THE EQUILIBRIUM AND THE STABILITY OF THE RIEMANN ELLIPSOIDS. I

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ABSTRACT

Riemann's problem is concerned with the ellipsoidal figures of equilibrium of homogeneous masses rotating uniformly with an angular velocity Ω and with internal motions having a uniform vorticity ζ_Ω in the rotating frame. In this paper, the equilibrium and the stability of these *Riemann ellipsoids* are considered in the special case the axes of rotation and vorticity coincide with a principal axis of the ellipsoid. It is shown that for the case considered (1) the equilibrium figures can be delineated into sequences—*the Riemann sequences*—along which the ratio $f = \zeta_\Omega/\Omega$ is a constant; (2) an ellipsoid which is a figure of equilibrium for some given f is also a figure of equilibrium for $f\dagger = (a_1^2 + a_2^2)^2/a_1^2a_2^2f$, where a_1 and a_2 are the semi-axes of the ellipsoid in the equatorial plane; (3) the two states of internal motion, corresponding to f and $f\dagger$, lead to configurations which are *adjoint* in the sense of a theorem due to Dedekind; (4) the first member of a Riemann sequence is a Maclaurin spheroid which is stable in the absence of any dissipative mechanism; (5) from each point of the stable part of the Maclaurin sequence two Riemann sequences bifurcate; (6) there exist two self-adjoint sequences along which $f = f\dagger = \pm(a_1^2 + a_2^2)/a_1a_2$ and that limit the domain of the Riemann sequences in the $(a_2/a_1, a_3/a_1) =$ plane; and (7) the bifurcation of the Jacobian and the Dedekind sequences from, what is usually called, *the* point of bifurcation is a special case of a much more general phenomenon.

The stability of the Riemann ellipsoids with respect to modes of oscillation belonging to the second and the third harmonics is also investigated. With respect to modes of oscillation belonging to the second harmonics it is shown that (1) the Riemann ellipsoids allow a non-trivial neutral mode of oscillation; (2) the characteristic frequencies of oscillation of an ellipsoid and its adjoint are the same; (3) the Riemann ellipsoids with $f \geq -2$ are stable with respect to these modes; and (4) instability by one of these modes arises along the sequences for $f < -2$. With respect to modes of oscillation belonging to the third harmonics, it is shown that along all Riemann sequences instability first arises by a mode which deforms the ellipsoid into a pear-shaped configuration. The points at which instability sets in along the different Riemann sequences and the loci, which separate the regions of stability from the regions of instability in the domain of the Riemann ellipsoids considered, are also determined.

I. INTRODUCTION

In 1860 Dedekind proved the following remarkable theorem: *if a homogeneous ellipsoid with semi-axes* a_1, a_2, *and* a_3 *is in gravitational equilibrium, with a prevalent motion* $\boldsymbol{u}^{(0)}$ *whose components in an inertial frame are given by*

$$\boldsymbol{u}^{(0)} = \begin{Vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{Vmatrix} \begin{Vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{Vmatrix} = A \begin{Vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{Vmatrix}, \tag{1}$$

then, the same ellipsoid will also be a figure of equilibrium if the prevalent motion is that derived from the transposed matrix $\mathbf{A}\dagger$, *i.e., for* $\boldsymbol{u}^{(0)}$ *given by*

$$\boldsymbol{u}^{(0)} = \begin{Vmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{Vmatrix} \begin{Vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{Vmatrix} = A\dagger \begin{Vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{Vmatrix}. \tag{2}$$

We shall call the configuration with the motion derived from $\boldsymbol{A}^\dagger$ as the *adjoint* of the configuration with the motion derived from $\boldsymbol{A}$. Love (1888) seems to have been the first to have pointed out that *the Jacobian and the Dedekind sequences*[1] *are adjoint* in this sense. This fact can be verified quite readily as follows.

The motion of a Jacobi ellipsoid (rotating uniformly with an angular velocity Ω about the x_3-axis) can be represented in the manner

$$\boldsymbol{u}^{(0)} = \begin{vmatrix} 0 & -\Omega a_2 & 0 \\ \Omega a_1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix} . \tag{3}$$

By Dedekind's theorem, the motion in the adjoint configuration will be given by

$$\boldsymbol{u}^{(0)} = \begin{vmatrix} 0 & \Omega a_1 & 0 \\ -\Omega a_2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix} , \tag{4}$$

or, in terms of components,

$$u^{(0)}{}_1 = \frac{\Omega a_1}{a_2} x_2 , \qquad u^{(0)}{}_2 = -\frac{\Omega a_2}{a_1} x_1 , \qquad \text{and} \qquad u_3 = 0 ; \tag{5}$$

and this motion clearly satisfies the condition

$$\boldsymbol{u}^{(0)} \cdot \text{grad} \left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 1 \right) = 0 \tag{6}$$

required for the preservation of the ellipsoidal boundary. Also, the motion represented by (4) is one of uniform vorticity ζ, about the x_3-axis, given by

$$\zeta = -\Omega \left(\frac{a_2}{a_1} + \frac{a_1}{a_2} \right) = -\frac{a_1^2 + a_2^2}{a_1 a_2} \Omega ; \tag{7}$$

and this result is in agreement with the known relation between the Jacobian and the Dedekind sequences (cf. Paper I, eq. [18]).

Pursuing the ideas underlying Dedekind's theorem along somewhat different lines, Riemann (1860) investigated the type of motions, in accordance with Dedekind's initial *assumption* (1), that *can* lead to an ellipsoidal figure of equilibrium. And Riemann showed that *the most general type of motion compatible with an ellipsoidal figure of equilibrium consists of a superposition of uniform rotation and internal motions of uniform vorticity about axes that lie in a principal plane of the ellipsoid.*

In spite of the interest that would appear to be attached to Riemann's investigation of Dedekind's problem, it does not seem to have attracted any attention from those who, like Poincaré, Darwin, and Jeans, have concerned themselves with the ellipsoidal figures of equilibrium of homogeneous liquid masses. Indeed, apart from a brief account of an expository nature by Basset (1888), the subject seems to have been simply ignored. In restituting to its proper place this long-neglected work of Riemann, we shall consider

[1] The Dedekind sequence consists of ellipsoids that are stationary in an inertial frame and whose ellipsoidal figures are maintained by internal motions of uniform vorticity. The equilibrium and the stability of these *Dedekind ellipsoids* have recently been considered (Chandrasekhar 1965; this paper will be referred to hereinafter as "Paper I").

in this paper the equilibrium and the stability of the *Riemann ellipsoids* in the special case when the axes of rotation and of internal vorticity coincide with a principal axis of the ellipsoid. It will appear that the delineation of the resulting *Riemann sequences* not only discloses several features of general interest, it also clarifies many aspects of the "classical sequences" of Maclaurin and Jacobi that have lain obscured. In a later paper we shall consider the case when the axes of rotation and of internal vorticity do not coincide but lie in a principal plane of the ellipsoid.

II. THE SECOND-ORDER VIRIAL THEOREM AND THE EQUATIONS DETERMINING THE EQUILIBRIUM OF THE RIEMANN ELLIPSOIDS

The problem that is to be considered in this paper is that of a homogeneous mass, rotating uniformly with an angular velocity $\boldsymbol{\Omega}$, with internal motions having a uniform vorticity $\boldsymbol{\zeta}_\Omega$ in the direction of Ω and in the frame of reference rotating with the angular velocity $\boldsymbol{\Omega}$. In treating this problem, we shall use the methods, based on the virial theorem and its extensions, which have been developed recently in various papers published in this *Journal* during the past four years.

Let the directions of $\boldsymbol{\Omega}$ and $\boldsymbol{\zeta}_\Omega$ be along the x_3-axis; and let the figure of equilibrium be an ellipsoid with semi-axes a_1, a_2, and a_3. Without entailing any loss of generality, we may suppose that

$$a_1 > a_2 \,. \tag{8}$$

The components of the internal motion, having the assigned vorticity ζ_Ω, that will preserve the ellipsoidal boundary can be expressed in the form (cf. Paper I, eqs. [5] and [6])

$$u_1 = Q_1 x_2, \qquad u_2 = Q_2 x_1, \qquad \text{and } u_3 = 0 \,, \tag{9}$$

where

$$Q_1 = -\frac{a_1^2}{a_1^2 + a_2^2}\,\zeta_\Omega \qquad \text{and } Q_2 = +\frac{a_2^2}{a_1^2 + a_2^2}\,\zeta_\Omega \,. \tag{10}$$

Some elementary relations which follow from these definitions and which we shall find useful are

$$a_1^2 Q_2 = -a_2^2 Q_1, \qquad a_1^2 Q_2^2 = -a_2^2 Q_1 Q_2, \qquad \text{and } a_2^2 Q_1^2 = -a_1^2 Q_1 Q_2 \,, \tag{11}$$

where

$$Q_1 Q_2 = -\left(\frac{a_1 a_2}{a_1^2 + a_2^2}\,\zeta_\Omega\right)^2 . \tag{12}$$

The internal motion specified by equation (9) is with respect to the rotating frame of reference; the components of the same motion in the inertial frame follow from the equation

$$\boldsymbol{u}^{(0)} = \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{x} \,, \tag{13}$$

or, explicitly,

$$u^{(0)}{}_1 = u_1 - \Omega x_2 = -\Omega\left(1 + \frac{a_1^2 f}{a_1^2 + a_2^2}\right) x_2 \,,$$

$$u^{(0)}{}_2 = u_2 + \Omega x_1 = +\Omega\left(1 + \frac{a_2^2 f}{a_1^2 + a_2^2}\right) x_1 \,, \tag{14}$$

and

$$u^{(0)}{}_3 = 0 \,,$$

where we have defined

$$f = \zeta_\Omega / \Omega \,. \tag{15}$$

We can rewrite the components of $\boldsymbol{u}^{(0)}$, in conformity with Dedekind's assumption (1), in the forms

$$u^{(0)}{}_1 = U_1 \frac{x_2}{a_2}, \qquad u^{(0)}{}_2 = U_2 \frac{x_1}{a_1}, \qquad \text{and} \qquad u^{(0)}{}_3 = 0\,, \tag{16}$$

where

$$U_1 = -a_2\Omega\left(1 + \frac{a_1^2 f}{a_1^2 + a_2^2}\right) \qquad \text{and} \qquad U_2 = +a_1\Omega\left(1 + \frac{a_2^2 f}{a_1^2 + a_2^2}\right). \tag{17}$$

To obtain the conditions that the ellipsoid is also in gravitational equilibrium, we shall make use of the second-order virial theorem. According to this theorem

$$\frac{d}{dt}\int_V \rho u_i x_j d\boldsymbol{x} = 2\mathfrak{T}_{ij} + \mathfrak{W}_{ij} + \Omega^2(I_{ij} - \delta_{i3}I_{3j}) + 2\Omega\epsilon_{il3}\int_V \rho u_l x_j d\boldsymbol{x} + \delta_{ij}\Pi\,, \tag{18}$$

where $\mathfrak{T}_{ij}$, $\mathfrak{W}_{ij}$, I_{ij}, and Π have their usual meanings. Under conditions of equilibrium, equation (18) gives

$$\begin{aligned} -\Pi = \mathfrak{W}_{33} &= 2\mathfrak{T}_{11} + \Omega^2 I_{11} + \mathfrak{W}_{11} + 2\Omega\int_V \rho u_2 x_1 d\boldsymbol{x} \\ &= 2\mathfrak{T}_{22} + \Omega^2 I_{22} + \mathfrak{W}_{22} - 2\Omega\int_V \rho u_1 x_2 d\boldsymbol{x}\,. \end{aligned} \tag{19}$$

(We need not consider the non-diagonal components of eq. [18]: these equations are trivially satisfied.)

For the internal motion specified by equation (9),

$$\mathfrak{T}_{11} = \tfrac{1}{2}Q_1^2 I_{22}\,, \qquad \int_V \rho u_2 x_1 d\boldsymbol{x} = Q_2 I_{11}\,,$$

and

$$\mathfrak{T}_{22} = \tfrac{1}{2}Q_2^2 I_{11}\,, \qquad \int_V \rho u_1 x_2 d\boldsymbol{x} = Q_1 I_{22}\,; \tag{20}$$

and equation (19) gives

$$\begin{aligned} \mathfrak{W}_{33} &= Q_1^2 I_{22} + (\Omega^2 + 2Q_2\Omega)I_{11} + \mathfrak{W}_{11} \\ &= Q_2^2 I_{11} + (\Omega^2 - 2Q_1\Omega)I_{22} + \mathfrak{W}_{22}\,. \end{aligned} \tag{21}$$

Making use of the known expressions for I_{ij} and $\mathfrak{W}_{ij}$ (Chandrasekhar and Lebovitz 1962, eqs. [57] and [58]), we can rewrite the foregoing equations in the forms

$$\begin{aligned} -2A_3 a_3^2 &= a_2^2 Q_1^2 + a_1^2(\Omega^2 + 2Q_2\Omega) - 2A_1 a_1^2 \\ &= a_1^2 Q_2^2 + a_2^2(\Omega^2 - 2Q_1\Omega) - 2A_2 a_2^2\,, \end{aligned} \tag{22}$$

where the index symbols A_i are so normalized that $\Sigma A_i = 2$ and Ω^2 and ζ_Ω^2 are measured in the unit $\pi G\rho$.

In view of equations (11), the relation expressed by the second equality in (22) becomes

$$-a_1^2 Q_1 Q_2 + a_1^2\Omega^2 - 2A_1 a_1^2 = -a_2^2 Q_1 Q_2 + a_2^2\Omega^2 - 2A_2 a_2^2\,, \tag{23}$$

or, alternatively,

$$\Omega^2 - Q_1 Q_2 = 2\,\frac{A_1 a_1^2 - A_2 a_2^2}{a_1^2 - a_2^2} = 2B_{12}\,. \tag{24}$$

This last equation is a generalization of relations known to be valid for the Jacobi ($Q_1 = Q_2 = 0$) and the Dedekind ($\Omega = 0$) ellipsoids (cf. Paper I, eq. [46]).

Returning to equations (22), we have the pair of equations

$$\Omega^2 - Q_1Q_2 + 2Q_2\Omega = 2\left(A_1 - \frac{a_3^2}{a_1^2}A_3\right) = 2\,\frac{a_1^2 - a_3^2}{a_1^2}B_{13} \tag{25}$$

and

$$\Omega^2 - Q_1Q_2 - 2Q_1\Omega = 2\left(A_2 - \frac{a_3^2}{a_2^2}A_3\right) = 2\,\frac{a_2^2 - a_3^2}{a_2^2}B_{23}\,. \tag{26}$$

From equation (25) it follows that

$$a_1 > a_3\,; \tag{27}$$

for, by substituting for Q_1 and Q_2 their expressions in terms of ζ_Ω, we find

$$\Omega^2 - Q_1Q_2 + 2Q_2\Omega = \left(\Omega + \frac{a_2^2}{a_1^2 + a_2^2}\zeta_\Omega\right)^2 + \frac{a_2^2(a_1^2 - a_2^2)}{(a_1^2 + a_2^2)^2}\zeta_\Omega^2\,, \tag{28}$$

and this expression is positive definite since $a_1 > a_2$ (by definition!). Thus, *the choice of* a_1 *as the longer of the two axes in the* (x_1,x_2)*-plane makes it also the longest.*

Next, we obtain, by addition and subtraction, from equations (25) and (26)

$$(\Omega - Q_1)(\Omega + Q_2) = A_1 + A_2 - \frac{a_3^2(a_1^2 + a_2^2)}{a_1^2a_2^2}A_3$$

and

$$(Q_1 + Q_2)\Omega = A_1 - A_2 + \frac{a_3^2(a_1^2 - a_2^2)}{a_1^2a_2^2}A_3\,. \tag{29}$$

Accordingly, for determining the geometry of the ellipsoid we may use the equation

$$\frac{(\Omega - Q_1)(\Omega + Q_2)}{\Omega(Q_1 + Q_2)} = \frac{A_1 + A_2 - a_3^2(a_1^2 + a_2^2)A_3/a_1^2a_2^2}{A_1 - A_2 + a_3^2(a_1^2 - a_2^2)A_3/a_1^2a_2^2}\,. \tag{30}$$

We shall now define a *Riemann sequence as one along which* $f = \zeta_\Omega/\Omega$ *is a constant.* According to this definition, *the Jacobian and the Dedekind sequences are Riemann sequences for* $f = 0$ *and* $f = \pm\infty$, *respectively.* Moreover, since, according to equation (13) the vorticity $\zeta^{(0)}$ in the inertial frame is related to the vorticity ζ_Ω in the rotating frame by

$$\zeta^{(0)} = \zeta_\Omega + 2\Omega = \Omega(2 + f)\,, \tag{31}$$

it follows that *the irrotational sequence,* $\zeta^{(0)} = 0$, *is a Riemann sequence for* $f = -2$.

In terms of f, we have the equations

$$Q_1 = -\frac{a_1^2\Omega}{a_1^2 + a_2^2}f\,,\quad Q_2 = +\frac{a_2^2\Omega}{a_1^2 + a_2^2}f\,,\quad\text{and}\quad Q_1Q_2 = -\frac{a_1^2a_2^2}{(a_1^2 + a_2^2)^2}\Omega^2f^2\,. \tag{32}$$

Making use of these relations, we can rewrite equations (24) and (30) in the forms

$$\Omega^2 = \frac{2B_{12}}{1 + a_1^2a_2^2f^2/(a_1^2 + a_2^2)^2} \tag{33}$$

and

$$\frac{[1 + a_1^2f/(a_1^2 + a_2^2)][1 + a_2^2f/(a_1^2 + a_2^2)]}{(a_1^2 - a_2^2)f/(a_1^2 + a_2^2)} = -\frac{A_1 + A_2 - a_3^2(a_1^2 + a_2^2)A_3/a_1^2a_2^2}{A_1 - A_2 + a_3^2(a_1^2 - a_2^2)A_3/a_1^2a_2^2}\,. \tag{34}$$

On simplifying this last equation, we find

$$a_1^2a_2^2\left[A_1-A_2+\frac{a_3^2(a_1^2-a_2^2)}{a_1^2a_2^2}A_3\right]f^2+2(a_1^2+a_2^2)(A_1a_1^2-A_2a_2^2)f$$
$$+(a_1^2+a_2^2)^2\left[A_1-A_2+\frac{a_3^2(a_1^2-a_2^2)}{a_1^2a_2^2}A_3\right]=0\,. \tag{35}$$

For a given f, equation (35) determines the ratios of the axes of the ellipsoids that are compatible with equilibrium; and the value of Ω^2 which is to be associated with a particular solution of equation (35), then, follows from equation (33).

III. THE ADJOINT RIEMANN ELLIPSOIDS AND DEDEKIND'S THEOREM

For reasons which will become apparent presently we shall let

$$x=\frac{a_1a_2}{a_1^2+a_2^2}f\,. \tag{36}$$

The relations given in equations (32) and (33) now take the forms

$$\Omega^2=\frac{2B_{12}}{1+x^2}\,,\qquad -Q_1Q_2=2B_{12}\frac{x^2}{1+x^2}\,,$$
$$Q_1\Omega=-2B_{12}\frac{a_1}{a_2}\frac{x}{1+x^2}\,,\qquad \text{and}\qquad Q_2\Omega=+2B_{12}\frac{a_2}{a_1}\frac{x}{1+x^2}\,; \tag{37}$$

and equation (35) becomes

$$\left(A_1-A_2+\frac{a_1^2-a_2^2}{a_1^2a_2^2}a_3^2A_3\right)x^2+\frac{2}{a_1a_2}(A_1a_1^2-A_2a_2^2)x$$
$$+\left(A_1-A_2+\frac{a_1^2-a_2^2}{a_1^2a_2^2}a_3^2A_3\right)=0\,. \tag{38}$$

Since the constant term and the coefficient of x^2 are the same in equation (38), it is clear that *if* x *is a root, then so is 1*/x. From equation (36) it now follows that *a Riemann ellipsoid belonging to a given* f *is equally a figure of equilibrium for*

$$f\dagger=\frac{(a_1^2+a_2^2)^2}{a_1^2a_2^2}\frac{1}{f}\,. \tag{39}$$

We shall call *a Riemann ellipsoid with internal motions corresponding to* f† *the adjoint of the one with internal motions corresponding to* f. We shall presently verify that this definition of adjointness is in accord with the earlier definition (§ I) based on Dedekind's theorem. But before verifying this fact, it is useful to notice the following features of the present definition.

i) The relationship between two adjoint configurations is a *reflexive* one.

ii) Since the Jacobian and the Dedekind sequences belong to $f=0$ and $f=\pm\infty$, respectively, it follows that the two sequences are adjoints of one another, also in the present sense.

iii) Since the constants appropriate to an adjoint configuration are obtained by simply replacing x by $1/x$ in the relevant formulae, it follows from equations (37) that *the values of* Ω^2 *and* $-Q_1Q_2$ *are simply interchanged when one passes from a configuration to its adjoint, while the values of* $Q_1\Omega$ *and* $Q_2\Omega$ *are unchanged.*

iv) A further relation between Ω and $\Omega\dagger$ (the angular velocity appropriate to the adjoint configuration) which follows from equations (37) is

$$(\Omega\dagger)^2 = 2B_{12}\frac{x^2}{1+x^2} = \Omega^2 x^2 ,$$

or (40)

$$\Omega\dagger = \pm\,\Omega x = \pm\frac{a_1 a_2}{a_1^2+a_2^2}\,\Omega f .$$

We now verify that the present definition of adjointness is in accord with Dedekind's theorem. For this verification, we should determine the components of the motion, in the adjoint configuration, in the inertial frame of reference; and these follow from equations (16) and (17) by simply replacing Ω and f by $\Omega\dagger$ and $f\dagger$. Thus

$$U_1\dagger = -\,a_2\Omega\dagger\left(1+\frac{a_1^2 f\dagger}{a_1^2+a_2^2}\right); \tag{41}$$

or making use of equations (39) and (40), we have

$$\begin{aligned} U_1\dagger &= \mp\frac{a_1 a_2^2}{a_1^2+a_2^2}\,\Omega f\left(1+\frac{a_1^2+a_2^2}{a_2^2}\frac{1}{f}\right) \\ &= \mp\,a_1\Omega\left(1+\frac{a_2^2 f}{a_1^2+a_2^2}\right) = \mp\,U_2 . \end{aligned} \tag{42}$$

Similarly,

$$U_2\dagger = \mp U_1 . \tag{43}$$

Hence, the matrix representing the motion in the adjoint configuration is the transposed of the matrix, representing the motion in the original configuration, or its negative. Since an alteration in the sign of Ω does not affect any of the conditions for equilibrium, it is clear that configurations which are adjoints of one another according to the present definition are also adjoints of one another in the sense of Dedekind's theorem.

We shall conclude this section by evaluating the angular momentum $\mathfrak{M}$ of a Riemann ellipsoid about the x_3-axis in the inertial frame. By definition

$$\mathfrak{M} = \int_V \rho\,[x_1 u^{(0)}{}_2 - x_2 u^{(0)}{}_1]\,d\boldsymbol{x} . \tag{44}$$

Inserting in this equation the expressions for $u^{(0)}{}_1$ and $u^{(0)}{}_2$ given in equations (14), we find

$$\mathfrak{M} = \tfrac{1}{5}M(a_1^2+a_2^2)\Omega\left[1+\frac{2a_1^2a_2^2}{(a_1^2+a_2^2)^2}\,f\right], \tag{45}$$

where M denotes the mass of the ellipsoid. The angular momentum, $\mathfrak{M}\dagger$, of the corresponding adjoint configuration is given by

$$\mathfrak{M}\dagger = \tfrac{1}{5}M(a_1^2+a_2^2)\Omega\dagger\left[1+\frac{2a_1^2a_2^2}{(a_1^2+a_2^2)^2}\,f\dagger\right], \tag{46}$$

or making use of equations (39) and (40), we have

$$\mathfrak{M}\dagger = \pm\tfrac{1}{5}Ma_1a_2\Omega f(1+2/f) . \tag{47}$$

In view of equation (31), we can also write

$$\mathfrak{M}\dagger = \pm\tfrac{1}{5}Ma_1a_2\zeta^{(0)} . \tag{48}$$

IV. THE STABLE MACLAURIN SPHEROIDS AS THE FIRST MEMBERS OF RIEMANN SEQUENCES

In this section we shall show how a Maclaurin spheroid can be considered as a limiting form of a Riemann ellipsoid. If a Maclaurin spheroid can be so considered, then the angular velocity Ω that would be ascribed to it will not necessarily be the same as the angular velocity Ω_{Mc} which would normally be attributed to it. On the other hand, since an object cannot be materially affected by simply viewing it from a different frame of reference, it is clear that, when viewed from a frame of reference rotating with an angular velocity Ω, a Maclaurin spheroid, rotating uniformly with an angular velocity Ω_{Mc} with respect to an inertial frame, will be described as having internal motions with a uniform vorticity ζ_Ω such that the net vorticity in the inertial frame is the same.

Now the vorticity associated with the uniform rotation Ω_{Mc} is

$$\zeta^{(0)} = 2\Omega_{Mc} ; \tag{49}$$

and from equation (31) it follows that, when viewed from a frame of reference rotating with an angular velocity Ω, the Maclaurin spheroid will be described as having internal motions with the vorticity

$$\zeta_\Omega = \zeta^{(0)} - 2\Omega = 2(\Omega_{Mc} - \Omega) . \tag{50}$$

At the same time, from equation (12) it follows that when $a_1 = a_2$ (as is the case for a Maclaurin spheroid)

$$Q_1 Q_2 = -\tfrac{1}{4}\zeta_\Omega^2 = -(\Omega_{Mc} - \Omega)^2 . \tag{51}$$

But in order that a Maclaurin spheroid may be considered as a Riemann ellipsoid, equation (24) must be satisfied. Hence, the angular velocity Ω which should be ascribed to a Maclaurin spheroid, when it is considered as a limiting form of a Riemann ellipsoid, is determined by the equation

$$\Omega^2 + (\Omega_{Mc} - \Omega)^2 = 2B_{11} , \tag{52}$$

where B_{11} is the index symbol determined by the eccentricity of the spheroid. Solving equation (52), we have

$$\Omega = \tfrac{1}{2}[\Omega_{Mc} \pm \sqrt{(4B_{11} - \Omega_{Mc}^2)}] . \tag{53}$$

The corresponding value of ζ_Ω is given by (cf. eq. [50])

$$\zeta_\Omega = \Omega_{Mc} \mp \sqrt{(4B_{11} - \Omega_{Mc}^2)} . \tag{54}$$

We now observe that *equation (54) determining ζ_Ω is the same as the characteristic equation that determines the natural frequencies of the toroidal modes of oscillation of a Maclaurin spheroid.*[2] *Consequently, only the stable members of the Maclaurin sequence can be considered as limiting forms of Riemann ellipsoids:* for the Maclaurin spheroid becomes unstable when $\Omega_{Mc}^2 > 4B_{11}$; and when this is the case, Ω and ζ_Ω given by equations (53) and (54) become complex and this is incompatible with their meanings as real quantities.

We may now state that a Maclaurin spheroid rotating uniformly with an angular velocity Ω_{Mc} may be considered as the first member of two Riemann sequences belonging to

$$f = 2\,\frac{\Omega_{Mc} - \sqrt{(4B_{11} - \Omega_{Mc}^2)}}{\Omega_{Mc} + \sqrt{(4B_{11} - \Omega_{Mc}^2)}} \quad \text{and} \quad f^\dagger = 2\,\frac{\Omega_{Mc} + \sqrt{(4B_{11} - \Omega_{Mc}^2)}}{\Omega_{Mc} - \sqrt{(4B_{11} - \Omega_{Mc}^2)}} . \tag{55}$$

There are several things to be noticed about this association of the Riemann sequences with the stable members of the Maclaurin sequence.

[2] Cf. Lebovitz (1961, eq. [189]). For a more direct comparison with equation (54), see Chandrasekhar (1964, p. 70, eq. [38]), where the oscillations of the Maclaurin spheroid are treated in the present notation.

i) Since $(a_1^2 + a_2^2)^2/a_1^2a_2^2 = 4$ when $a_1 = a_2, f$ and $f\dagger$, as defined by equations (55), are consistent with the general relation (39), which now requires that

$$f\dagger = 4/f\,. \tag{56}$$

ii) When

$$\Omega_{\mathrm{Mc}}^2 = 2B_{11}, \quad f = 0, \quad \Omega = \Omega_{\mathrm{Mc}}, \quad \text{and} \quad \zeta_\Omega = 0\,;$$
$$\text{and} \quad f\dagger = \infty, \quad \Omega\dagger = 0, \quad \text{and} \quad \zeta_\Omega\dagger = 2\Omega_{\mathrm{Mc}}\,. \tag{57}$$

In other words, the Maclaurin spheroid at the usual point of bifurcation is the first member of both the Jacobian and the Dedekind sequences. This result is, of course, in accord with what we know about these sequences.

iii) When

$$\Omega_{\mathrm{Mc}} = 0, \quad f = f\dagger = -2, \quad \Omega^2 = B_{11}, \quad \text{and } \zeta_\Omega^2 = 4B_{11}\,,$$

where

$$B_{11} = \tfrac{4}{15} \tag{58}$$

is the value appropriate for a sphere. *The first member of the irrotational sequence,* $\zeta^{(0)} = 0$ *and* f $= -2$, *is the sphere.* However, the sphere considered as the first member of the irrotational sequence is ascribed an angular velocity[3] $\Omega = \sqrt{(4\pi G\rho/15)}$ and internal motions with the vorticity, $\zeta_\Omega = \sqrt{(16\pi G\rho/15)}$, which exactly cancels the vorticity due to the rotation.

TABLE 1

THE PARAMETERS TO BE ASSOCIATED WITH THE MACLAURIN SPHEROIDS WHEN CONSIDERED AS THE FIRST MEMBERS OF RIEMANN SEQUENCES

e	f	Ω	$f\dagger$	$\Omega\dagger$	e	f	Ω	$f\dagger$	$\Omega\dagger$
0	−2 00000	0 51640	− 2 0000	−0 51640	0 80	−0 05014	0 61812	− 79 780	−0 01550
0.20 ...	−1 49613	58136	− 2 6736	− 43490	.81.	−0 01083	61316	−369 20	− 00332
25 . .	−1 38358	59502	− 2 8910	− 41163	81267	0	61174	± ∞	0
.30 ...	−1 27419	60757	− 3 1392	− 38708	.82	+0 03055	60764	+130 93	+ 00928
.35 ...	−1 16686	.61891	− 3 4280	− 36109	.83 .	+0 07434	60148	+ 53 806	+ 02236
.40 ..	−1 06059	62893	− 3 7715	− .33352	84	+0 12094	59460	+ 33 074	+ 03596
45 ...	−0 95434	63748	− 4 1914	− .30419	85	+0 17086	58691	+ 23 411	+ 05014
.50 ...	−0 84694	.64433	− 4 7229	− 27285	.86 .	+0 22475	57828	+ 17 798	+ 06498
.55 ..	−0 73698	64918	− 5 4276	− .23921	88	+0 34824	55749	+ 11 486	+ .09707
.60 ...	−0 62269	65159	− 6 4237	− .20287	90	+0 50327	53029	+ 7 9480	+ 13344
.65 ...	−0 50169	65096	− 7 9731	− .16329	92	+0 71695	49279	+ 5 5792	+ 17666
70 ...	−0 37044	64635	−10 798	− 11972	.94..	+1 08248	43421	+ 3 6952	+ 23501
0 75 ...	−0 22331	0 63626	−17 912	−0 07104	0 95289	+2 00000	0 33175	+ 2 0000	+0 33175

iv) The Maclaurin spheroids with angular velocities in the range $0 \le \Omega_{\mathrm{Mc}}^2 < 2B_{11}$ are the first members of Riemann sequences belonging to values of f in the ranges $-2 \le f < 0$ and $-2 \ge f > -\infty$; and the Maclaurin spheroids with angular velocities in the ranges $2B_{11} < \Omega_{\mathrm{Mc}}^2 \le 4B_{11}$ are the first members of Riemann sequences belonging to values of f in the ranges $0 < f \le 2$ and $2 \le f < +\infty$. Since every value of f has been accounted for in this enumeration of the first members, it is clear that no Riemann sequence has been left out. In other words, *there is no Riemann sequence which does not begin with a stable Maclaurin spheroid.*

The variation of f and $f\dagger$ along the Maclaurin sequence is exhibited in Figure 1; and the relevant numerical data are given in Table 1.

[3] Restoring the factor $\pi G\rho$, which had hitherto been suppressed.

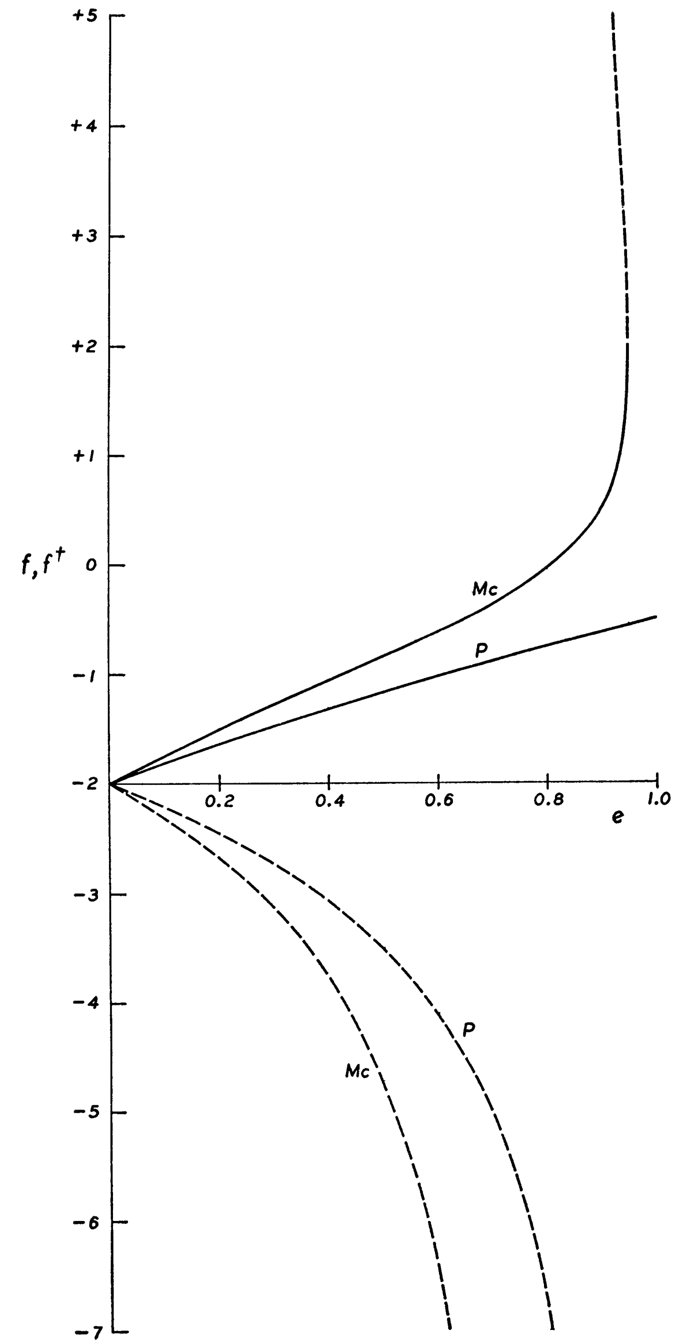

FIG. 1.—The variation of f (*full-line curves*) and $f\dagger$ (*dashed curves*) along the Maclaurin sequence (labeled "Mc") and the sequence of the prolate spheroids (labeled "P").

V. THE DELINEATION OF THE RIEMANN SEQUENCES

The Riemann sequences for different values of f are best delineated in an $(a_2/a_1, a_3/a_1)$-plane, where a_2 and a_3 are the two minor axes of the ellipsoid (see Fig. 2). In this plane the sphere is represented by the point $S = (1, 1)$. And the Maclaurin sequence is represented by points along the line $a_2/a_1 = 1$; on this line the point of bifurcation (where $\Omega_{\text{Mc}}^2 = 2B_{11}$) occurs at M_2, where $a_3/a_1 = 0.58272$ and $e = 0.81267$; and the point where $\Omega_{\text{Mc}}^2 = 4B_{11}$ and the Maclaurin spheroid becomes unstable by overstable oscillations occurs at O_2 where $a_3/a_1 = 0.30333$ and $e = 0.95289$. As we have seen in § IV, all Riemann sequences adjoin some member of the Maclaurin sequence represented by points on the segment O_2S of the line $a_2/a_1 = 1$. In particular, the Jacobian and the Dedekind sequences considered as Riemann sequences (belonging to $f = 0$ and $\pm\infty$, respectively) branch off at M_2 and join the origin becoming tangential to the line $a_2 = a_3$ at this point. We now proceed to a consideration of the Riemann sequences for some representative values of f.

a) *The Irrotational Sequence,* $\zeta^{(0)} = 0$ *and* $f = -2$

The Riemann sequence for $f = -2$ is of particular interest since its first member is a sphere; its generalization to compressible masses may be expected to have practical applications for astrophysics (see a forthcoming paper by Chandrasekhar and Clement where this matter is considered).

As we have already remarked, when $f = -2$ the vorticity of the internal motions exactly cancels the vorticity due to the rotation. It is on this account that the sequence is designated irrotational. It further follows from equation (48) that the adjoint to an irrotational ellipsoid has zero angular momentum.[4] *There exists then a sequence of zero angular momentum ellipsoids; and this sequence is the adjoint of the irrotational sequence.*

Turning to a more detailed consideration of the irrotational ellipsoids, we first observe that according to equations (32) we now have

$$Q_1 = \frac{2a_1^2\Omega}{a_1^2 + a_2^2} \quad \text{and} \quad Q_2 = -\frac{2a_2^2\Omega}{a_1^2 + a_2^2}. \tag{59}$$

These values of Q_1 and Q_2 when inserted into equations (25) and (26) give

$$\frac{a_1^2(a_1^2 - a_2^2)(a_1^2 + 3a_2^2)}{(a_1^2 + a_2^2)^2}\,\Omega^2 = 2(A_1a_1^2 - A_3a_3^2) = 2(a_1^2 - a_3^2)B_{13} \tag{60}$$

and

$$\frac{a_2^2(a_1^2 - a_2^2)(a_2^2 + 3a_1^2)}{(a_1^2 + a_2^2)^2}\,\Omega^2 = 2(A_3a_3^2 - A_2a_2^2) = 2(a_3^2 - a_2^2)B_{23}. \tag{61}$$

Since the left-hand side of equation (61) is necessarily positive (since $a_1 > a_2$ by definition) it follows that

$$a_3 > a_2 . \tag{62}$$

The irrotational ellipsoids are therefore prolate.[5] In particular, the axis in the direction of rotation is *not* the least axis.

The equation determining the geometry of the ellipsoid can be obtained by setting $f = -2$ in equation (35), or, more directly, by taking the ratio of equations (60) and (61). We find

$$a_1^2a_2^2[A_1(a_2^2 + 3a_1^2) + A_2(a_1^2 + 3a_2^2)] = A_3a_3^2(a_1^4 + 6a_1^2a_2^2 + a_2^4)\ ; \tag{63}$$

[4] This fact was first noticed by Norman R. Lebovitz.

[5] We shall call an ellipsoid *prolate* if $a_1 > a_3 > a_2$ and *oblate* if $a_1 > a_2 > a_3$.

and the value of Ω^2 to be associated with a solution of equation (63) is given by (cf. eq. [33])

$$\Omega^2 = \frac{2(a_1^2 + a_2^2)^2}{a_1^4 + 6a_1^2 a_2^2 + a_2^4} B_{12} . \tag{64}$$

Equations, essentially equivalent to the two foregoing, were derived by Greenhill (1879, 1880) in his investigations of this subject.

Equation (63) has been solved for a number of initially assigned values of the ratio a_2/a_1. The results of the calculations are summarized in Table 2. The irrotational sequence is delineated in Figure 2; and the variation of Ω^2 along the sequence is exhibited in Figure 3.

In many respects the most striking feature of the irrotational sequence is the extreme sensitivity of the figure to very small changes of Ω^2 as we progress along the sequence from its initial spherical form. Thus Ω^2 changes by less than 1 per cent from its

TABLE 2*

THE PROPERTIES OF THE IRROTATIONAL ELLIPSOIDS

ϕ . . θ	0°	30° 48°.828	45° 54°.2405	55° 59°.822	57° 61°.190	59° 62°.656	60° 63°.4267
a_2/a_1	1 00000	0 86603	0 70711	0 57358	0 54464	0 51504	0 50000
a_3/a_1	1 00000	0 92647	0 81899	0 70608	0 67822	0 64829	0 63252
A_1 .	0 66667	0 60869	0 52665	0 44363	0 42361	0 40231	0 39117
A_2	0 66667	0 72345	0 79948	0 87005	0 88587	0 90211	0 91035
A_3	0 66667	0 66786	0 67387	0 68632	0 69052	0 69558	0 69848
Ω^2	0 26667	0 26715	0 26875	0 26882	0 26811	0 26686	0 26598
Q_1Q_2	−0 26667	−0 26170	−0 23889	−0 20029	− 0 18921	− 0 17688	− 0 17023
Ω	0 51640	0 51687	0 51841	0 51848	0 51779	0 51659	0 51573
Q_1	0 51640	0 59070	0 69121	0 78026	0 79867	0 81657	0 82517
Q_2	−0 51640	−0 44303	−0 34561	−0 25670	− 0 23691	− 0 21661	− 0 20629
$f\dagger$.	−2 00000	−2 04167	−2 25000	−2 68430	− 2 83391	− 3 01755	− 3 12500
$\Omega\dagger$	0 51640	0 51156	0 48876	0 44754	0 43499	0 42057	0 41259
$Q_1\dagger$	0 51640	0 59682	0 73314	0 90394	0 95071	1 00301	1 03147
$Q_2\dagger$	−0 51640	−0 44762	−0 36657	−0 29739	− 0 28201	− 0 26606	− 0 25787

* Cos $\phi = a_2/a_1$; sin $\theta = [(a_1^2 - a_3^2)/(a_1^2 - a_2^2)]^{1/2}$; A_1, A_2, and A_3 are so normalized that $\Sigma A_i = 2$; Ω, Q_1, and Q_2 are measured in the unit $\sqrt{(\pi G\rho)}$; and $f\dagger$, $\Omega\dagger$, etc , refer to the adjoint configurations having the same figure

ϕ . . . θ .	70° 72°.618	71° 73°.6775	72° 74°.756	75° 78°.074	78° 81°.4035	80° 83°.535	85° 87°.991
a_2/a_1	0 34202	0 32557	0 30902	0 25882	0 20791	0 17365	0 08716
a_3/a_1	0 44248	0 42025	0 39752	0 32685	0 25417	0 20603	0 09389
A_1	0 26022	0 24522	0 22992	0 18264	0 13452	0 10313	0 03475
A_2 . .	0 99102	0 99801	1 00458	1 02095	1 03063	1 03239	1 01973
A_3	0 74876	0 75677	0 76550	0 79641	0 83485	0 86449	0 94552
Ω^2	0 23768	0 23164	0 22475	0 19825	0 16228	0 13332	0 05285
Q_1Q_2	−0 08914	−0 08029	−0 07153	−0 04666	− 0 02578	− 0 01515	− 0 00158
Ω	0 48753	0 48129	0 47407	0 44525	0 40283	0 36513	0 22988
Q_1	0 87294	0 87034	0 86550	0 83459	0 77228	0 70888	0 45630
Q_2	−0 10211	−0 09225	−0 08265	−0 05591	− 0 03338	− 0 02138	− 0 00347
$f\dagger$	−5 33280	−5 77022	−6 28381	−8 49760	−12 58839	−17 59680	−66 82684
$\Omega\dagger$	0 29856	0 28335	0 26745	0 21601	0 16057	0 12310	0 03977
$Q_1\dagger$	1 42543	1 47832	1 53414	1 72030	1 93752	2 10269	2 63759
$Q_2\dagger$	−0 16674	−0 15669	−0 14650	−0 11524	− 0 08375	− 0 06340	− 0 02004

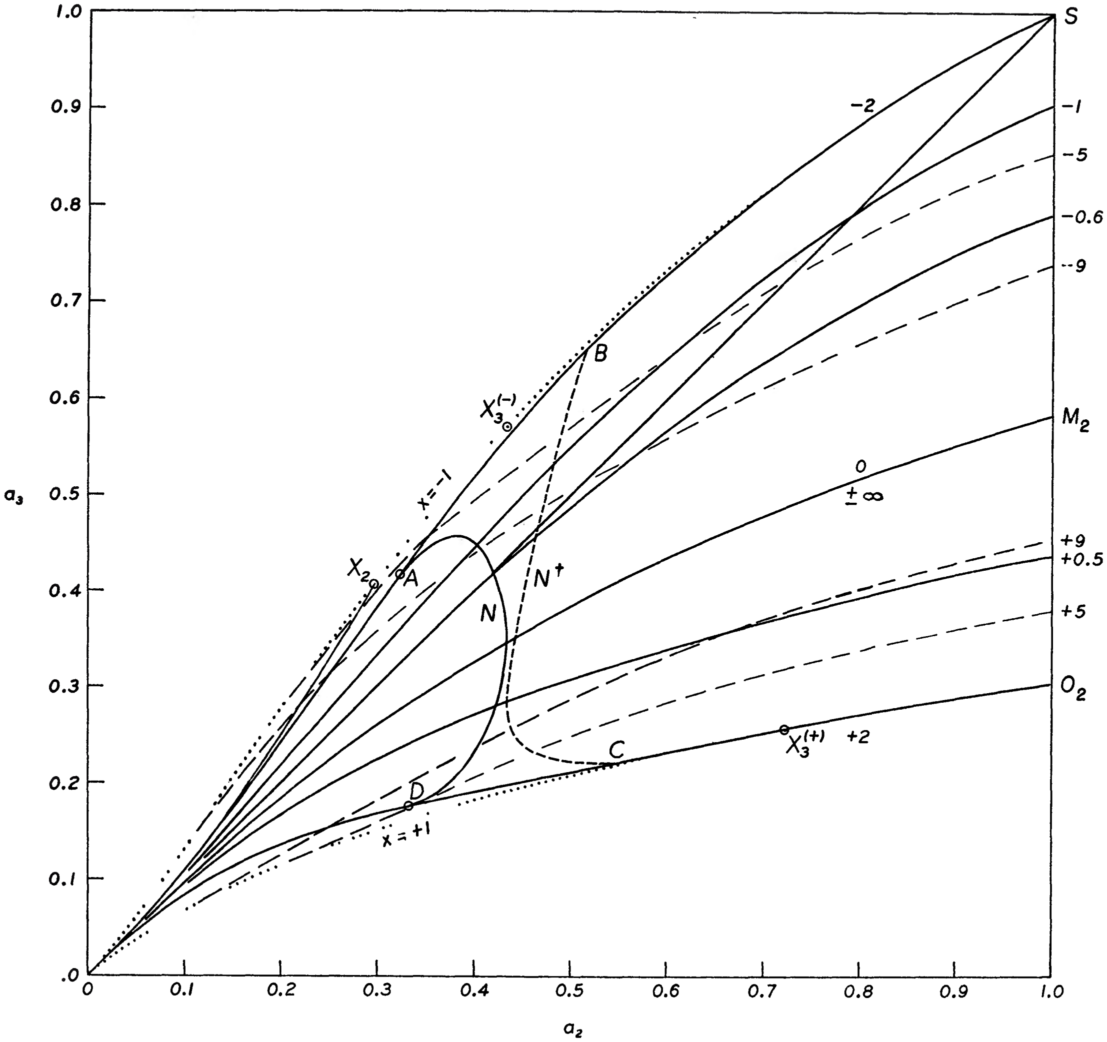

FIG. 2.—The delineation of the Riemann sequences in the (a_2, a_3)-plane (a_1 has been set equal to 1) The stable part of the Maclaurin sequence is represented by the segment O_2S of the line $a_2 = 1$. At O_2 the Maclaurin spheroid becomes unstable by overstable oscillations and at M_2 the Jacobian and the Dedekind sequences bifurcate (labeled by "0, $\pm\infty$").

The different Riemann sequences are labeled by the values of f to which they belong; these sequences are bounded by the two self-adjoint sequences (the dotted curves labeled $x = -1$ and $x = +1$) along which $f = f\dagger = \mp(a_1^2 + a_2^2)/a_1a_2$. The sequences belonging to f in the range $-2 \leq f \leq +2$ form a non-intersecting family of continuous curves which join points on the line O_2S to the origin. The sequences belonging to $f < -2$ and $f > +2$ are represented by curves which consist of two parts: a part which joins a point on the line SM_2 (or M_2O_2) to a point of the self-adjoint sequence for $x = -1$ (or $x = +1$) and a part which joins the point on the self-adjoint sequence to the origin. Along the self-adjoint sequence $x = -1$, instability by a mode of oscillation belonging to the second harmonics sets in at the point indicated by X_2 and the locus of points at which instability by this mode sets in, is the curve which joins X_2 to the origin. The curve labeled AND is the locus of neutral points, belonging to the third harmonics, along the Riemann sequences for $-2 \leq f \leq +2$; and the curve labeled $BN\dagger C$ is the corresponding locus for configurations adjoint to the Riemann ellipsoids represented in the domain included between the same sequences $f = -2$ and $f = +2$ The continuations of the loci AND and $BN\dagger C$ into the domains included between the sequences $x = -1$ and $f = -2$ (and, similarly, between the sequences $x = +1$ and $f = +2$) are represented by curves (not shown) joining the points A and B to $X_3^{(-)}$ on the sequence $x = -1$ (and, similarly, by curves joining the points D and C to the point $X_3^{(+)}$ on the sequence $x = +1$); $X_3^{(-)}$ and $X_3^{(+)}$ are the neutral points, belonging to the third harmonics, along the self-adjoint sequences $x = -1$ and $x = +1$, respectively.

initial value $4\pi G\rho/15$ for figures of equilibrium comprised in the range $1 \geq a_2/a_1 \geq 0.5$ and $1 \geq a_3/a_1 \geq 0.6325$.

b) Prolate Spheroids among the Riemann Ellipsoids

We have seen that the irrotational sequence is entirely prolate (i.e., $a_1 > a_3 > a_2$); but the Jacobian sequence, which is a Riemann sequence for $f = 0$, is entirely oblate (i.e., $a_1 > a_2 > a_3$). Moreover, since the Riemann sequences for $f \neq -2$, all have for their first members Maclaurin spheroids with $\Omega_{\mathrm{Mc}}^2 > 0$, it is clear that these sequences begin as oblate objects. Indeed, it follows from remark (iv) in § IV (p. 898) that *all* Riemann ellipsoids belonging to positive values of f must be oblate since they are represented by points below the Jacobian sequence in the (a_2, a_3)-plane. However, Riemann sequences for at least some negative values of f *must* end as prolate objects. And the question arises as to which among the Riemann sequences belonging to negative values

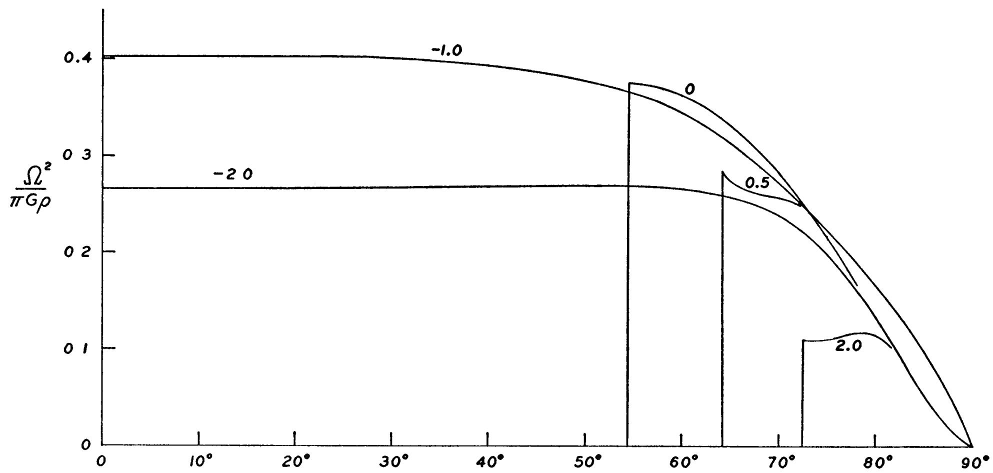

FIG. 3.—The variation of $\Omega^2/\pi G\rho$ along the various Riemann sequences; the curves are labeled by the values of f to which they belong. The abscissa is $\cos^{-1}(a_2/a_1)$ for the sequences $f = -2$ and -1 and $\cos^{-1}(a_3/a_1)$ for the sequences $f = 0$ (the Jacobian sequence), 0.5, and 2.

of f include prolate ellipsoids and in particular *prolate spheroids*. This question can be answered with the aid of equation (35); thus, setting $a_2 = a_3$ and $A_2 = A_3$ in this equation, we obtain

$$a_1^2 a_2^2 f^2 + 2a_1^2(a_1^2 + a_2^2)f + (a_1^2 + a_2^2)^2 = 0 . \tag{65}$$

Solving equation (65) for f, we have

$$f = -\left(1 + \frac{a_1^2}{a_2^2}\right)\left[1 \pm \left(1 - \frac{a_2^2}{a_1^2}\right)^{1/2}\right]. \tag{66}$$

In terms of the eccentricity

$$e = \sqrt{(1 - a_2^2/a_1^2)} , \tag{67}$$

the solution for f takes the simple form

$$f = -\frac{2 - e^2}{1 \pm e} . \tag{68}$$

The two solutions for f,

$$f = -\frac{2 - e^2}{1 + e} \quad \text{and} \quad f\dagger = -\frac{2 - e^2}{1 - e} , \tag{69}$$

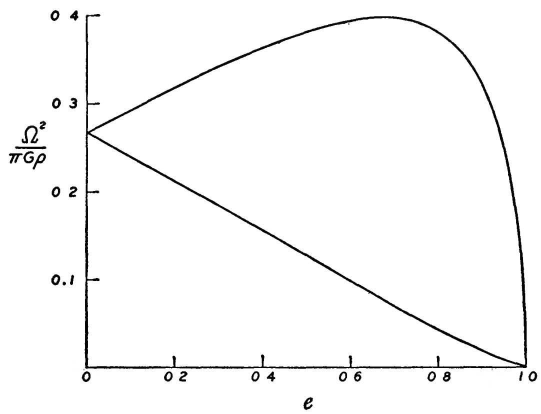

FIG. 4.—The variation of $\Omega^2/\pi G\rho$ along the sequence of prolate spheroids for the two adjoint configurations associated with each figure.

TABLE 3

THE PHYSICAL PARAMETERS OF THE PROLATE SPHEROIDS INCLUDED AMONG THE RIEMANN ELLIPSOIDS

e	f	Ω^2	Q_1	Q_2	$f\dagger$	$(\Omega\dagger)^2$	$Q_1\dagger$	$Q_2\dagger$
0	−2 00000	0 26667	0 51640	−0 51640	− 2 00000	0 26667	0 51640	−0 51640
0 05	−1 90238	28018	50411	− 50285	− 2 10263	25350	0 52999	− 52866
.10	−1 80909	29289	49199	− 48707	− 2 21111	23963	0 54391	− 53847
.15	−1 71957	30566	48075	− 46994	− 2 32647	22592	0 55919	− 54661
20	−1 63333	31810	47000	− 45120	− 2 45000	21207	0 57563	− 55261
25	−1 55000	.33016	45968	− 43095	− 2 58333	19810	0 59344	− 55635
30	−1 46923	34177	44970	− 40923	− 2 72857	18403	0 61284	− 55769
.35	−1 39074	35285	44001	− 38611	− 2 88846	.16989	0 63412	− .55644
.40	−1 31429	36324	43050	− 36162	− 3 06667	15568	0 65760	− 55238
45	−1 23966	37281	42109	− 33582	− 3 26818	14141	0 68372	− 54526
50	−1 16667	38131	.41167	− 30875	− 3 50000	12710	0 71303	− 53477
.55	−1 09516	.38846	.40211	− .28047	− 3 77222	11278	0 74628	− 52053
60	−1 02500	.39386	.39224	− 25103	− 4 10000	09846	0 78448	− 50206
65	−0 95606	.39692	38183	− 22051	− 4 50714	08420	0 82904	− 47877
70	−0 88824	39682	37055	− .18898	− 5 03333	07003	0 88209	− 44986
75	−0 82143	39225	35789	− 15658	− 5 75000	05604	0 94688	− 41426
.80	−0 75556	.38112	34297	− 12347	− 6 80000	04235	1 02892	− 37041
.82	−0 72945	37409	33606	− 11009	− 7 37556	03700	1 06860	− 35007
84	−0 70348	.36508	32838	− 09668	− 8 09000	03175	1 11360	− 32784
.86	−0 67763	35364	31972	− 08325	− 9 00286	02662	1 16536	− 30346
88	−0 65191	.33908	.30974	− 06988	−10 21333	02164	1 22597	− 27658
.90	−0 62632	32045	29794	− 05661	−11 90000	01687	1 29868	− 24675
.92	−0 60083	.29628	.28350	− 04354	−14 42000	01234	1 38884	− 21333
94	−0 57546	26412	26491	− 03084	−18 60667	00817	1 50636	− 17534
96	−0 55020	21944	23900	− 01874	−26 96000	00448	1 67300	− 13116
0 98	−0 52505	0 15146	0 19656	−0 00778	−51 98000	0 00153	1 95570	−0 07745

given by equation (68) define adjoint configurations; for, as can be readily verified, f and $f\dagger$ as defined are consistent with the general relation (39) which now requires that

$$f = \frac{(2-e^2)^2}{1-e^2}\frac{1}{f\dagger}. \tag{70}$$

The solution for f given by equation (68) allows all values of $f < -\frac{1}{2}$ (see Fig. 1). Therefore, along all Riemann sequences for $f < -\frac{1}{2}$, a prolate spheroid occurs; therefore, in the (a_2, a_3)-plane the curves representing these sequences must intersect the line $a_2 = a_3$. We conclude then that *the Riemann sequences for* f $< -\frac{1}{2}$ *start as oblate ellipsoids and end as prolate ellipsoids, while the Riemann sequences for* f $> -\frac{1}{2}$ *consist entirely of oblate ellipsoids.*

The values of Ω^2 that are to be associated with a prolate spheroid of eccentricity e follow from equation (33). We find

$$\Omega^2 = (1 \pm e)B_{12}(e)\ ; \tag{71}$$

the variation of Ω^2 with e predicted by this equation is illustrated in Figure 4.

The occurrence of a sequence of prolate spheroids among the Riemann ellipsoids seems to have been first noticed by Greenhill (1879, 1880).

Finally, in Table 3 we list some of the physical parameters which characterize the sequence of prolate spheroids.

c) *The Self-adjoint Sequences* $x = \pm 1$

From equations (36)–(38) it is clear that any set of values a_1, a_2 ($<a_1$ by definition), and a_3 consistent with these equations provides a solution. The only restriction these equations impose is the requirement that equation (38) allows real roots for x; and the reality of x requires that

$$A_1 a_1^2 - A_2 a_2^2 \geq a_1 a_2 \left| A_1 - A_2 + \frac{a_1^2 - a_2^2}{a_1^2 a_2^2} a_3^2 A_3 \right| . \tag{72}$$

On simplification, this condition leads to the inequalities

$$\frac{a_1 + a_2}{a_1 a_2} a_3^2 A_3 \leq a_2 A_2 + a_1 A_1 \qquad \text{(case i)}$$

and

$$\frac{a_1 - a_2}{a_1 a_2} a_3^2 A_3 \geq a_2 A_2 - a_1 A_1 \qquad \text{(case ii)}. \tag{73}$$

When these inequalities degenerate into equalities, equation (38) allows two equal roots ($x = -1$ and $x = +1$, respectively), and we are led to two *self-adjoint sequences* along which

$$x = -1, f = f\dagger = -(a_1^2 + a_2^2)/a_1 a_2, \qquad \Omega^2 = -Q_1 Q_2 = B_{12},$$
$$Q_1 \Omega = +a_1 B_{12}/a_2, \qquad \text{and} \qquad Q_2 \Omega = -a_2 B_{12}/a_1\ ;$$

and

$$x = +1, f = f\dagger = +(a_1^2 + a_2^2)/a_1 a_2, \qquad \Omega^2 = -Q_1 Q_2 = B_{12},$$
$$Q_1 \Omega = -a_1 B_{12}/a_2, \qquad \text{and} \qquad Q_2 \Omega = +a_2 B_{12}/a_1. \tag{74}$$

The properties of the Riemann ellipsoids along these two self-adjoint sequences are listed in Table 4. The sequences are further delineated in Figure 2. It is clear that these self-adjoint sequences limit the domain of occupancy of the Riemann sequences in the $(a_2/a_1, a_3/a_1)$-plane.

d) *The Riemann Sequences for* $f = -1.0, -0.6, +0.5,$ *and* 2.0

The equations governing the equilibrium of the Riemann ellipsoids belonging to $f = -1.0, -0.6, +0.5$, and 2.0 have been solved numerically for a sufficient number of cases to determine their behavior along the respective sequences. The results of the calculations are summarized in Tables 5, 6, 7, and 8. The sequences are delineated in Figure 2; and the variation of Ω^2 along them is further exhibited in Figure 3.

And finally, in Table 9 we list the properties of a few ellipsoids which occur in the

TABLE 4

THE PROPERTIES OF THE RIEMANN ELLIPSOIDS ALONG THE SELF-ADJOINT SEQUENCES

a) The Sequence $x = -1$

a_2/a_1	0 68127	0 56284	0 51014	0 46101	0 37135	0 29066	0 21645	0 14641	0 077470
a_3/a_1	0 80	0 70	0 65	0 60	0 50	0 40	0 30	0 20	0 10
A_1	0 51198	0 43747	0 40070	0 36423	0 29223	0 22169	0 15342	0 08918	0 03320
A_2	0 81312	0 87786	0 90841	0 93764	0 99217	1 04057	1 08076	1 10810	1 10988
A_3 .	0 67490	0 68468	0 69089	0 69813	0 71561	0 73774	0 76582	0 80272	0 85692
$f = f^\dagger$ . .	−2 1491	−2 3395	−2 4704	−2 6302	−3 0642	−3 7311	−4 8364	−6 9766	−12 986
$\Omega^2 = -Q_1Q_2$	0 25115	0 23326	0 22209	0 20948	0 18027	0 14612	0 10783	0 06686	0 02670
$\Omega = \Omega^\dagger$.	0 50115	0 48298	0 47127	0 45769	0 42458	0 38226	0 32838	0 25858	0 16340
$Q_1 = Q_1^\dagger$	0 73561	0 85810	0 92381	0 99279	1 14335	1 31514	1 51711	1 76611	2 1092
$Q_2 = Q_2^\dagger$	−0 34142	−0 27184	−0 24041	−0 21100	−0 15767	−0 11111	−0 07108	−0 03786	− 0 01266

b) The Sequence $x = +1$

a_2/a_1	0 75	0 70	0 60	0 55	0 50	0 40	0 30	0 20	0 10
a_3/a_1	0 26129	0 25172	0 23105	0 21986	0 20798	0 18181	0 15147	0.11513	0 068941
A_1	0 27694	0 26229	0 23091	0 21412	0 19653	0 15887	0 11787	0 074102	0 030198
A_2	0 40912	0 42560	0 46236	0 48299	0 50536	0 55641	0 61842	0 69642	0 80166
A_3 .	1 31395	1 31212	1 30672	1 30290	1 29811	1 28472	1 26370	1 22948	1 16814
$f = f^\dagger$ . .	2 0833	2 1286	2 2667	2 3682	2 5000	2 9000	3 6333	5 2000	10 1000
$\Omega^2 = -Q_1Q_2$	0 10699	0 10538	0 10072	0 097508	0.093583	0 083143	0 068368	0 048172	0 022406
$\Omega = \Omega^\dagger$	0 32709	0 32462	0 31737	0 31226	0 30591	0 28834	0 26147	0 21948	0 14969
$Q_1 = Q_1^\dagger$	−0 43612	−0 46375	−0 52894	−0 56775	−0 61183	−0 72086	−0 87157	−1 0974	− 1 4969
$Q_2 = Q_2^\dagger$	0 24532	0 22724	0 19042	0 17174	0 15296	0 11534	0 078442	0 043896	0 014969

TABLE 5*

THE PROPERTIES OF THE RIEMANN ELLIPSOIDS ALONG THE SEQUENCES FOR $f = -1\ 0$

ϕ . θ	. .	38°1727 90°	60° 74°627	61° 75°026	62° 75°460	66° 77°492	67° 78°063	68° 78°656
a_2/a_1 . .	1 00000	0 78615	0 50000	0 48481	0 46947	0 40674	0 39073	0 37461
a_3/a_1 . .	0 90352	0 78615	0 55019	0 53489	0 51917	0 45231	0 43465	0 41664
A_1 . .	0 63925	0 54342	0 36507	0 35374	0 34210	0 29259	0 27951	0 26616
A_2	0 63925	0 72829	0 86116	0 86822	0 87528	0 90340	0 91030	0 91709
A_3 . .	0 72151	0 72829	0 77377	0 77804	0 78262	0 80401	0 81020	0 81674
Ω^2 . .	0 40196	0 39527	0 34433	0 33907	0 33339	0 30578	0 29753	0 28867
Q_1Q_2	−0 10049	−0 09331	−0 05509	−0 05225	−0 04934	−0 03724	−0 03419	−0 03115
Ω .	0 63400	0 62870	0 58679	0 58230	0 57740	0 55298	0 54546	0 53728
Q_1 . .	0 31700	0 38856	0 46943	0 47148	0 47312	0 47448	0 47321	0 47116
Q_2 .	−0 31700	−0 24014	−0 11736	−0 11082	−0 10428	−0 07850	−0 07225	−0 06612
$f^\dagger$	−4 00000	−4 23607	−6 25000	−6 48963	−6 75754	−8 21012	−8 70271	−9 26638
$\Omega^\dagger$. .	0 31700	0 30547	0 23472	0 22858	0 22212	0 19299	0 18490	0 17650
$Q_1^\dagger$	0 63400	0 79972	1 17358	1 20108	1 22988	1 35954	1 39599	1 43424
$Q_2^\dagger$	−0 63400	−0 49426	−0.29340	−0 28230	−0 27107	−0 22492	−0 21313	−0 20127

* Cos $\phi = a_2/a_1$; sin $\theta = [(a_1^2 - a_3^2)/(a_1^2 - a_2^2)]^{1/2}$; and the remaining symbols have the same meanings as explained at the bottom of Table 2 The first and the second columns refer to configurations that are on the lines $a_2/a_1 = 1$ and $a_2 = a_3$, respectively

domains bounded by the self-adjoint sequences $x = -1$ and $x = +1$ and the sequences $f = -2$ and $f = +2$, respectively.

e) The Riemann Sequences for $-2 > f > -\infty$ and $+2 < f < +\infty$

We have seen that Riemann sequences for f in the range $-2 \leq f \leq +2$ are represented in the $(a_2/a_1, a_3/a_1)$-plane by a continuous family of non-intersecting curves which join the origin to points along the line O_2S (see Fig. 2). The Riemann sequences for f outside this range are included in the domain bounded by the self-adjoint sequences for

TABLE 6*

THE PROPERTIES OF THE RIEMANN ELLIPSOIDS ALONG THE SEQUENCES FOR $f = -0.6$

ϕ	37°5628	52°	67°0221	72°
θ	0°	71°085	90°	87°400
a_2/a_1	1 00000	0 66655	0 39038	0 30902
a_3/a_1	0 79269	0 61566	0 39038	0 31201
A_1	0 60299	0 44796	0 26272	0 19916
A_2	0 60299	0 74086	0 86864	0 90508
A_3	0 79402	0 81118	0 86864	0 89576
Ω^2	0 42476	0 39711	0 29537	0 24232
Q_1Q_2	−0 03823	−0 03045	− 0 01220	− 0 00694
Ω	0 65173	0 63017	0 54347	0 49226
Q_1	0 19552	0 26179	0.28296	0 26961
Q_2	−0 19552	−0 11631	− 0 04312	− 0 02575
f†	−6 66667	−7 82511	−14 52393	−20 94605
Ω†	0 19552	0 17450	0 11046	0 08331
Q_1†	0 65173	0 94541	1 39218	1 59299
Q_2†	−0 65173	−0 42004	− 0 21216	− 0 15212

* In the first two columns $\cos\phi = a_3/a_1$ and $\sin\theta = [(a_1^2 - a_2^2)/(a_1^2 - a_3^2)]^{1/2}$; and in the second two columns $\cos\phi = a_2/a_1$ and $\sin\theta = [(a_1^2 - a_3^2)/(a_1^2 - a_2^2)]^{1/2}$; and the remaining symbols have the same meanings as explained at the bottom of Table 2 And the first and the third columns refer to configurations on the lines $a_2/a_1 = 1$ and $a_2 = a_3$, respectively

TABLE 7*

THE PROPERTIES OF THE RIEMANN ELLIPSOIDS ALONG THE SEQUENCES FOR $f = 0$ 5

ϕ	64°1095	69°	72°	73°	74°	75°
θ	0°	52°478	65°214	68°543	71°498	74°120
a_2/a_1	1 00000	0 67212	0 50444	0 45591	0 41113	0 36992
a_3/a_1	0 43665	0 35837	0 30902	0 29237	0 27564	0 25882
A_1	0 43545	0 32896	0 26099	0 23886	0 21726	0 19631
A_2	0 43545	0 55436	0 64248	0 67264	0 70249	0 73169
A_3	1 12910	1 11668	1 09652	1 08850	1 08026	1 07200
Ω^2	0 28182	0 25875	0 25141	0 24147	0 23000	0 21711
Q_1Q_2	−0 01761	−0 01387	− 0 01016	− 0 00860	− 0 00711	− 0 00575
Ω	0 53086	0 50868	0 50141	0 49139	0 47958	0 46595
Q_1	−0 13272	−0 17520	− 0 19985	− 0 20342	− 0 20512	− 0 20493
Q_2	0 13272	0 07914	0 05085	0 04228	0 03467	0 02804
f†	8 00000	9 33075	12 36863	14 03782	16 17040	18 88910
Ω†	0 13272	0 11775	0 10081	0 09274	0 08433	0 07581
Q_1†	−0 53086	−0 75682	− 0 99399	− 1 07783	− 1 16649	− 1 25960
Q_2†	0 53086	0 34189	0 25293	0 22403	0 19717	0 17237

* $\cos\phi = a_3/a_1$; $\sin\theta = [(a_1^2 - a_2^2)/(a_1^2 - a_3^2)]^{1/2}$; and the remaining symbols have the same meanings as explained at the bottom of Table 2

$x = -1$ and $x = +1$; and they can be sketched in with the information already at our disposal by following the procedure outlined below.

The adjoint of an ellipsoid along a given sequence belongs to some other sequence; and the value of f to which the adjoint belongs is $f\dagger$ given by equation (39). Along the sequence for a given f, the value of $f\dagger$ varies; and this variation, for the sequences we have constructed, is shown in Figure 5. Figure 5 also includes the variation of $f(= f\dagger)$ along the self-adjoint sequences for $x = -1$ and $x = +1$. These variations together with those along the Maclaurin sequence and the sequence of the prolate spheroids (shown in Fig. 1) will enable us to delineate in Figure 2 the Riemann sequences for f outside the range $-2 \leq f \leq +2$. The sequences for $f = -5, -9, +9$, and $+5$ (determined graphically with the aid of Figs. 1 and 5) have been sketched in Figure 2.

We observe that the sequences for $f < -2$ and $f > +2$ are represented in the $(a_2/a_1, a_3/a_1)$-plane by curves which consist of two parts: a part which joins a point on the line

TABLE 8*

THE PROPERTIES OF THE RIEMANN ELLIPSOIDS ALONG THE SEQUENCES FOR $f = 2\ 0$

ϕ θ	72°3425 0°	75° 44°516	76° 51°732	77° 57°936	79° 68°632	80° 73°340	83° 83°401
a_2/a_1	1 00000	0 73578	0 64781	0 56406	0 40538	0 33146	0 16692
a_3/a_1	0 30333	0 25882	0 24192	0 22495	0 19081	0 17365	0 12187
A_1	0 34132	0 27299	0 24681	0 22028	0 16591	0 13851	0 06863
A_2	0 34132	0 41391	0 44501	0 47975	0 56815	0 62574	0 81114
A_3	1 31737	1 31309	1 30818	1 29997	1 26594	1 23575	1 12024
Ω^2	0 11005	0 11159	0 11292	0 11453	0 11693	0 11554	0 08565
Q_1Q_2	−0 11005	−0 10171	−0 09405	−0 08389	−0 05670	−0 04122	− 0 00904
Ω	0 33175	0 33405	0 33604	0 33843	0 34196	0 33991	0 29267
Q_1	−0 33175	−0 43344	−0 47341	−0 51348	−0 58739	−0 61252	− 0 56946
Q_2	0 33175	0 23465	0 19867	0 16337	0 09653	0 06730	0 01587
$f\dagger$	2 00000	2 19426	2 40127	2 73061	4 12481	5 60587	18 95916
$\Omega\dagger$	0 33175	0 31892	0 30668	0 28963	0 23811	0 20303	0 09506
$Q_1\dagger$	−0 33175	−0 45400	−0 51873	−0 59999	−0 84355	−1 02547	− 1 75332
$Q_2\dagger$	0 33175	0 24579	0 21769	0 19089	0 13862	0 11267	0 04885

* Cos $\phi = a_3/a_1$; sin $\theta = [(a_1^2 - a_2^2)/(a_1^2 - a_3^2)]^{1/2}$; and the remaining symbols have the same meanings as explained at the bottom of Table 2

TABLE 9

THE PROPERTIES OF THE RIEMANN ELLIPSOIDS WHICH OCCUR IN THE DOMAINS BOUNDED BY THE SEQUENCES

$x = \mp 1$ AND $f = \mp 2$

	Ellipsoids in the Domain Bounded by the Sequences $x = -1$ and $f = -2$				Ellipsoids in the Domain Bounded by the Sequences $x = +1$ and $f = +2$			
a_2/a_1	0 30769	0 23077	0 16667	0 15385	0 3	0 2	0 2	0 1
a_3/a_1	0 4	0 3	0 2	0 2	0 15789	0 12500	0 11765	0 076923
A_1	0 23009	0 16041	0 098128	0 092535	0 12169	0 079039	0 075378	0 033011
A_2	1 00943	1 04656	1 04051	1.08239	0 63502	0 73231	0 70580	0 85354
A_3	0 76049	0 79303	0 86136	0 82508	1 24328	1 18865	1 21882	1 11345
f	−5 9801	−8 3140	−17 232	−11 8405	5 3534	9 6687	7 0137	25 359
Ω^2	0 077686	0 051209	0 016167	0 032899	0 044732	0 023252	0 034840	0 006770
Q_1Q_2	−0 21949	−0 16993	− 0 12624	− 0.10418	−0 097113	−0 080387	−0 063381	− 0 042676
Ω	0 27872	0 22629	0 12715	0 18138	0 21150	0 15249	0 18666	0 082279
Q_1	1 5226	1.7863	2 1318	2 0980	−1 03877	−1 4176	−1 2588	− 2 0658
Q_2	−0 14415	−0 095128	− 0.059217	− 0 049656	0 093489	0 056705	0 050351	0 020658
$f\dagger$	−2 1166	−2.5055	− 2 2068	− 3 7392	2 4659	2 7967	3 8553	4 0227
$\Omega\dagger$	0 46850	0 41222	0 35530	0 32277	0 31163	0 28353	0 25176	0 20658
$Q_1\dagger$	0 90585	0 98061	0 76289	1 1790	−0 70500	−0 76243	−0 93327	− 0 82279
$Q_2\dagger$	−0 085760	−0 052222	− 0 021191	− 0 02790	0 063450	0 030497	0 037331	0 008228

SM_2 (or M_2O_2) to a point on the self-adjoint sequence for $x = -1$ (or $x = +1$) and a part which joins the point on the self-adjoint sequence to the origin. It follows that the sequences for $f < -2$ intersect one another in the domain bounded by the sequences $f = -2$ and $x = -1$; similarly, the sequences for $f > +2$ intersect one another in the domain bounded by the sequences $f = +2$ and $x = +1$; and these facts are consistent with Dedekind's theorem which requires that each figure of equilibrium (except when it is self-adjoint) allows two distinct states of internal motions.

VI. THE SECOND-ORDER VIRIAL EQUATIONS GOVERNING SMALL OSCILLATIONS ABOUT EQUILIBRIUM

Suppose that an equilibrium ellipsoid determined consistently with respect to equations (33) and (35) is slightly perturbed. Let the ensuing motions be described in terms

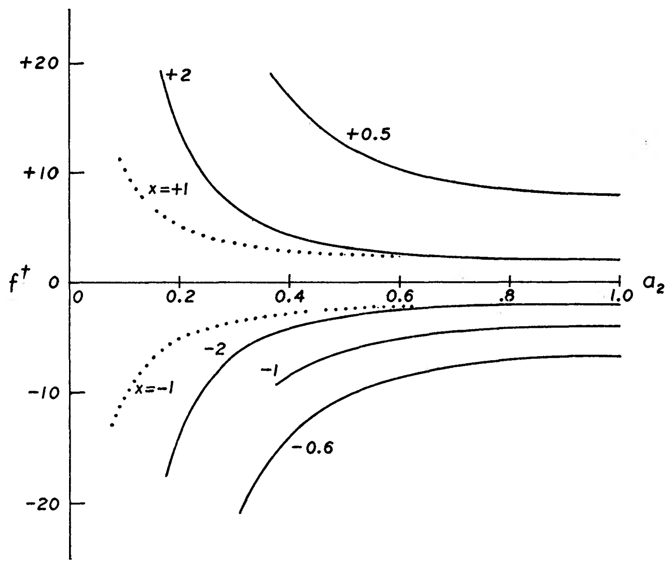

FIG. 5.—The variation of $f\dagger$ along the different Riemann sequences: the curves are labeled by the values of f to which they belong. The dotted curves represent the variation of $f = f\dagger$ along the two self-adjoint sequences for $x = +1$ and $x = -1$.

of a Lagrangian displacement $\boldsymbol{\xi}$. Then a fluid element originally at $\boldsymbol{x}$ with a velocity $\boldsymbol{u}$ will find itself at $\boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t)$ with a velocity $\boldsymbol{u} + \Delta \boldsymbol{u}$ where (to the first order in $\boldsymbol{\xi}$)

$$\Delta u_i = \frac{\partial \xi_i}{\partial t} + u_j \frac{\partial \xi_i}{\partial x_j}, \tag{75}$$

where u_j denotes the velocity in the equilibrium state. The virial equation (18) linearized to the first order in $\boldsymbol{\xi}$, gives

$$\begin{aligned}\frac{d}{dt}\int_V \rho(\Delta u_i x_j + u_i \xi_j)\, d\boldsymbol{x} - \int_V \rho(\Delta u_i u_j + \Delta u_j u_i)\, d\boldsymbol{x}& \\ - 2\Omega\epsilon_{il3}\int_V \rho(\Delta u_l x_j + u_l \xi_j)\, d\boldsymbol{x} = \Omega^2(V_{ij} - \delta_{i3}V_{3j}) + \delta\mathfrak{W}_{ij} + \delta_{ij}\delta\Pi\,,&\end{aligned} \tag{76}$$

where

$$V_{ij} = V_{i;j} + V_{j;i} = \int_V \rho\xi_i x_j\, d\boldsymbol{x} + \int_V \rho\xi_j x_i\, d\boldsymbol{x}\,. \tag{77}$$

Also, it follows from the definitions of the various quantities that (cf. Paper I, eq. [27])

$$\frac{dV_{i;j}}{dt}=\int_V \rho\Delta u_i x_j dx+\int_V \rho\,\xi_i u_j dx\,. \tag{78}$$

In the further reduction of equation (73) we shall assume that in the equilibrium configuration

$$u_j=Q_{jl}x_l\,, \tag{79}$$

where the Q_{jl}'s are certain constants. On this assumption equation (78) gives

$$\frac{dV_{i;j}}{dt}=\int_V \rho\Delta u_i x_j dx+Q_{jl}V_{i;l}\,. \tag{80}$$

A further application of this last equation gives

$$\int_V \rho\Delta u_i u_j dx=Q_{jl}\int_V \rho\Delta u_i x_l dx=Q_{jl}\frac{dV_{i;l}}{dt}-Q^2{}_{jl}V_{i;l}\,, \tag{81}$$

where Q^2 denotes the square of the matrix Q. Making use of the foregoing relations in equation (76) and simplifying, we obtain

$$\begin{aligned}\frac{d^2V_{i;j}}{dt^2}-2Q_{jl}\frac{dV_{i;l}}{dt}+Q^2{}_{jl}V_{i;l}+Q^2{}_{il}V_{j;l}&=\Omega^2(V_{ij}-\delta_{i3}V_{3j})\\&+2\Omega\epsilon_{il3}\left(\frac{dV_{l;j}}{dt}-Q_{jk}V_{l;k}+Q_{lk}V_{j;k}\right)+\delta\mathfrak{W}_{ij}+\delta_{ij}\delta\Pi\,.\end{aligned} \tag{82}$$

We shall now suppose that the Lagrangian displacement is of the form

$$\xi(x,t)=e^{\lambda t}\xi(x)\,, \tag{83}$$

where λ is a parameter whose characteristic values are to be determined. Equation (82) then gives

$$\begin{aligned}&\lambda^2V_{i;j}-2\lambda Q_{jl}V_{i;l}-2\lambda\Omega\epsilon_{il3}V_{l;j}+Q_{jl}{}^2V_{i;l}+Q_{il}{}^2V_{j;l}\\&-2\Omega\epsilon_{il3}(Q_{lk}V_{j;k}-Q_{jk}V_{l;k})=\Omega^2(V_{ij}-\delta_{i3}V_{3j})+\delta\mathfrak{W}_{ij}+\delta_{ij}\delta\Pi\,.\end{aligned} \tag{84}$$

And this equation must be supplemented by the condition,

$$\frac{V_{11}}{a_1{}^2}+\frac{V_{22}}{a_2{}^2}+\frac{V_{33}}{a_3{}^2}=0\,, \tag{85}$$

required by the solenoidal character of ξ.

For the particular case of the Riemann ellipsoids, the matrices Q and Q^2 have the simple forms

$$Q=\left|\begin{matrix}0 & Q_1 & 0\\ Q_2 & 0 & 0\\ 0 & 0 & 0\end{matrix}\right| \quad\text{and}\quad Q^2=\left|\begin{matrix}Q_1Q_2 & 0 & 0\\ 0 & Q_1Q_2 & 0\\ 0 & 0 & 0\end{matrix}\right|\,. \tag{86}$$

Also, it is known that $\delta\mathfrak{W}_{ij}$ can be expressed in terms of the symmetrized virials V_{ij}. Thus (cf. Chandrasekhar and Lebovitz 1963*b*, eqs. [47] and [48])

$$\delta\mathfrak{W}_{ij} = -2B_{ij}V_{ij} \qquad (i \neq j) \tag{87}$$

and

$$\delta\mathfrak{W}_{ii} = -(2B_{ii} - a_i^2 A_{ii})V_{ii} + a_i^2 \sum_{l \neq i} A_{il}V_{ll} \tag{88}$$

(no summation over repeated indices in eqs. [87] and [88]).

(In writing eqs. [87] and [88] a common factor $\pi G\rho a_1 a_2 a_3$ has been suppressed. The suppression of this factor is equivalent to the normalization of the index symbols to give $\Sigma A_i = 2$ and the measurement of Ω^2 and ζ_Ω^2 in the unit $\pi G\rho$—conventions which we have consistently adopted.)

We shall now write down the explicit forms which the different components of equation (84) take in view of the special forms of the matrices $\boldsymbol{Q}$ and $\boldsymbol{Q}^2$. The five equations even in the index 3 are:

$$\tfrac{1}{2}\lambda^2 V_{33} = \delta\mathfrak{W}_{33} + \delta\Pi\,, \tag{89}$$

$$(\tfrac{1}{2}\lambda^2 + Q_1Q_2 - \Omega^2)V_{11} - 2\lambda Q_1 V_{1;2} - 2\lambda\Omega V_{2;1} - \Omega(Q_2V_{11} - Q_1V_{22}) = \delta\mathfrak{W}_{11} + \delta\Pi\,, \tag{90}$$

$$(\tfrac{1}{2}\lambda^2 + Q_1Q_2 - \Omega^2)V_{22} - 2\lambda Q_2 V_{2;1} + 2\lambda\Omega V_{1;2} + \Omega(Q_1V_{22} - Q_2V_{11}) = \delta\mathfrak{W}_{22} + \delta\Pi\,, \tag{91}$$

$$\lambda^2 V_{1;2} - \lambda Q_2 V_{11} + Q_1Q_2V_{12} - \lambda\Omega V_{22} = \delta\mathfrak{W}_{12} + \Omega^2 V_{12} = -(2B_{12} - \Omega^2)V_{12}\,, \tag{92}$$

$$\lambda^2 V_{2;1} - \lambda Q_1 V_{22} + Q_1Q_2V_{12} + \lambda\Omega V_{11} = \delta\mathfrak{W}_{12} + \Omega^2 V_{12} = -(2B_{12} - \Omega^2)V_{12}\,, \tag{93}$$

where in equations (92) and (93) we have substituted for $\delta\mathfrak{W}_{12}$ in accordance with equation (87). Similarly, the four equations odd in the index 3 are:

$$\begin{aligned}
\lambda^2 V_{1;3} - 2\lambda\Omega V_{2;3} + Q_1Q_2V_{3;1} - 2\Omega Q_2V_{3;1} &= \delta\mathfrak{W}_{13} + \Omega^2V_{13} = -(2B_{13} - \Omega^2)V_{13}\,,\\
\lambda^2 V_{2;3} + 2\lambda\Omega V_{1;3} + Q_1Q_2V_{3;2} + 2\Omega Q_1V_{3;2} &= \delta\mathfrak{W}_{23} + \Omega^2V_{23} = -(2B_{23} - \Omega^2)V_{23}\,,\\
\lambda^2 V_{3;1} - 2\lambda Q_1V_{3;2} + Q_1Q_2V_{3;1} &= \delta\mathfrak{W}_{13} = -2B_{13}V_{13}\,,\\
\lambda^2 V_{3;2} - 2\lambda Q_2V_{3;1} + Q_1Q_2V_{3;2} &= \delta\mathfrak{W}_{23} = -2B_{23}V_{23}\,.
\end{aligned} \tag{94}$$

VII. THE CHARACTERISTIC FREQUENCIES OF OSCILLATION OF THE RIEMANN ELLIPSOIDS BELONGING TO THE SECOND HARMONICS

Considering first equations (89)–(93) governing the even modes, we observe that, in view of the relation (24), equations (92) and (93) become

$$\lambda^2 V_{1;2} - \lambda Q_2 V_{11} - \lambda\Omega V_{22} = 0 \tag{95}$$

and

$$\lambda^2 V_{2;1} - \lambda Q_1 V_{22} + \lambda\Omega V_{11} = 0\,. \tag{96}$$

Excluding the possibility that λ may be zero—a possibility to which we shall return presently—we may conclude from equations (95) and (96) that

$$\lambda V_{1;2} = Q_2 V_{11} + \Omega V_{22} \tag{97}$$

and

$$\lambda V_{2;1} = Q_1 V_{22} - \Omega V_{11}\,. \tag{98}$$

Eliminating $V_{1;2}$ and $V_{2;1}$ from equations (90) and (91) with the aid of equations (97) and (98), we obtain

$$(\tfrac{1}{2}\lambda^2 + \Omega^2 - Q_1Q_2 - \Omega Q_2)V_{11} - 3\Omega Q_1 V_{22} = \delta\mathfrak{W}_{11} + \delta\Pi \tag{99}$$

and

$$(\tfrac{1}{2}\lambda^2 + \Omega^2 - Q_1Q_2 + \Omega Q_1)V_{22} + 3\Omega Q_2 V_{11} = \delta\mathfrak{W}_{22} + \delta\Pi\,. \tag{100}$$

Next eliminating $\delta\Pi$ from equations (99) and (100) with the aid of equation (89), we obtain the pair of equations

$$\begin{aligned}(\tfrac{1}{2}\lambda^2 + 2B_{12} - \Omega Q_2)V_{11} - 3\Omega Q_1 V_{22} - \tfrac{1}{2}\lambda^2 V_{33} &= \delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{33} \\ &= -(3B_{11} - B_{13})V_{11} + (B_{23} - B_{12})V_{22} + (3B_{33} - B_{13})V_{33}\end{aligned} \tag{101}$$

and

$$\begin{aligned}(\tfrac{1}{2}\lambda^2 + 2B_{12} + \Omega Q_1)V_{22} + 3\Omega Q_2 V_{11} - \tfrac{1}{2}\lambda^2 V_{33} &= \delta\mathfrak{W}_{22} - \delta\mathfrak{W}_{33} \\ &= -(3B_{22} - B_{23})V_{22} + (B_{13} - B_{12})V_{11} + (3B_{33} - B_{23})V_{33}\,,\end{aligned} \tag{102}$$

where we have made use of the relation (24) and further substituted for $\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{33}$ and $\delta\mathfrak{W}_{22} - \delta\mathfrak{W}_{33}$ in accordance with equation (88).

Equations (101) and (102) must be supplemented by equation (85) which expresses the solenoidal condition on $\boldsymbol{\xi}$; and these three equations lead to the following characteristic equation for λ^2:

$$\begin{bmatrix} \tfrac{1}{2}\lambda^2+2B_{12}-\Omega Q_2+3B_{11}-B_{13} & B_{12}-B_{23}-3\Omega Q_1 & -\tfrac{1}{2}\lambda^2-3B_{33}+B_{13} \\ B_{12}-B_{13}+3\Omega Q_2 & \tfrac{1}{2}\lambda^2+2B_{12}+\Omega Q_1+3B_{22}-B_{23} & -\tfrac{1}{2}\lambda^2-3B_{33}+B_{23} \\ \dfrac{1}{a_1^2} & \dfrac{1}{a_2^2} & \dfrac{1}{a_3^2} \end{bmatrix} = 0\,. \tag{103}$$

A somewhat simpler form of equation (103) is

$$\begin{bmatrix} \tfrac{1}{2}\lambda^2+2B_{12}-\Omega Q_2+3B_{11}-B_{13} & B_{12}-B_{23}-3\Omega Q_1 & 3(B_{12}+B_{11}-B_{33})-B_{23}-\Omega(Q_2+3Q_1) \\ B_{12}-B_{13}+3\Omega Q_2 & \tfrac{1}{2}\lambda^2+2B_{12}+\Omega Q_1+3B_{22}-B_{23} & 3(B_{12}+B_{22}-B_{33})-B_{13}+\Omega(Q_1+3Q_2) \\ \dfrac{1}{a_1^2} & \dfrac{1}{a_2^2} & \dfrac{1}{a_1^2}+\dfrac{1}{a_2^2}+\dfrac{1}{a_3^2} \end{bmatrix} = 0\,. \tag{104}$$

Returning to equations (95) and (96) and the possibility of a non-trivial root $\lambda = 0$, we observe that we do indeed have such a root belonging to a proper solution associated with the possibility

$$V_{11} = V_{22} = V_{33} = 0 \quad \text{and} \quad V_{12} \neq 0\,. \tag{105}$$

In other words, *all Riemann ellipsoids allow a non-trivial neutral mode*. The Jacobian and the Dedekind ellipsoids share this property, as indeed, they must as special cases.

Turning next to equations (94) governing the odd modes of oscillation, we first rewrite them in the forms

$$(\lambda^2 + 2B_{13} - \Omega^2)V_{1;3} + (2B_{13} - \Omega^2 + Q_1Q_2 - 2\Omega Q_2)V_{3;1} - 2\lambda\Omega V_{2;3} = 0\,, \tag{106}$$

$$(\lambda^2 + 2B_{23} - \Omega^2)V_{2;3} + (2B_{23} - \Omega^2 + Q_1Q_2 + 2\Omega Q_1)V_{3;2} + 2\lambda\Omega V_{1;3} = 0\,, \tag{107}$$

$$(\lambda^2 + 2B_{13} + Q_1Q_2)V_{3;1} + 2B_{13}V_{1;3} - 2\lambda Q_1 V_{3;2} = 0\,, \tag{108}$$

$$(\lambda^2 + 2B_{23} + Q_1Q_2)V_{3;2} + 2B_{23}V_{2;3} - 2\lambda Q_2 V_{3;1} = 0\,. \tag{109}$$

On the other hand, according to equations (25) and (26)

$$2B_{13} - (\Omega^2 - Q_1Q_2 + 2\Omega Q_2) = 2\frac{a_3^2}{a_1^2}B_{13} \tag{110}$$

and

$$2B_{23} - (\Omega^2 - Q_1Q_2 - 2\Omega Q_1) = 2\frac{a_3^2}{a_2^2}B_{23}\,. \tag{111}$$

With this simplification of the coefficients of $V_{3;1}$ and $V_{3;2}$ in equation (106) and (107), equations (106)–(109) lead to the following characteristic equation:

$$\begin{bmatrix} \lambda^2 + 2B_{13} - \Omega^2 & 2a_3^2B_{13}/a_1^2 & -2\lambda\Omega & 0 \\ 2B_{13} & \lambda^2 + 2B_{13} + Q_1Q_2 & 0 & -2\lambda Q_1 \\ 2\lambda\Omega & 0 & \lambda^2 + 2B_{23} - \Omega^2 & 2a_3^2B_{23}/a_2^2 \\ 0 & -2\lambda Q_2 & 2B_{23} & \lambda^2 + 2B_{23} + Q_1Q_2 \end{bmatrix} = 0\,. \tag{112}$$

By a judicious manipulation of this determinant, it can be shown that equation (112) allows the roots

$$\lambda^2 = -\Omega^2 \quad \text{and} \quad \lambda^2 = Q_1Q_2\,; \tag{113}$$

and on factoring out $(\lambda^2 + \Omega^2)(\lambda^2 - Q_1Q_2)$, we find that the characteristic equation reduces to

$$\lambda^4 + (4B_{13} + 4B_{23} + 2B_{12})\lambda^2 + (4B_{13} - \Omega Q_1)(4B_{23} + \Omega Q_2) = 0\,. \tag{114}$$

The most striking feature of the characteristic equations (104) and (114) is that, apart from constants that depend only on the semi-axes of the ellipsoid, the equations involve Ω, Q_1, and Q_2 only in the combinations ΩQ_1 and ΩQ_2; and these combinations, as we have noted (see p. 895), have the same values for a configuration and its adjoint. Hence, the roots of equations (104) and (114) are unchanged as we pass from a configuration to its adjoint; and moreover, the roots $-\Omega^2$ and Q_1Q_2, which the characteristic equation (112) allows, are simply interchanged in the process. Thus, *all the characteristic frequencies of oscillation of a Riemann ellipsoid, belonging to the second harmonics, are the same for the two states of internal motions that can prevail in accordance with Dedekind's theorem.* This theorem generalizes the result established in Paper I for the special case of the Jacobi and the Dedekind ellipsoids.

An additional fact of some interest may be noted here. For an irrotational ellipsoid, it can be shown that one of the roots of equation (114) is again Q_1Q_2 so that in this case the four characteristic roots belonging to the odd modes of oscillation are

$$\lambda^2 = -\Omega^2, \qquad \lambda^2 = Q_1Q_2 \text{ (double root)}, \qquad \lambda^2 = -(4B_{13} + 4B_{23} + \Omega^2)\,. \tag{115}$$

In Table 10 we list the characteristic frequencies of oscillation, determined in accordance with equations (104) and (114), of the Riemann ellipsoids for which the equilibrium parameters have been determined; and in Figure 6 the variation of these frequencies along the sequences are illustrated for the two cases, $f = -2$ and $f = +2$.

TABLE 10*

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE SECOND HARMONICS

(σ^2 Is Listed in the Unit $\pi G\rho$)

ϕ	σ_1^2	σ_2^2	σ_3^2	σ_4^2
	$f=-2$			
0°	4 2667	1 0667	2 4000	0 2667
30°	4 2164	1 0667	2 4001	2617
45°	3 9852	1 0662	2 3998	2389
55°	3 5944	1 0614	2 3975	2003
57°	3 4830	1 0584	2 3962	1892
59°	3 3596	1 0537	2 3943	1769
60°	3 2936	1 0505	2 3931	1702
70°	2 5523	0 9227	2 3595	0891
71°	2 4852	0 8904	2 3523	0803
72°	2 4233	0 8524	2 3439	0715
75°	2 2761	0 7026	2 3107	.0467
78°	2 1880	0 5119	2 2627	0258
80°	2 1525	0 3787	2 2217	0152
85°	2 0810	0 1074	2 0928	0 0016

ϕ	σ_1^2	σ_2^2	σ_3^2	σ_4^2
	$f=-1$			
25°3753	3 6177	1 1959	2 3033	0 3879
60°	2 7215	1 1468	2 3050	2820
61°	2 6701	1 1350	2 3007	2776
62°	2 6186	1 1208	2 2959	2728
66°	2 4226	1 0347	2 2723	2508
67°	2 3785	1 0040	2 2653	2444
68°	2 3375	0 9692	2 2576	0 2377
	$f=-0\ 6$			
37°5628	2 8714	1 3419	2 2648	0 4383
52°	2 6173	1 3026	2 2486	0 4134

ϕ	σ_1^2	σ_2^2	σ_3^2	σ_4^2
	$f=+0\ 5$			
64°1095	1 5131	0 6341	1 6737	0 8020
69°	1 5416	6481	1 7127	7493
72°	1 6152	5896	1 7522	6795
73°	1 6459	5663	1 7711	6454
74°	1 6790	5374	1 7903	6081
75°	1 7132	0 5036	1 8097	0 5683
	$f=+2$			
72°3425	1 2612	0	1 0797	1 0797
75°	1 2550	0 0125	1 1776	0 9767
76°	1 2514	0256	1 2189	0 9319
77°	1 2504	0454	1 2658	0 8820
79°	1 2817	1090	1 3902	0 7590
80°	1 3373	1460	1 4736	0 6807
83°	1 6632	0 1532	1 7399	0 3990

e	σ_1^2	σ_2^2	σ_3^2	σ_4^2
	Sequence of the Prolate Spheroids			
0	4 2667	1 0667	2 4000	0 2667
0 20	4 1982	1 0800	2 3959	2737
30	4 1082	1 0969	2 3903	2825
40	3 9738	1 1211	2 3818	2947
50	3 7850	1 1527	2 3692	3101
55	3 6659	1 1710	2 3609	3189
60	3 5273	1 1906	2 3508	.3281
65	3 3666	1 2106	2 3383	3375
70	3 1810	1 2292	2 3229	3467
80	2 7295	1 2405	2 2785	3602
85	2 4757	1 1998	2 2455	3597
90	2 2437	1 0624	2 1998	3453
0 95	2 0939	0 7249	2 1321	0 2924

a_3/a_1	σ_1^2	σ_2^2	σ_3^2	σ_4^2
	Along the Self-adjoint Sequence $x=-1$			
0 80	3 9326	1 0644	2 4000	+0 2320
70	3 5844	1 0543	2 4000	+ 1917
.65	3 3870	1 0430	2 3995	+ 1664
60	3 1833	1 0250	2 3987	+ 1378
.50	2 7896	0 9555	2 3953	+ 0712
.40	2 4688	0 8178	2 3875	− 0059
.30	2 2671	0 6003	2 3719	− 0894
20	2 1782	0 3412	2 3395	− .1675
0 10	2 1531	0 1127	2 2679	−0 2061

a_2/a_1	σ_1^2	σ_2^2	σ_3^2	σ_4^2
	Along the Self-adjoint Sequence $x=+1$			
0 60	1 2449	0 0297	1 2361	0 9077
55	1 2411	0389	1 2609	8776
50	1 2376	0494	1 2874	8446
40	1 2359	0728	1 3471	7678
30	1 2546	0932	1 4195	6716
20	1 3274	0950	1 5140	5457
0 10	1 5206	0 0575	1 6556	0 3671

* The roots σ_1^2 and σ_2^2 belong to the even modes (eq. [104]) and the roots σ_3^2 and σ_4^2 belong to the odd modes (eq [114]).

TABLE 10—*Continued*

a_3/a_1	a_2/a_1	σ_1^2	σ_2^2	σ_3^2	σ_4^2	a_2/a_1	a_3/a_1	σ_1^2	σ_2^2	σ_3^2	σ_4^2
	Ellipsoids Included between the Sequences $f=-2$ and $x=-1$						Ellipsoids Included between the Sequences $f=+2$ and $x=+1$				
0 4	0 308	2 4364	0 8510	2 3514	+0 0609	0 3 .	0 158	1 3102	0 1230	1 4638	0 6580
3	231	2 2518	.6225	2 3242	− 0093	2	125	1 4547	.1314	1 5973	5112
2	167	2 1547	3603	2 2292	− 0047	2	118	1 3610	1044	1 5359	5372
0 2	0 154	2 1746	0 3491	2 2976	−0 1038	0 1 .	0 077	1 6958	0 0746	1 7642	0 2990

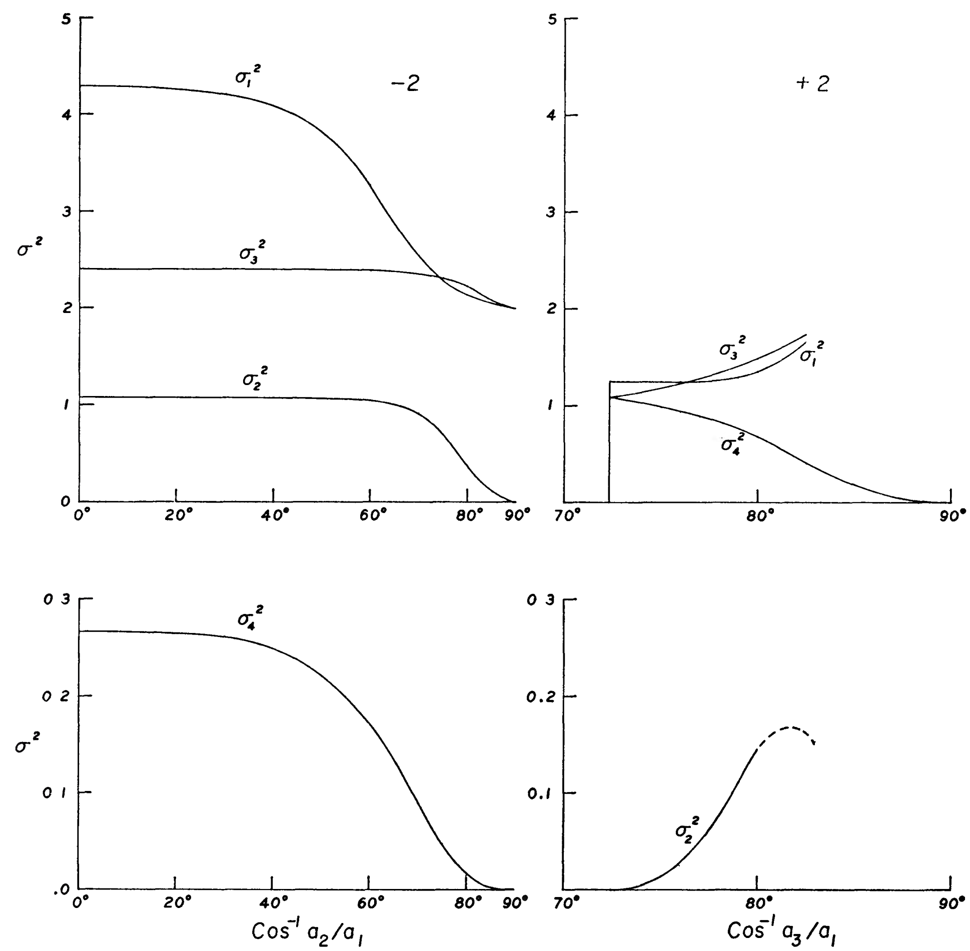

FIG. 6 —The squares of the characteristic frequencies belonging to the second harmonics for the Riemann sequences belonging to $f = -2$ (the curves on the left) and for $f = +2$ (the curves on the right). The roots denoted by σ_1^2 and σ_2^2 belong to the even modes (eq. [104]) and the roots denoted by σ_3^2 and σ_4^2 belong to the odd modes (eq. [114]).

It emerges from the foregoing calculations that all Riemann ellipsoids with $f \geq -2$ are stable with respect to oscillations belonging to the second harmonics. But instability by one of the odd modes sets in along the sequences for $f < -2$. Thus, along the self-adjoint sequence for $x = -1$, instability sets in where

$$a_2/a_1 = 0.29633\,, \qquad a_3/a_1 = 0.40733\,, \qquad f = f\dagger = -3.6710\,,$$

$$\Omega^2 = -Q_1Q_2 = 0.14878\,, \qquad Q_1 = Q_1\dagger = 1.3017\,, \qquad \text{and} \qquad Q_2 = Q_2\dagger = -0.1143\,. \tag{116}$$[6]

This is the point denoted by X_2 in Figure 2; and the locus of points in $(a_2/a_1, a_3/a_1)$-plane along which instability sets in by this odd mode of oscillation is the curve which joins X_2 to the origin.

VIII. THE ISOLATION OF THE NEUTRAL POINTS BELONGING TO THE THIRD HARMONICS ALONG THE RIEMANN SEQUENCES

We have seen in § VII that the Riemann ellipsoids for $f \geq -2$ are stable with respect to all modes of oscillation belonging to the second harmonics. We may therefore expect, in analogy with the Jacobian and the Dedekind sequences, that instability along these Riemann sequences will be manifested, first, by a mode of oscillation belonging to the third harmonics. In this section we shall show how the various integral properties provided by the third-order virial theorem enable us to determine the locus of points, in the domain of the Riemann ellipsoids, that separate the regions of stability from the regions of instability.

Now the third-order virial theorem gives

$$\frac{d}{dt}\int_V \rho u_i x_j x_k d\mathbf{x} = 2(\mathfrak{T}_{ij;k} + \mathfrak{T}_{ik;j}) + \Omega^2(I_{ijk} - \delta_{i3}I_{3jk}) + \mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j}$$
$$+ 2\Omega\epsilon_{il3}\int_V \rho u_l x_j x_k d\mathbf{x} + \Pi_k\delta_{ij} + \Pi_j\delta_{ik}\,, \tag{117}$$

where

$$\mathfrak{T}_{ij;k} = \tfrac{1}{2}\int_V \rho u_i u_j x_k d\mathbf{x}\,, \qquad \mathfrak{W}_{ij;k} = -\tfrac{1}{2}\int_V \rho \mathfrak{V}_{ij}\, x_k d\mathbf{x}\,,$$
$$I_{ijk} = \int_V \rho x_i x_j x_k d\mathbf{x}\,, \qquad \text{and} \qquad \Pi_j = \int_V p x_j d\mathbf{x}\,. \tag{117'}$$

Under conditions of equilibrium, equation (117) gives

$$2(\mathfrak{T}_{ij;k} + \mathfrak{T}_{ik;j}) + \Omega^2(I_{ijk} - \delta_{i3}I_{3jk}) + \mathfrak{W}_{ij;k} + \mathfrak{W}_{ik;j}$$
$$+ 2\Omega\epsilon_{il3}\int_V \rho u_l x_j x_k d\mathbf{x} = -\Pi_k\delta_{ij} - \Pi_j\delta_{ik}\,. \tag{118}$$

[6] It follows from eqs. (74) and (114) that instability along the sequence $x = -1$ sets in where

$$4B_{13} = \Omega Q_1 = a_1 B_{12}/a_2\,;$$

and, making use of eq. (25), which now takes the form

$$B_{13} = \frac{a_1(a_1 - a_2)}{a_1^2 - a_3^2} B_{12}\,,$$

we find that at the point where instability sets in,

$$a_1 - a_3 = 2a_2\,.$$

We observe that this relation is satisfied by the values quoted.

At a neutral point belonging to the third harmonics, the first variations of all the integral relations provided by equation (118) must vanish for a Lagrangian displacement which leads to a set of third-order virials

$$V_{i;jk} = \int_V \rho \xi_i x_j x_k d\boldsymbol{x} \tag{119}$$

that are not all zero.

The first variation of equation (118) gives

$$\begin{aligned} &2(\delta\mathfrak{T}_{ij;k} + \delta\mathfrak{T}_{ik;j}) + \Omega^2(V_{ijk} - \delta_{i3}V_{3jk}) + \delta\mathfrak{W}_{ij;k} + \delta\mathfrak{W}_{ik;j} \\ &+ 2\Omega\epsilon_{il3}\left(\int_V \rho\Delta u_l x_j x_k d\boldsymbol{x} + \int_V \rho u_l \xi_j x_k d\boldsymbol{x} + \int_V \rho u_l x_j \xi_k d\boldsymbol{x}\right) \\ &= -(\delta\Pi_k \delta_{ij} + \delta\Pi_j \delta_{ik}), \end{aligned} \tag{120}$$

where $\delta\mathfrak{T}_{ij;k}$, etc., are the first-order changes in the respective quantities induced by an appropriate Lagrangian displacement and V_{ijk} is the symmetrized virial

$$V_{ijk} = V_{i;jk} + V_{j;ki} + V_{k;ij} (= \delta I_{ijk}) . \tag{121}$$

Equation (120) must be supplemented by the three further conditions

$$\sum_{j=1}^{3} \frac{V_{ijj}}{a_j^2} = 0 \qquad (i = 1, 2, 3) \tag{122}$$

(summation only over the index indicated),

which express the solenoidal character of the displacement.

In an earlier paper (Chandrasekhar and Lebovitz 1963*a*, Table 1) it has been shown that first variations of $\delta\mathfrak{W}_{ij;k}$ can be expressed as linear combinations of the symmetrized virials V_{ijk}. And in Paper I it has been shown how, for a quasi-static deformation, $\delta\mathfrak{T}_{ij;k}$ can be expressed in terms of the virials $V_{i;jk}$ if the motion in the equilibrium configuration is linear in the coordinates and is of the form given by equation (76); thus (cf. Paper I, eq. [90])

$$2\delta\mathfrak{T}_{ij;k} = -Q_{jl}(Q_{ln}V_{i;kn} + Q_{kn}V_{i;ln}) - Q_{il}(Q_{ln}V_{j;kn} + Q_{kn}V_{j;ln}) + Q_{im}Q_{jn}V_{k;mn} . \tag{123}$$

This equation follows from the relation (Paper I, eq. [85])

$$\int_V \rho\Delta u_i x_l x_m d\boldsymbol{x} = -(Q_{ln}V_{i;mn} + Q_{mn}V_{i;ln}), \tag{124}$$

which obtains for any quasi-static deformation. By a further application of this last relation, the Coriolis term in equation (120) can also be expressed in terms of the virials; thus

$$2\Omega\epsilon_{il3}\delta\int_V \rho u_l x_j x_k d\boldsymbol{x} = 2\Omega\epsilon_{il3}[Q_{ln}(V_{j;kn} + V_{k;jn}) - Q_{jn}V_{l;kn} - Q_{kn}V_{l;jn}] . \tag{125}$$

Equations (123) and (125), together with the known expansions of the $\delta\mathfrak{W}_{ij;k}$'s in terms of the virials, enable us to express the right-hand side of equation (120) as linear combinations of the virials. For our present purposes of isolating the neutral point, it will suffice to consider the five equations which are odd in the index 1 and even in the indices 2 and 3; these equations are

$$4\delta\mathfrak{T}_{11;1} + 2\delta\mathfrak{W}_{11;1} + \Omega^2 V_{111} + 4\Omega(Q_2 V_{1;11} - Q_1 V_{2;12}) = -2\delta\Pi_1 , \tag{126}$$

$$2\delta\mathfrak{T}_{12;2} + 2\delta\mathfrak{T}_{22;1} + \delta\mathfrak{W}_{12;2} + \delta\mathfrak{W}_{22;1} + \Omega^2 V_{122} + 2\Omega(Q_2 V_{1;11} - Q_1 V_{2;12}) = -\delta\Pi_1 , \tag{127}$$

$$2\delta\mathfrak{T}_{13;3} + 2\delta\mathfrak{T}_{33;1} + \delta\mathfrak{W}_{13;3} + \delta\mathfrak{W}_{33;1} = -\delta\Pi_1 , \tag{128}$$

$$4\delta\mathfrak{T}_{12;2} + 2\delta\mathfrak{W}_{12;2} + \Omega^2 V_{122} = 0 , \tag{129}$$

$$4\delta\mathfrak{T}_{13;3} + 2\delta\mathfrak{W}_{13;3} + \Omega^2 V_{133} + 4\Omega Q_2 V_{3;31} = 0 . \tag{130}$$

Eliminating $\delta\Pi_1$ appropriately from these equations, we obtain, in addition to equations (129) and (130) the pair of equations

$$2\delta R_{122} + \delta S_{122} + \Omega^2(V_{111} - 3V_{122}) = 0 \tag{131}$$

and

$$2\delta R_{133} + \delta S_{133} + \Omega^2(V_{111} - V_{133}) + 4\Omega Q_2 V_{1;11} - 4\Omega Q_1 V_{2;12} - 4\Omega Q_2 V_{3;13} = 0 , \tag{132}$$

where

$$\delta S_{ijj} = -4\delta\mathfrak{W}_{ij;j} - 2\delta\mathfrak{W}_{jj;i} + 2\delta\mathfrak{W}_{ii;i}$$

and

$$\delta R_{ijj} = -4\delta\mathfrak{T}_{ij;j} - 2\delta\mathfrak{T}_{jj;i} + 2\delta\mathfrak{T}_{ii;i} \tag{133}$$

(no summation over repeated indices).

And we may note here for reference the following explicit expressions for the particular quantities which occur in equations (129)–(132) (cf. Paper I, eqs. [95]–[98])

$$-4\delta\mathfrak{T}_{12;2} = 2Q_1Q_2(V_{2;21} + V_{1;22}) + 2Q_2^2 V_{1;11} , \tag{134}$$

$$-4\delta\mathfrak{T}_{13;3} = 2Q_1Q_2V_{3;31} , \tag{135}$$

$$+2\delta R_{122} = (2Q_2^2 - 4Q_1Q_2)V_{1;11} + (4Q_1Q_2 - 2Q_1^2)V_{1;22} + 12Q_1Q_2V_{2;12} , \tag{136}$$

$$+2\delta R_{133} = -4Q_1Q_2V_{1;11} - 2Q_1^2V_{1;22} + 4Q_1Q_2V_{3;31} . \tag{137}$$

Equations (129)–(132), together with the solenoidal condition,

$$\frac{V_{111}}{a_1^2} + \frac{V_{122}}{a_2^2} + \frac{V_{133}}{a_3^2} = 0 , \tag{138}$$

provide five homogeneous equations for the five virials $V_{1;11}$, $V_{1;22}$, $V_{1;33}$, $V_{2;12}$, and $V_{3;13}$. The condition for the occurrence of a neutral point is that the determinant of the system vanishes. The neutral points along the Jacobian and the Dedekind sequences have already been determined by making use of this condition (Chandrasekhar 1962, 1963, and Paper I). The neutral points along the other Riemann sequences were similarly determined; and by considering the adjoint configurations along the same sequences the places where the neutral points occur for these configurations were also determined. In the same way, the two neutral points along the sequence of the prolate spheroids, as well as the two self-adjoint sequences, were determined. The results of all these calculations are summarized in Table 11. And in Figure 2 the loci separating the regions of stability from the regions of instability, in the domain of these Riemann ellipsoids, are drawn.

IX. CONCLUDING REMARKS

In some respects, the most important result which has emerged from the present study is the fact that from every point of the stable part of the Maclaurin sequence two Riemann sequences branch off; and, further, that the branching of these sequences is, in no essential way, different from the branching of the Jacobian sequence. Indeed, from the present vantage point, the branching of the Jacobian sequence from the Maclaurin sequence does not appear as a unique or an isolated phenomenon: it appears as the manifestation of a cause which is operative at *every point* of the Maclaurin sequence. The restrictiveness of the common view which ascribes a special significance to the branching of the Jacobian sequence is nowhere more apparent than in its disregard for the simultaneous branching of the Dedekind sequence from the same "point of bifurcation." In view of the contrariness of these remarks, it may be useful to formulate the present interpretation in a manner in which its physical basis becomes transparent.

Consider a Maclaurin spheroid rotating uniformly with an angular velocity Ω_{Mc} in an

TABLE 11

THE NEUTRAL POINTS ALONG THE RIEMANN SEQUENCES BELONGING TO THE THIRD HARMONICS

	Neutral Points along the Riemann Sequences for					Along the Sequence of the Prolate Spheroids	
	$f=-2$	$f=-1$	$f=0$*	$f=0\ 5$	$f=2$	$f=-0\ 61563$	$f=-10\ 65076$
a_2/a_1	+0 32098	+0 40152	0 43216	+0 42253	+0 33197	+0 41815	+0 46386
a_3/a_1	+ 41397	+ 44657	34503	+ 28004	+ .17378	+ 41815	+0 46386
Ω^2	+ .22981	+ 30315	0 28400	+ 23315	+ 11556	+ 31111	+0 02021
$-Q_1Q_2$	+ 07784	+ 03624	0	+ 00749	+ 04133	+ 01494	+0 33406
Q_1	+ 86922	+ 47415	0	− 20486	− 61240	+ 29228	+1 24601
Q_2	−0 08955	−0 07644	0	+0 03657	+0 06749	−0 05110	−0 26810

* The entries in this column refer to the Jacobian sequence.

	Neutral Points for the Adjoint Configurations along the Riemann Sequences for				
	$f=-2$	$f=-1$	$f=0$*	$f=0\ 5$	$f=2$
$f^\dagger$	−3 02418	−6 53927	$\pm\infty$	+15 02537	+2 84745
a_2/a_1	+0 51407	+0 48185	+0 44133	+ 0 43336	+0 54226
a_3/a_1	+0 64729	+0 53188	+0 35041	+ 0 28412	+0 22038
$(\Omega^\dagger)^2$	+0 17645	+0 05169	0	+ 0 00785	+0 08075
$-Q_1^\dagger Q_2^\dagger$	+0 26681	+0 33800	+0 28782	+ 0 23599	+0 11496
$Q_1^\dagger$	+1 00481	+1 20657	−1 21560	− 1 12099	−0 62527
$Q_2^\dagger$	−0 26553	−0 28014	+0 23677	+ 0 21052	+0 18386

* The entries in this column refer to the Dedekind sequence.

	Neutral Points along the Self-adjoint Sequences for	
	$x=-1$	$x=+1$
$f=f^\dagger$	−2 7279	2 1067
a_2/a_1	0 43640	0 7224
a_3/a_1	0 57370	0 25606
$\Omega^2=(\Omega^\dagger)^2=-Q_1Q_2$	0 20230	0 10615
$Q_1=Q_1^\dagger$	1 03067	−0 45101
$Q_2=Q_2^\dagger$	−0 19628	0 23536

inertial frame. When viewed from a frame of reference rotating with this same angular velocity Ω_{Mc}, the spheroid will appear as in hydrostatic equilibrium with no internal motions. However, when viewed from a frame of reference rotating with an angular velocity Ω different from Ω_{Mc}, the spheroid will appear as having internal motions with the vorticity

$$\zeta_\Omega = 2(\Omega_{Mc} - \Omega) . \tag{139}$$

Since the transverse sections of the spheroid are circular, the motion associated with ζ_Ω is purely rotational and is exactly the difference between Ω and Ω_{Mc}; the components of the motion, in the chosen frame, are, in fact, (cf. eq. [9])

$$u_1 = -(\Omega_{Mc} - \Omega)x_2 = Q_1 x_2 \text{ (say)}$$

and

$$u_2 = +(\Omega_{Mc} - \Omega)x_1 = Q_2 x_1 \text{ (say)} . \tag{140}$$

We now ask the question whether the Maclaurin spheroid, so described, can be deformed quasi-statistically into a triaxial ellipsoid without in any way affecting its equilibrium *as viewed* from the frame of reference rotating with the angular velocity Ω. We shall now show that such a deformation can be accomplished only if the angular velocity Ω of the frame is chosen properly.

Now an infinitesimal displacement $\boldsymbol{\xi}$ that will deform the spheroid into a triaxial ellipsoid without affecting its angular momentum is given by

$$\xi_1 = \alpha x_2, \qquad \xi_2 = \beta x_1, \qquad \text{and} \qquad \xi_3 = 0 , \tag{141}$$

where α and β are two infinitesimal constants. From the equations derived in § VI, the condition that the displacement (141) will have the properties requisite for a neutral mode of oscillation can be written down. But it should be noted, first, that, in deriving the linearized virial equations (89)–(94), no restrictive assumptions regarding the underlying equilibrium were made; the only assumption that was made was that the object, prior to its perturbation and in the chosen frame of reference, was in a steady state with internal motions compatible with the general form (9) and (79).

For the deformation represented by equation (141), the only non-vanishing virials are $V_{1;2}$ and $V_{2;1}$; therefore,

$$V_{i;j} \neq 0 \textit{ only for } i = 1, j = 2 \textit{ and } i = 2, j = 1 . \tag{142}$$

Now setting $\lambda = 0$ (as required for a neutral mode) in equations (89)–(94), we observe that these equations (and eq. [85] expressing the solenoidal character of $\boldsymbol{\xi}$) are consistent with the requirements expressed in (142) only if

$$\Omega^2 - Q_1 Q_2 = 2B_{11} . \tag{143}$$

The condition in this particular form follows from equations (92) and (93); the remaining equations are satisfied identically.

With the present definitions of Q_1 and Q_2 (see eqs. [140]), the condition (143) gives

$$\Omega^2 + (\Omega_{Mc} - \Omega)^2 = 2B_{11} . \tag{144}$$

But this last equation is the same as equation (52) derived in § IV for determining the particular Riemann sequences which branch off from a given point of the Maclaurin sequence.

The foregoing arguments, when carried out in the inertial frame ($\Omega = 0$) or in the frame in which the Maclaurin spheroid appears as in hydrostatic equilibrium ($\Omega = \Omega_{Mc}$), lead to the same condition

$$\Omega_{Mc}{}^2 = 2B_{11} ; \tag{145}$$

and this result is in agreement with the known fact that the Jacobian and the Dedekind sequences bifurcate from the Maclaurin sequence at the same point. The non-uniqueness of this phenomenon from the general point of view leading to equation (144) is apparent.

Finally, it should be stated that the investigation of the Riemann ellipsoids, in the case where the directions of $\mathbf{\Omega}$ and $\boldsymbol{\zeta}_\Omega$ do not coincide, leads to a point of view which is even more general than the one described in this section.[7]

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[7] The arguments of this section can be stated more generally. Consider the oscillations of a Maclaurin spheroid in a frame of reference rotating with an angular velocity Ω different from Ω_{Mc} It can be shown that the frequencies of oscillation of the toroidal (σ_e) and the transverse sheer (σ_o) modes are then given by

$$\sigma_e = 2\Omega - \Omega_{\mathrm{Mc}} \pm \sqrt{(4B_{11} - \Omega_{\mathrm{Mc}}^2)} \tag{i}$$

and

$$2\sigma_o = 2\Omega - \Omega_{\mathrm{Mc}} \pm \sqrt{(16B_{13} + \Omega_{\mathrm{Mc}}^2)} \,. \tag{ii}$$

Accordingly, the even mode can be neutralized by choosing Ω so that

$$2\Omega - \Omega_{\mathrm{Mc}} = \sqrt{(4B_{11} - \Omega_{\mathrm{Mc}}^2)} \,; \tag{iii}$$

and similarly the odd mode can be neutralized by choosing Ω so that

$$2\Omega - \Omega_{\mathrm{Mc}} = \sqrt{(16B_{13} + \Omega_{\mathrm{Mc}}^2)} \,. \tag{iv}$$

It can be readily verified that condition (iii) is the same as that given by eq. (144) and leads to the two manners of bifurcation described in the text. The choice provided by condition (iv) leads to two further manners of bifurcation corresponding to the existence of another class of Riemann ellipsoids in which the directions of $\mathbf{\Omega}$ and $\boldsymbol{\zeta}_\Omega$ do not coincide. We shall return to these matters in greater detail in a further paper on the Riemann ellipsoids; but it is important to notice meantime that *every point of the Maclaurin sequence is a point of bifurcation in four different ways.*