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Modified Encke Special Perturbation Method

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The classical Encke method for integrating the equations of motion of a near-earth satellite is modified so that the difference between the nominal and true orbits does not contain the first-order effects of the earth's oblateness. The nominal trajectory is well defined for all inclinations and all bounded orbits. Numerical tests of the new method indicated that drag-free orbits can be integrated for at least 100 revolutions before the length of the difference vector exceeds 65 km.

I. INTRODUCTION

The purpose of this paper is to discuss a modification of the classical Encke method for computing satellite orbits. Our goal is the accurate integration of

$$(d^2\mathbf{r}/dt^2)+\mu(\mathbf{r}/r^3)=\mathbf{F}, \tag{1}$$

where the perturbing force $\mathbf{F}$ is primarily due to the earth's oblateness. The assumed potential function of the earth is

$$V=-[(\mu/r)+U], \tag{2}$$

where

$$U=\frac{\mu}{r}\left[-\sum_{n=2}^{n_1} J_n\left(\frac{a_e}{r}\right)^n P_n(\sin\Phi) -\sum_{n=2}^{n_2}\sum_{m=1}^{n} J_{nm}\left(\frac{a_e}{r}\right)^n P_n{}^m(\sin\Phi)\cos m(\lambda-\lambda_{nm})\right],$$

and μ is the product (GM) of the Newtonian gravitational constant and mass of the earth, $\mathbf{r}$ is the position vector relative to the center of mass (with magnitude r), Φ, λ are geocentric latitude and (east) longitude, a_e is the mean equatorial radius of the earth, J_n, J_{nm} are numerical coefficients, P_n is the Legendre polynomial of the first kind of degree $n\leq n_1$, $P_n{}^m$ is the Legendre associated function of the first kind of degree $n\leq n_2$ and order m, λ_{nm} are longitudes associated with the J_{nm}. The force $\mathbf{F}$ is grad$U$ plus nongravitational forces.

An Encke method integrates

$$\boldsymbol{\xi}=\mathbf{r}-\mathbf{r}_n, \tag{3}$$

the deviation from an accurately nominal (or intermediate) trajectory, and adds the correction to $\mathbf{r}_n$ to obtain $\mathbf{r}$. In effect, one thereby increases the work length of the computer and gains accuracy. However, this advantage is present only if $\boldsymbol{\xi}$ remains small, i.e., only if the true and nominal trajectories are close together. To ensure this, the nominal orbit used here is an ellipse (we only consider bounded orbits) which is slowly rotating in space, the rate of rotation being selected so that the difference between the nominal and true orbits will not contain the first-order secular effects of the earth's oblateness. The new method has been compared with the classical Encke method, and for all orbits tested, a significant improvement was established. For a typical drag-free orbit, the classical method required rectification ($|\boldsymbol{\xi}|$ exceeded 638 km) after 5.3 revolutions, while the new method was used for 212 revolutions and the maximum value of $|\boldsymbol{\xi}|$ during the last revolution was about 64 km. With drag, the rectification ratio has been at least two for those orbits tested. Section III contains the details of the numerical tests.

II. MODIFIED ENCKE METHOD

In the classical Encke method, the nominal trajectory is on an ellipse whose shape and orientation in space are fixed. It is the solution of

$$(d^2\mathbf{r}/dt^2)+\mu(\mathbf{r}/r^3)=0. \tag{4}$$

The exact solution of (1) is on a twisted space curve which can be thought of as being generated by a particle which is traversing an ellipse which is slowly rotating and pulsing in a plane (the orbital plane) which in turn is slowly rotating in space. Therefore this instantaneous ellipse begins to rotate away from the nominal ellipse, both because of the rotation of the plane and the rotation in the plane. Furthermore, the perturbation typically changes the period of the satellite so that even if the nominal and instantaneous ellipses were coincident, $|\boldsymbol{\xi}|$ would grow, and the advantage over other methods would be destroyed.

Because of this, a modification of the classical Encke method has been developed. The basic idea is to force the nominal and instantaneous ellipses to remain close for many revolutions by rotating the nominal ellipse slowly in a plane which is rotating slowly in space. Two rate parameters are needed for this: η, the rotation rate of the major axis of the nominal ellipse, and τ, the rotation rate of the line of nodes of the nominal plane. The nominal inclination and eccentricity are held constant. In addition, a third parameter γ is introduced to control the nominal anomalistic mean motion $\bar{n}$.

It is defined by

$$\bar{n}=n_0(1-\gamma), \tag{5}$$

where n_0 is the mean motion at epoch. As usual, the nominal anomalistic period is $2\pi/\bar{n}$. In detail, we set

$$\mathbf{r}_n=R(\Omega_n,i_n,u_n)\begin{bmatrix} r_n \\ 0 \\ 0 \end{bmatrix}, \tag{6}$$

where (subscripts zero denote initial values and subscripts n denote nominal values), $\Omega_n=\tau(f_n-f_0)+\Omega_0$, the longitude of the ascending node, $i_n=i_0$, the inclination, $u_n=f_n+\omega_n$, the argument of latitude $\omega_n=\eta(f_n-f_0)+\omega_0$, the argument of perigee,

$$r_n=\frac{a_0(1-e_0^2)}{1+e_0\cos f_n},$$

the length of $\mathbf{r}_n$, $a_n=a_0$, the semimajor axis, and $e_n=e_0$, the eccentricity. The nominal true anomaly f_n is given as a function of time by means of a modified Kepler's equation

$$n_0(1-\gamma)t+M_0=E_n-e_0\sin E_n, \tag{7}$$

where as usual

$$\tan(\tfrac{1}{2}f_n)=\left(\frac{1+e_0}{1-e_0}\right)^{\frac{1}{2}}\tan(\tfrac{1}{2}E_n), \tag{8}$$

M_0 is the initial value of the mean anomaly, and $n_0^2a_0^3=\mu$. The rotation matrix is $R=D(\Omega)C(i)B(u)$, where

$$D=\begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$C=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix},$$

$$B=\begin{bmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The nominal trajectory $\mathbf{r}_n(t)$ has been completely specified. It depends on nine parameters: six initial values $(a_0,e_0,i_0,\omega_0,\Omega_0,M_0)$ and three arbitrary rate parameters (γ,η,τ). If the rate parameters are set equal to zero, we have the classical nominal trajectory. For near-earth satellites, the first-order secular effects are easy to derive (Sterne 1960):

$$\begin{aligned} \eta&=\tfrac{3}{4}J_2\left(\frac{a_e}{a_0}\right)^2\frac{1}{(1-e_0^2)^2}(4-5\sin^2 i_0),\\ \tau&=-\tfrac{3}{2}J_2\left(\frac{a_e}{a_0}\right)^2\frac{\cos i_0}{(1-e_0^2)^2}, \\ \bar{n}&=n_0\left[1+\tfrac{3}{2}J_2\left(\frac{a_e}{a_0}\right)^2\left(\frac{a}{r}\right)_0^3(1-3\sin^2 i_0\sin^2 u_0)\right] \end{aligned} \tag{10}$$

[this formula for the anomalistic mean motion is a first-order approximation of the Cunningham integral (Sterne 1960)]

$$\gamma=-\tfrac{3}{2}J_2\left(\frac{a_e}{a_0}\right)^2\left(\frac{a}{r}\right)_0^3(1-3\sin^2 i_0\sin^2 u_0).$$

Note that the nominal trajectory is well defined for all inclinations and for all bounded orbits.

It is straightforward to verify that $\mathbf{r}_n$ satisfies the differential equation

$$(d^2\mathbf{r}_n/dt^2)+\mu(\mathbf{r}_n/r_n^3)=\mathbf{L}(t), \tag{11}$$

where

$$\begin{aligned} \mathbf{L}=\frac{\mu}{r_n^3}(1+e_0\cos f_n)\bigg\{&\tau^2(1-\gamma)^2\frac{d^2D}{d\Omega_n^2}CB \\ &+2\tau(1-\gamma)^2(1+\eta)\frac{dD}{d\Omega_n}C\frac{dB}{du_n} \\ &-(1-\gamma)^2(\eta^2+2\eta)R\bigg\}\begin{bmatrix} r_n \\ 0 \\ 0 \end{bmatrix}-(\gamma^2-2\gamma)\frac{\mu}{r_n^3}\mathbf{r}_n. \end{aligned} \tag{12}$$

Therefore, the deviation vector satisfies

$$\frac{d^2\xi}{dt^2}=\mu\left(\frac{\mathbf{r}_n}{r_n^3}-\frac{\mathbf{r}}{r^3}\right)+\mathbf{F}-\mathbf{L}, \tag{13}$$

where we replace r by $\mathbf{r}_n+\xi$. In order to avoid cancellation error, we employ the O. K. Smith development (Conte 1962)

$$\frac{\mathbf{r}_n}{r_n^3}-\frac{\mathbf{r}}{r^3}=\frac{1}{r_n^3}\left\{\left(1+\frac{\zeta^2}{1+\zeta}\right)[(\mathbf{r}+\mathbf{r}_n)\cdot\xi]\frac{\mathbf{r}}{r^2}-\xi\right\} \tag{14}$$

where

$$\zeta=r_n/r.$$

A similar device could be used with $\mathbf{F}-\mathbf{L}$, but careful numerical tests show this is not needed. The differential equation (13) was integrated using both t and f_n as independent variables. If f_n is the independent variable, the left side of (13) should be replaced by

$$\left(\frac{df_n}{dt}\right)^2\frac{d^2\xi}{df_n^2}+\frac{d^2f_n}{dt^2}\frac{d\xi}{df_n},$$

where

$$\begin{aligned} df_n/dt&=[\bar{n}/(1-e_0^2)^{\frac{3}{2}}](1+e_0\cos f_n)^2, \\ d^2f_n/dt^2&=-[2\bar{n}^2e_0/(1-e_0^2)^3](1+e_0\cos f_n)^3\sin f_n. \end{aligned} \tag{15}$$

In the next section, some of the numerical tests of this modified Encke method are discussed.

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TABLE I. Comparison of MES/CES trajectory computation results.

Test case[a]	Applicable parameter values	Comments
1	a =6908 e =0.05 i =0 Ω =0 ω =30 M =0 h_p =185 C_DA/W=0	CES rectified after 493 steps (t=451 min). MES terminated after 10 675 steps (1 week), when $\lvert\xi\rvert=30$.
2	Same as Case 1 except i =5	CES rectified after 497 steps (455 min). MES terminated after 10 674 steps (1 week), when $\lvert\xi\rvert=40$.
3	Same as Case 1 except i =45	CES rectified after 1005 steps (936 min). MES terminated after 10 647 steps (1 week), when $\lvert\xi\rvert=15$.
4	Same as Case 1 except i =45 M =60	CES rectified after 1900 steps (1785 min). MES terminated after 10 647 steps (1 week), when $\lvert\xi\rvert=8$.
5	Same as Case 1 except i =90	CES rectified after 4282 steps (4039 min). MES terminated after 10 661 steps (1 week), when $\lvert\xi\rvert=10$.
6	Same as Case 1 except a =13126 e =0.5 i =45	CES rectified after 295 steps (725 min). MES terminated after 4098 steps (1 week), when $\lvert\xi\rvert=35$.
7	Same as Case 1 except a =13126 e =0.5 i =63.434947	CES rectified after 408 steps (989 min). MES terminated after 4088 steps (1 week), when $\lvert\xi\rvert=26$.
8	Same as Case 1 except M =60	CES rectified after 548 steps (503 min). MES terminated after 21 328 steps (2 weeks), when $\lvert\xi\rvert=62$.
9	Same as Case 1 except a =13126 e =0.5 i =63.434947 M =60	CES rectified after 3475 steps (5031 min). MES terminated after 8126 steps (2 weeks), when $\lvert\xi\rvert=6$.
10	Same as Case 1 except i =45 ω =0 C_DA/W=0.0041	Both drag and the effect of J_2 are perturbations. CES rectified after 705 steps (507 min). MES rectified after 3720 steps (3503 min).
11	a =21472577 e =0.1×10^{-7} i =45 Ω =0 ω =182.99169 M =177.00830 h_p =152 C_DA/W=0.0041	Same perturbations as Case 10. CES rectified after 598 steps (507 min). MES rectified after 1090 steps (935 min).
12	a =6908 e =0.05 i =45 Ω =0 ω =0 M =0 h_p =185 C_DA/W=0	Perturbations included effects of sun and moon and presence of zonal coefficients thru J_6 and tesseral coefficients through J_{44} in the potential. CES rectified after 714 steps (660 min). MES terminated after 10 675 steps (1 week), when $\lvert\xi\rvert=8$.
13	a =8751 e =0.25 i =90 Ω =0 ω =0 M =60 h_p =185 C_DA/W=0	Same perturbations as Case 12. CES rectified after 383 steps (510 min). MES terminated after 22194 steps (3 weeks), when $\lvert\xi\rvert=13$.

[a] Test Cases 1 through 9 are perturbed only by the presence of the zonal coefficent J_2 in the potential model.

III. NUMERICAL TESTS

Since the absolute accuracy of a special perturbation method depends on both its theoretical basis and on the particular computer programs employed, we emphasize comparisons with a classical Encke scheme which uses the same integration and coordinate transformation subroutines. For comparison purposes the classical Encke scheme (CES) was integrated until rectification and the same trajectory was then generated by the

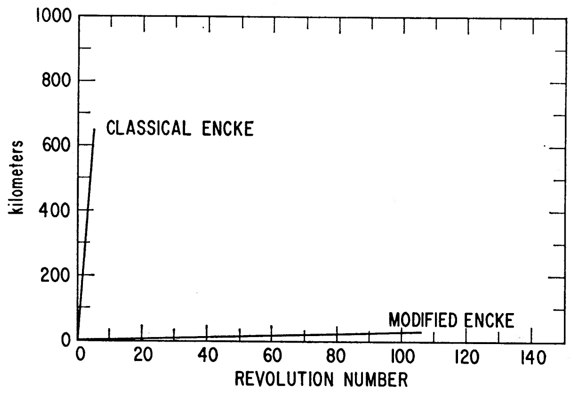

FIG. 1. Magnitude of XI at perigee—Case 1.

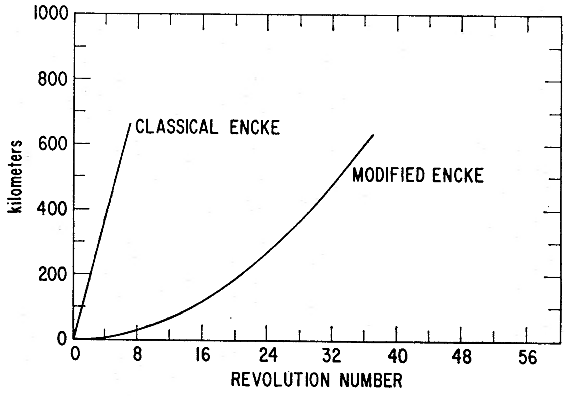

FIG. 2. Magnitude of XI at perigee—Case 10.

modified Encke scheme (MES) until either rectification occurred or a prescribed cutoff time was reached. The results presented here are for the current version of the method. During its development a large number of trajectories were computed, and while our earlier attempts increased the rectification interval, it was not until the anomalistic period was used in the computation of the rate parameters that a striking advantage was obtained.

A test orbit is described by six initial conditions a, e, i, Ω, ω, M and the force model. In addition, the perigee height h_p is given. For most of the tests, the potential function (2) was truncated so that the second harmonic was the only perturbation. While this is the most favorable setting for our method, the presence of the higher harmonics does not seem to affect the results. In theory, they are usually second-order perturbations, and for the orbits tested, they are dominated by the second harmonic. The atmospheric drag perturbation is proportional to C_DA/W, where C_D is the drag coefficient (dimensionless), A is the vehicle cross-sectional area, and W is the vehicle weight. The dimensions for the quantities referred to in the following cases are kilometers, degrees, and meters^2/kilogram.

Figures 1, 2, and 3 are plots of the growth of $|\xi|$ for case 1, a typical drag-free orbit, and for cases 10 and 11, the two tests with drag present. In all cases, the classical Encke scheme is represented by the curve which attains the rectification value (638 km) before the sixth revolution. The modified Encke scheme is represented by the other curve.

Although results of absolute accuracy have not been the primary concern of this paper it is worth mentioning that (using equivalent numerical precision and common integration and coordinate transformation subroutines) the accuracy of the modified Encke formulation exceeded that obtained with the classical Encke formulation and the Cowell formulation. This is illustrated by Figs. 4, 5, and 6, which are plots of the error in magnitude of the position vector, the error tangent to the orbit, and the magnitude of the total error for a drag-free trajectory with $a=7251.$, $e=0.1$, $i=75$, $\omega=0$, $M=0$, $\Omega=45$. The second harmonic is the only perturbation. The trajectory was computed three times: with modified Encke, with Cowell, and with a complete double precision Cowell program. The error is defined to be the deviation from the double precision trajectory. In both graphs, the larger error is that of the Cowell program.

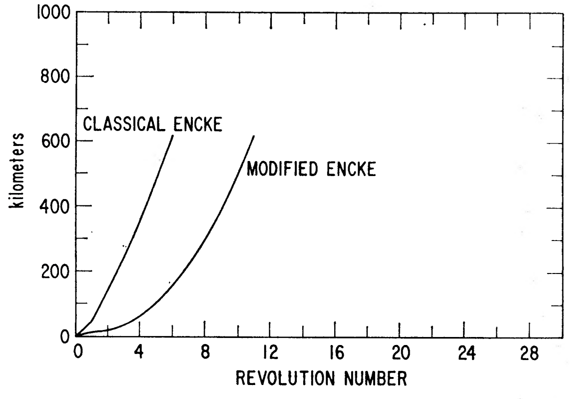

FIG. 3. Magnitude of XI at perigee—Case 11.

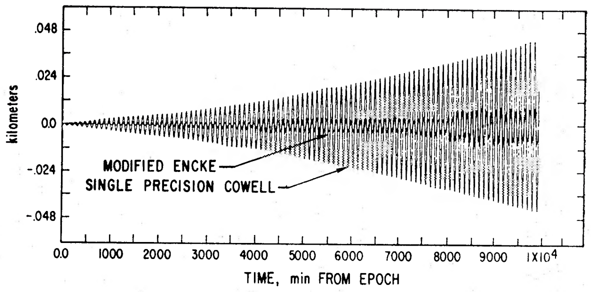

FIG. 4. Error in magnitude of position vector.

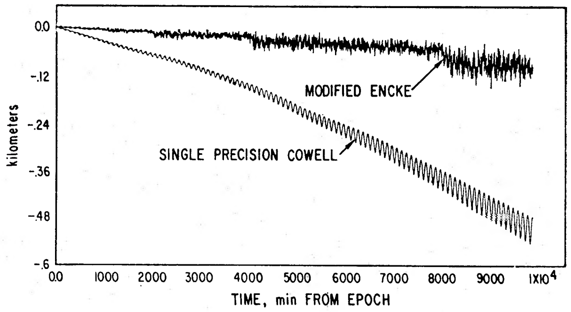

FIG. 5. Error in in-track direction.

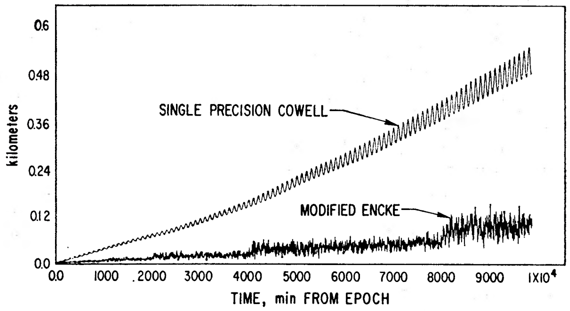

FIG. 6. Magnitude of error vector.

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