\frac{dW}{dt} \simeq \left[\frac{\mu^2 c\beta_0}{L(t)} W\left(1-\frac{m_0{}^2 c^4}{W^2}\right)\right]\left(\frac{1+\beta_0/\mu\beta_p}{1-\beta_0{}^2}\right), \tag{I.4}$$

where $\beta_p = v_p/c = (1 - m_0^2c^4/W^2)^{1/2}$. We are chiefly interested in the limit $\beta_0/\beta_p \ll 1$ and $\beta_0^2 \ll 1$, so that

$$\frac{dW}{dt} \simeq \frac{\mu^2 c\beta_0}{L(t)} W\left(1-\frac{m_0{}^2 c^4}{W^2}\right). \tag{I.5}$$

APPENDIX II

SOLUTION OF THE PARTICLE-CONSERVATION EQUATION

A formal Green's function solution to equation (3) is readily obtained. First, define a new time variable $\psi = \psi_0 - (1/\beta_0 c) \ln [L(t)/L_0]$. Then equation (3) can be written

$$\frac{\partial n}{\partial \psi} + aW\left(1-\frac{1}{W^2}\right)\frac{\partial n}{\partial W} + \left[b\left(1-\frac{1}{W^2}\right)^{1/2} + a\left(1+\frac{1}{W^2}\right)\right]n = S^*(W,\psi), \tag{II.1}$$

where W is written in units of m_0c^2, $S^*(W,\psi) = L[t(\psi)]S[W,t(\psi)]$, and where the above equation for ψ is easily inverted to give $t(\psi)$. Equation (II.1) is to be solved for $\psi > \psi_0$, $W > 1$ subject to the boundary conditions that $n(W,\psi_0) = n(1,\psi) = 0$. Replacing $S^*(W,\psi)$ by $\delta(W - W_1)$ $\delta(\psi - \psi_1)$ in equation (II.1) and solving by the Laplace transform technique, one eventually obtains Green's function for equation (II.1),

$$\begin{aligned} n_g(W,\psi;W_1,\psi_1) &= \theta(W-W_1)\,\delta\left[(\psi-\psi_1) - \frac{1}{2a}\ln\left(\frac{W^2-1}{W_1{}^2-1}\right)\right] \\ &\quad\times \frac{W}{a(W^2-1)}\left[\frac{W+\sqrt{(W^2-1)}}{W_1+\sqrt{(W_1{}^2-1)}}\right]^{-b/a}. \end{aligned} \tag{II.2}$$

The general solution to equation (II.1) is then

$$n(W,\psi) = \int_1^\infty dW_1 \int_{\psi_0}^\infty d\psi_1 n_g(W,\psi;W_1,\psi_1) S^*(W_1,\psi_1). \tag{II.3}$$

The ψ_1 integral is trivial; one obtains

$$\begin{aligned} n(W,\psi) &= \frac{W[W+\sqrt{(W^2-1)}]^{-b/a}}{a(W^2-1)} \int_1^\infty dW_1 \theta(W-W_1)\,\theta\Big[\psi-\psi_0 \\ &-\frac{1}{2a}\ln\left(\frac{W^2-1}{W_1{}^2-1}\right)\Big][W_1+\sqrt{(W_1{}^2-1)}]^{b/a} S^*\left[W_1,\psi-\frac{1}{2a}\ln\left(\frac{W^2-1}{W_1{}^2-1}\right)\right]. \end{aligned} \tag{II.4}$$

In spite of the complexity, some useful limiting results may be drawn from equation (II.4) as is discussed in the text.

APPENDIX III

THE DIFFUSION EQUATION WITH CONVECTION OF SCATTERING CENTERS

We consider the equation

$$\frac{\partial \rho}{\partial t} = K\frac{\partial^2 \rho}{\partial x^2} - c\beta_1\frac{\partial \rho}{\partial x}$$

to be solved for $\rho(x,t,W)$ in the region $0 < x < \infty$ subject to the boundary conditions that $\rho(x,0,W) = 0$ and $\rho(0,t,W) = g(t,W)$ is a known function. The general solution is easily found by standard Laplace transform methods, and only the final result for the general case is presented. One obtains

$$\rho(x,t,W) = \mathfrak{L}^{-1}[F(x,s,W)], \tag{III.1}$$

where $\mathfrak{L}^{-1}(f)$ is the inverse Laplace transform of f and where

$$F(x,s,W) = \exp\left\{\left[\frac{c\beta_1}{2K} - \left(\frac{c^2\beta_1^2}{4K^2} + \frac{s}{K}\right)^{1/2}\right]x\right\}\int_0^\infty g(t,W)\,e^{-st}dt. \tag{III.2}$$

For the special case discussed in the text where $g(t,W)$ can be expressed as $\rho_0(W)\theta(t)$, then

$$\rho(x,t,W) = \rho_0(W)\left[1 - \frac{2}{\pi}\exp\left(\frac{c\beta_1 x}{2K} - \frac{c^2\beta_1^2}{4K}t\right)\right. \\ \left.\times\int_0^\infty dz\,\frac{z^2}{z^2 + c^2\beta_1^2/4K}\sin\frac{xz}{\sqrt{K}}\exp(-tz^2)\right]. \tag{III.3}$$

A quantity of interest, the average velocity at $x = 0$, is obtainable in terms of familiar functions. Setting

$$(v_D)_{x=0} = c\beta_1 - K\left(\frac{\partial\rho}{\partial x}\right)_{x=0}\frac{1}{\rho_0}, \tag{III.4}$$

one obtains

$$(v_D)_{x=0} = \left(\frac{K}{\pi t}\right)^{1/2}\exp\left(-\frac{c^2\beta_1^2 t}{4K}\right) - \frac{c\beta_1}{2}\,erfc\left[\frac{c\beta_1}{2}\sqrt{\left(\frac{t}{K}\right)}\right] + c\beta_1. \tag{III.5}$$

Clearly, as $K/c^2\beta_1^2 t < 1$, equation (III.5) rapidly approaches $c\beta_1$ for $\beta_1 > 0$.

APPENDIX IV

TRANSFORMATION TO THE "PREFERRED" REFERENCE SYSTEM FOR A POINT ON THE EARTH'S BOW SHOCK

If the magnetic field has a component normal to a plane-shock front, a reference system can always be found for which all flow velocities are parallel or antiparallel to the magnetic field and for which the shock is stationary. Such a transformation can be made for each point on the Earth's bow shock and will now be derived. Begin with a reference system in which the bow shock is at rest and in which the angles χ and α are defined as in the text. We assume non-relativistic transformation so that $\boldsymbol{B}' = \boldsymbol{B}$. If initially v_w is along the z-axis and if $\boldsymbol{B}_0$ is in the x-z-plane, the equations to be satisfied are

$$\frac{v_{wx}'}{v_{wz}'} = \frac{B_{0x}}{B_{0z}} = \tan\chi\,(\text{i.e.}, \boldsymbol{v}_w' \| \boldsymbol{B}) \tag{IV.1}$$

and

$$v_{wz}' = v_w - v_{wx}'\sin\alpha\,. \tag{IV.2}$$

These can be readily solved to yield

$$c\beta_0 = \frac{1}{1 + \sin\alpha\tan\chi}\,\frac{v_w}{\cos\chi}, \tag{IV.3}$$

which is the desired result.

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