A STATISTICAL MODEL OF THE FORMATION OF STARS AND INTERSTELLAR CLOUDS

GEORGE B. FIELD AND WILLIAM C. SASLAW
Princeton University Observatory

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ABSTRACT

Oort's (1954) model for the formation and destruction of interstellar clouds is formulated mathematically with simplifying assumptions. An equation for the time-dependent cloud-mass spectrum is obtained and solved. It is found that the rate of star formation is proportional to the square of the gas density, in agreement with Schmidt's (1959) empirical model, and that the calculated proportionality constant agrees approximately with the observed value. The form of the cloud-mass spectrum, approximately a $m^{-3/2}$ law, also agrees roughly with observation.

I. INTRODUCTION

A comprehensive theory of interstellar gas dynamics is lacking, owing to the difficulty of dealing with the non-linear phenomena known to occur in interstellar space. The roles played by shock waves, radiative processes, gravitation, angular momentum, and magnetic fields have been studied in idealized models (Spitzer 1965), but so far it has not been possible to incorporate all such phenomena into a single theory.

It is clear that an over-all theory of the interstellar medium must be statistical in nature. An early attempt at a statistical description of interstellar gas dynamics is that by Oort (1954). He took as the basic element of his scheme the interstellar "cloud"—a region of high density that supposedly moves freely through an intercloud medium of low density and high temperature. Such clouds frequently appear in descriptive models of interstellar gas, as they are consistent with a variety of observations of interstellar neutral hydrogen, calcium ions, and dust particles (Spitzer 1965). Oort assigns to them a dynamical significance. First, each cloud is supposed to be in approximate hydrostatic equilibrium, its internal pressure balancing external pressure plus self-gravitation. In this way, a cloud maintains its identity for long periods. Second, the clouds undergo collisions with each other that tend to change their mass, momentum, and energy.

A cloud starts its life cycle near an expanding H II region as a small region of high density compressed by surrounding hot gas. It is so small that self-gravitation is negligible. It is accelerated outward by the expanding H II region, and undergoes inelastic collisions with other clouds in the vicinity, engendering new clouds of larger mass through coalescence. Through repeated coalescence processes, the mass of the cloud increases to the point that self-gravitation becomes significant and ultimately dominant. At this point the cloud becomes gravitationally unstable, undergoes rapid collapse, and becomes subject to gravitational fragmentation. The final result is a group of young stars, the brightest of which ionize gas remaining in the cloud to form H II regions and thereby regenerate small clouds as before. These small clouds are then available for a new cycle of activity.

Oort's scheme is consistent with these observations: (*a*) the masses of associations and clusters significantly exceed those of typical interstellar clouds, but are significant fractions of the calculated critical mass for gravitational instability; (*b*) small clouds near associations are often found to be moving away from the hot stars of the association with their accompanying H II regions; and (*c*) the kinetic energy of interstellar clouds is very considerable in spite of the inevitable dissipation occurring during cloud collisions, indicating that some acceleration process must be at work.

Note that the life cycle of a cloud in Oort's model constitutes a feedback loop in that the hot stars produced at one point in the cycle are responsible for regenerating new clouds needed to start the next cycle. Suppose for a moment that this feedback did not exist—that the young stars did not react back on the gas in any way. Then there would be no source of small clouds, and the masses of all clouds would increase exponentially with time through coalescence processes, with a time constant about equal to the cloud collision time, 6×10^6 years (Spitzer 1965). Most clouds would approach gravitational instability in about 5 collision times, since then the typical mass would be about 100 times that of an average cloud, a value about equal to that of the observed unstable clouds. Unless some other effect intervened, the gas would turn to stars in about 3×10^7 years, rather than in 8×10^9 years, the observed time scale of star formation according to Schmidt (1959).

It appears that the Oort scheme without feedback would not agree with observation. It is true that we have not included in the above discussion the time which a massive unstable cloud takes to form stars after it has begun contracting. The purely gravitational theory of this process, called fragmentation (Hunter 1962, 1964), shows that while the cloud as a whole is freely falling to high density on a time scale of $(3\pi/32\, G\rho)^{1/2} \simeq 2 \times 10^7$ years, density fluctuations within the cloud can become unstable and separate out into protostars. This theory suggests that the whole process would be over in about 2×10^7 years, only $\frac{2}{3}$ of the time calculated above for massive clouds to form in the first place; this certainly would not extend the time scale of star formation significantly.

It has been pointed out by many authors that centrifugal and magnetic forces ignored in the analysis of pure gravitational instability will play an important role in the fragmentation process, as such forces become relatively more important as the density rises. Much current research is aimed at discovering whether these forces really do inhibit star formation significantly or not (see Spitzer 1965 for a review). If we agree with Oort that star formation does begin primarily through the instability of large clouds, there is observational indication that such forces cannot inhibit star formation in the required degree without additional help from another mechanism such as the feedback in Oort's scheme. For if they did, star formation in a given massive cloud would have to sputter along for some 8×10^9 years. This would result in an enormous range of ages for the stars which are formed in a single cloud. Instead, one characteristically finds a cluster or association all of whose members are of the same age within narrow limits. Indeed, the spread of ages may approach the absolute minimum, some fraction of the free-fall time.

There seems to be no escape from the conclusion that the Oort scheme is a workable hypothesis only if the inhibition of star formation by the dispersal of a considerable fraction of an unstable cloud into small stable clouds is effective enough to explain the present low rate of star formation in the solar neighborhood. This paper is aimed at calculating the rate of star formation from Oort's model. We shall find that the rate calculated using reasonable estimates for the necessary parameters is low in agreement with observation. In the process we shall derive an expression for the mass spectrum of interstellar clouds, and an expression for the dependence of the rate of star formation on the density. The latter is similar to one put forward on empirical grounds by Schmidt.

II. MODEL OF INTERSTELLAR CLOUDS

A statistical-mechanical description of Oort's scheme would be based on $N(M, \boldsymbol{v}, t)$, the number of density of clouds of mass between M and $M + dM$ and in velocity range $\boldsymbol{v}$ to $\boldsymbol{v} + d\boldsymbol{v}$ at time t. In this preliminary study we shall assume that all clouds have the same speed and are isotropically distributed in velocity, thereby suppressing the velocity variable and making $N(M, t)$ the quantity of interest. We shall further assume that the process of cloud creation described above always yields small clouds of the same mass, and that cloud collisions are completely inelastic, so that collisions invariably lead to the coalescence of the collision partners. While these assumptions are not very accurate (see

§ VII), they lead to helpful simplifications. In particular, it follows that all clouds must have masses which are some integral multiple of the mass of the small newly created clouds. We denote this integral multiple by m.

Three processes contribute to $\partial N(m, t)/\partial t$. First we have a positive contribution from coalescence of smaller clouds of masses m' and $m - m'$ to form a cloud of mass m, for which the expression is

$$\frac{1}{2}\sum_{m'=1}^{m-1} a(m', m-m')N(m')N(m-m'), \tag{1}$$

where

$$a(m, m') = \sigma(m, m')\,\langle|v(m) - v(m')|\rangle \tag{2}$$

is the rate coefficient in terms of the effective collision cross-section σ. Equation (1) is correct only for binary collisions, the dominant process when the fraction of volume filled by clouds is $\ll 1$. The actual fraction is estimated to be about 7 per cent (Spitzer 1965) so that this assumption is fairly good.

Similarly, there is a negative contribution from collisions of clouds of mass m with all others, for which we have the expression,

$$-N(m)\sum_{m'=1}^{m_1} a(m, m')N(m'), \tag{3}$$

where m_1 refers to the largest stable cloud present. Rapid collapse of clouds of mass $\geq m_1 + 1$ is taken care of by simply specifying that $N(m) = 0$ for such masses. It is useful to have an expression for the amount of mass per unit volume per unit time going into such unstable clouds:

$$U = \frac{1}{2}\sum_{m'=1}^{m_1}\ \sum_{m''=m_1-m'+1}^{m_1} a(m', m'')N(m')N(m'')(m'+m''). \tag{4}$$

Finally there is an input of small clouds of unit mass which accompanies star formation in massive clouds. Evidently this input is

$$\eta U\delta(m, 1), \tag{5}$$

where δ is the Kronecker delta and η is the mass fraction of a massive cloud that is sent back into the medium in the form of unit clouds ($\eta \leq 1$).

The fundamental equation is therefore

$$\begin{aligned}\frac{\partial N(m, t)}{\partial t} = {} & \frac{1}{2}\sum_{m'=1}^{m-1} a(m', m-m')N(m')N(m-m') \\ & - N(m)\sum_{m'=1}^{m_1} a(m, m')N(m') + \eta U\delta(m, 1).\end{aligned} \tag{6}$$

In principle, equation (6) could be solved numerically for any prescribed initial values, $N(m, t = 0)$. It is possible, however, to find solutions which are separable in m and t; these are discussed in the following section. It is shown in Appendix II that a particular solution of this type (for $m_1 \to \infty$) must be attained asymptotically as $t \to \infty$, no matter what the initial conditions are.

As a practical matter we have considered only the simple case $a(m, m') =$ const. Such an assumption implies that all clouds have the same cross-section and speed, irrespective of their mass. This assumption can undoubtedly be improved. The case in which all clouds have the same density and are in equipartition is formulated in Appendix III. The equation anologous to equation (6) is derived but not solved.

III. SEPARABLE SOLUTIONS

A separable solution can be written

$$N(m, t) = A(t)p(m) , \tag{7}$$

where $A(t)$ is the total number density of clouds, so that

$$\sum_{m=1}^{m_1} p(m) = 1 . \tag{8}$$

Equation (6) can then be written

$$\begin{aligned} \frac{1}{\alpha A^2}\frac{\partial A}{\partial t} = \frac{1}{p(m)}\Bigg[& \tfrac{1}{2}\sum_{m'=1}^{m-1} p(m')p(m-m') - p(m) \\ & + \tfrac{1}{2}\eta\delta(m, 1)\sum_{m'=1}^{m_1}\sum_{m''=m_1-m'+1}^{m_1} p(m')p(m'')(m'+m'')\Bigg], \end{aligned} \tag{9}$$

using equation (4). (Note that α = const.) The separation of equation (9) with the (dimensionless) separation constant $-a$ leads to

$$\frac{dA}{dt} = -a\alpha A^2 , \tag{10}$$

to which the solution is

$$A(t) = \frac{A(0)}{1 + \alpha a A(0)t} . \tag{11}$$

Since αA is the cloud collision frequency, a can be interpreted as the fractional loss of clouds (and therefore mass) in one collision time. The problem formulated at the end of § I can therefore be stated: Does Oort's model predict that a is about 10^{-3}?

The term a is determined as an eigenvalue of the equation for $p(m)$,

$$\begin{aligned} -ap(m) = & \tfrac{1}{2}\sum_{m'=1}^{m-1} p(m')p(m-m') - p(m) \\ & + \tfrac{1}{2}\eta\delta(m, 1)\sum_{m'=1}^{m_1}\sum_{m''=m_1-m'+1}^{m_1} p(m')p(m'')(m'+m'') . \end{aligned} \tag{12}$$

Evidently a is a function of η and m_1, which are natural physical parameters of the theory. Equation (12) is simplified by the introduction of

$$p_*(m) = \frac{p(m)}{1-a} \tag{13}$$

and

$$\beta = \eta\sum_{m'=1}^{m_1}\sum_{m''=m_1-m'+1}^{m_1} p_*(m')p_*(m'')(m'+m'') , \tag{14}$$

to

$$\tfrac{1}{2}\sum_{m'=1}^{m-1} p_*(m')\,p_*(m-m') - p_*(m) + \tfrac{1}{2}\beta\,\delta(m,1) = 0\,. \tag{15}$$

In this form, the equation can be solved recursively by numerical methods, since $p_*(m)$ can be calculated from its values for $m' \le m$. In fact we employed this numerical approach first, and the analytic solutions developed below were found only after computer solutions were available.

Before proceeding to the solution, we sketch the method by which the eigenvalue $a\,(m_1, \eta)$ is determined. Given a value of β we solve equation (15) for $p_*(m;\beta)$. Then for a given value of m_1, $a(m_1, \beta)$ is determined from equation (13):

$$a(m_1, \beta) = 1 - \frac{1}{\sum_{m=1}^{m_1} p_*(m;\beta)}\,. \tag{16}$$

The term $\eta\,(m_1, \beta)$ is determined from

$$\eta(m_1, \beta) = \left\{1 + \frac{2a(m_1,\beta)}{[1-a(m_1,\beta)]\beta}\sum_{m=1}^{m_1} m\,p_*(m;\beta)\right\}^{-1}, \tag{17}$$

which is shown in Appendix I to be equivalent to equation (14). The elimination of β between equations (16) and (17) yields $a(m_1, \eta)$.

We now proceed to the analytic solution of equation (15) for $p_*(m)$, given values of m_1 and β. This is accomplished through Laplace transformation of functions of m. Define

$$\overline{p_*(z)} = \sum_{m=1}^{\infty} z^m p_*(m)\,. \tag{18}$$

For those values of z for which the above series converges, we have the transformed version of equation (15):

$$\overline{p_*{}^2(z)^2} - 2\langle p_*(z)\rangle + \beta z = 0\,. \tag{19}$$

Equation (19), of course, has two roots, one of which is irrelevant because it does not approach zero as z approaches zero, as required by equation (18). The other root is

$$\overline{p_*(z)} = 1 - (1 - \beta z)^{1/2}\,, \tag{20}$$

which may be expanded in a power series in z and compared with equation (18). The comparison yields the solution for $p_*(m)$:

$$p_*(m) = \frac{1}{2\sqrt{\pi}}\,\frac{(m-\frac{3}{2})!}{m!}\,\beta^m\,. \tag{21}$$

We note that $p_*(m;\beta)$, the solution for arbitrary β, is related to that for $\beta = 1$, $p_*\,(m;1)$, by

$$p_*(m;\beta) = p_*(m;1)\beta^m\,. \tag{22}$$

We therefore consider the case $\beta = 1$ in more detail. Since

$$\sum_{m=1}^{m_1} p_*(m;1) \le \sum_{m=1}^{\infty} p_*(m;1)\,, \tag{23}$$

and the right-hand side is by equation (18) equal to $\overline{p_*(z)}$ for $z=1$ and $\beta=1$, which in turn equals unity by equation (20), we conclude that $a\,(m_1$ finite; $\beta = 1)$ is negative and that $a(m_1 = \infty; \beta = 1) = 0$, from equation (16). From physical reasoning we know that a is not negative since there is a net loss of mass to stars. Hence the only solution with $\beta = 1$ corresponds to $m_1 = \infty$ and $a = 0$, which represents a steady state attained with the unstable cloud having infinite mass. This steady state is best understood as the limit of a system with m_1 large but finite, and $a \neq 0$. Evidently in this system the number of collisions required to increase the cloud mass to the unstable value is enormous so the mass drain from the system per unit time becomes very small.

From Stirling's formula

$$p_*(m; 1) = \frac{1}{2\sqrt{\pi}} \frac{(m-\frac{3}{2})!}{m!} \rightarrow \frac{1}{2\sqrt{\pi}}\left(1+\frac{3}{2m}+\ldots\right) m^{-3/2}, \quad m \rightarrow \infty . \tag{24}$$

The accuracy of the asymptotic formula can be judged from the exact result in Figure 2 (see below).

If m_1 is finite but large (the case of interest), we expect β to be only slightly different from 1. $\beta < 1$ is of no interest, because $p_*(m; \beta)$ is then $< p_*(m; 1)$ from equation (22),

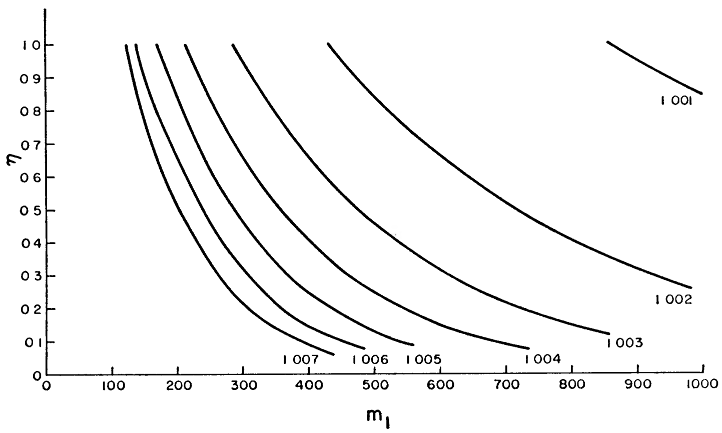

FIG 1.—The parameter β, which determines the shape of the mass spectrum, as a function of the mass of an unstable cloud, m_1, and the efficiency of ejection, η.

and the sum even up to $m_1 = \infty$ is perforce < 1, contradicting the requirement that $a \geq 0$. If $\beta > 1$, on the other hand, $p_*(m; \beta)$ exceeds $p_*(m; 1)$ to an increasing degree as m increases, permitting the sum of all terms up to a finite value of m to exceed unity as required with $a > 0$. Evidently the mass spectrum will show an upturn for sufficiently large values of m. Using equations (22) and (24) we conclude that the upturn occurs for

$$m = \frac{3}{2 \ln \beta} \simeq \frac{3}{2(\beta - 1)}; \tag{25}$$

m_1 is of this order also.

IV. NUMERICAL RESULTS

The term $p_*(m; \beta)$ was evaluated numerically from equation (21) for $1.000 \leq \beta \leq 1.007$ and $1 \leq m \leq 3000$; $a(m_1, \beta)$ was computed from equation (16), $\eta(m_1, \beta)$ from equation (17), and $a(m_1, \eta)$ from the combination of the two.

Curves of constant β in the m_1, η-plane are shown in Figure 1. A given choice of m_1 and η determines β, and this can be used to choose one of the mass spectra $p_*(m_1; \beta)$ in Figure 2. For example, $m_1 = 400$ and $\eta = 0.4$ corresponds to $\beta = 1.004$ from Figure 1, and this gives the spectrum second from the top in Figure 2. The values shown are appropriate only for $m \leq 400$ in this example; to be converted to $p(m_1; \beta)$ they must be multiplied by $1 - a(m_1, \beta)$.

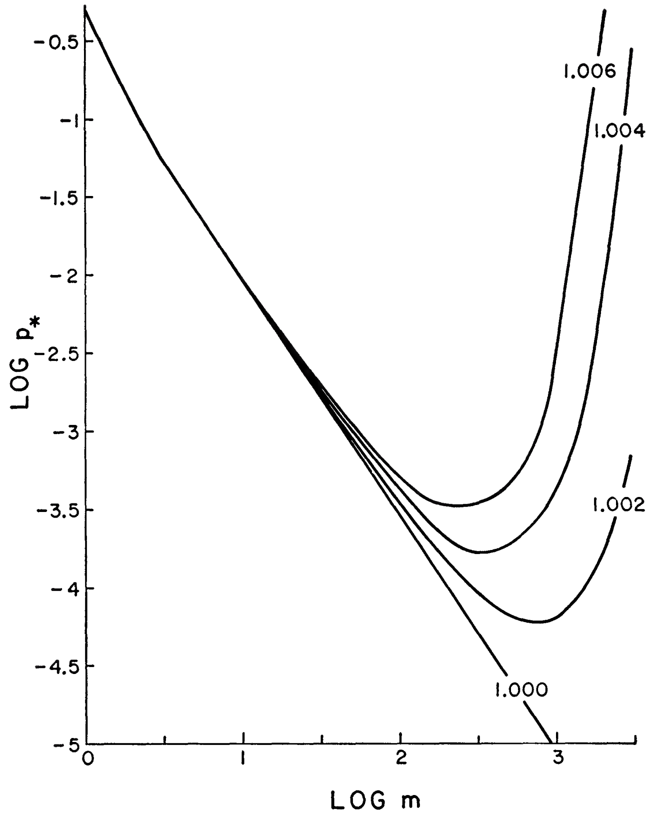

FIG. 2.—Mass spectra for various values of β. Note approach to $-\frac{3}{2}$ slope for $\beta = 1$. Slight adjustments in the vertical scale are necessary to read p rather than p_* (see text).

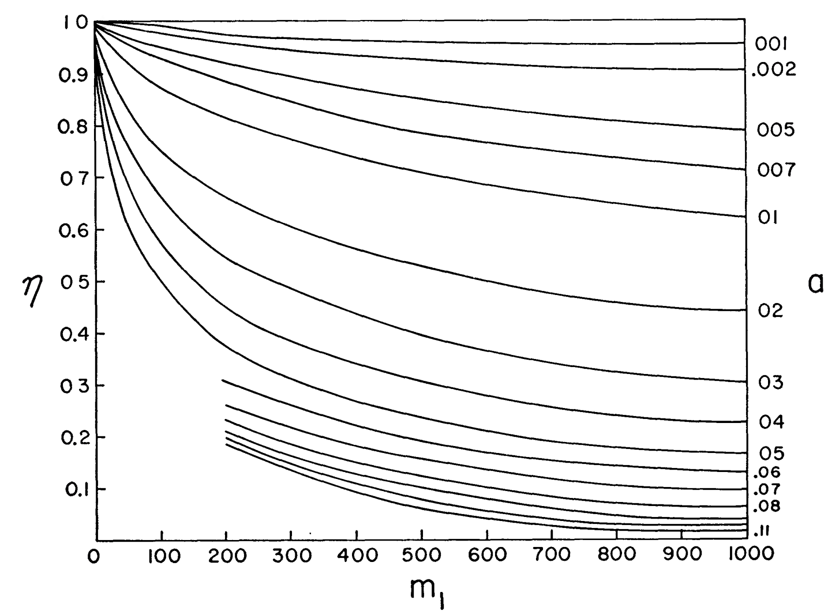

FIG. 3.—The normalized rate of star formation, a as a function of the mass of an unstable cloud, m_1, and the efficiency of ejection, η. Note inverse relationship to both parameters.

The relation $a(m_1, \eta)$ is shown in Figure 3. Evidently a varies inversely with both m_1 and η. This is expected since increasing m_1 inhibits conversion of gas into stars by making the unstable mass large and therefore inaccessible via coalescence processes, while increasing η decreases the fraction of mass of a large cloud going into stars.

It will be noted that the summation in equation (17) is simply related to $\langle m \rangle (m_1, \beta)$, which can be converted to $\langle m \rangle (m_1, \eta)$. The latter relation is shown in Figure 4. It is easily shown from equation (24) that

$$\langle m \rangle (m_1 \rightarrow \infty, \eta) = \sqrt{\frac{m_1}{\pi}}, \tag{26}$$

a result plotted in Figure 4.

Calculations were checked in various ways. The identity in Appendix I was found to be accurate to 1 per cent in most cases.

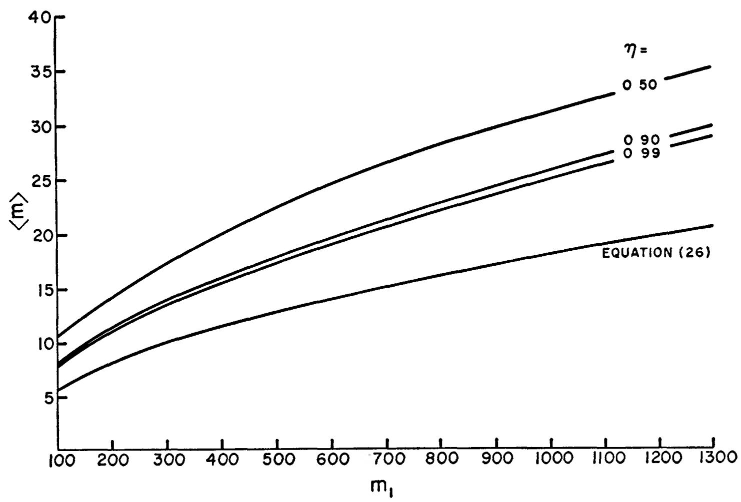

FIG. 4 —Mean cloud mass $\langle m \rangle$ as a function of the mass of an unstable cloud, m_1, for various values of the ejection efficiency, η. The curve labeled "equation (26)" is an asymptotic result for $m_1 \rightarrow \infty$.

V. RELATION TO OBSERVATIONAL DATA ON STAR FORMATION

We may begin with equation (10), which states that the loss of clouds to star formation is proportional to the square of the number of clouds. Such a law is a natural consequence of the assumption of binary collisions. It provides the basis for the interpretation of Schmidt's (1959) empirical law of star formation:

$$\frac{d\rho}{dt} = -K\rho^2, \tag{27}$$

where K is a constant ≥ 0, and ρ is the mean density of gas. Now

$$\rho = A\langle m \rangle M_0, \tag{28}$$

where M_0 is the mass of a unit cloud. If M_0, M_1 $(= m_1 M_0)$, and η are independent of time (being determined by relatively constant physical processes), then m_1 and hence $\langle m \rangle$ are also independent of time for the separable solution. Equation (27) is then equivalent to

$$\frac{dA}{dt} = -K\langle m \rangle M_0 A^2, \tag{29}$$

which is identical with our equation (10) if

$$K = \frac{a\alpha}{\langle m \rangle M_0}. \tag{30}$$

Thus Schmidt's empirical law of star formation can be readily interpreted using the Oort model as formulated here.

Equation (30) in principle provides a way to determine Schmidt's constant K theoretically. Instead of using K, we may work with a time constant for star formation at the present epoch,

$$\tau_S = \left(-\frac{1}{\rho} \frac{d\rho}{dt} \right)^{-1}. \tag{31}$$

From equations (27), (28), and (30), we find that

$$a = \frac{1}{\alpha A \tau_S}. \tag{32}$$

Equation (32) supports the previous interpretation of a as the fractional loss of mass in one collision time. As mentioned previously, Schmidt (1959) has estimated that $\tau_S = 8 \times 10^9$ years, while Spitzer (1965) states that the mean collision time is 6×10^6 years. Therefore, the value of K will agree with that implied by Schmidt if $a = 8 \times 10^{-4}$.

The term a can be estimated theoretically if the basic physical parameters of our theory, m_1 and η, can be derived either from theory or from observation. We attempt to estimate these parameters below.

M_1, the mass of an unstable cloud where star formation is occurring, has been given by Oort (1954) as about $4 \times 10^4\ M_\odot$. Similar values have been shown theoretically to be on the verge of instability by Spitzer (1965). The mass M_0 of unit clouds emerging from regions of star formation is not known, but it may be assumed to be somewhat smaller than that of typical clouds. Oort considers that $M_0 = 30\ M_\odot$ is a reasonable estimate. Hence $m_1 = 1300$. Now η can be estimated very roughly as the fraction of the collapsing cloud which still remains when the first star forms which is capable of ionizing the entire cloud. By reference to Strömgren's (1948) tables (as corrected by Pottasch 1960), it appears that an O5 star or brighter is required to ionize a cloud of $4 \times 10^4\ M_\odot$ if $n_H = 20\ \mathrm{cm}^{-3}$. Using the Salpeter stellar mass function ($M^{-2.34}$) between the limits 0.1 and 100 $M_\odot$, we find that about 4 per cent of the matter goes into initial main-sequence stars of mass $\geq 40\ M_\odot$ (O5). Therefore to obtain one O5 star, slightly more than $10^3\ M_\odot$ of gas is necessary. This is about 3 per cent of the mass of the cloud, so η would be about 97 per cent. This estimate is based on the assumption that most of the gas remaining when the first O5 star is formed is ejected and forms unit clouds. If a single fainter star capable of ionizing only a fraction of the gas were adequate to disrupt the cloud (through shock waves from its H II region), η would be larger. On the other hand, if several O stars are formed simultaneously, even though not needed, η would be smaller. We note that the mass of the cloud going into stars agrees roughly with the masses of clusters and associations.

Figure 5 is a theoretical plot of a versus η for various values of m_1. The region of interest is in the lower right-hand corner. If $m_1 = 1300$ and $\eta = 0.97$, one obtains $a = 6 \times 10^{-4}$, in remarkable agreement with the value derived from observational data. In view of the great uncertainty of our estimates of m_1 and η (not to mention the approximate nature of the theory) this shows only that the theory is not badly inconsistent with observational data on star formation.

The very small value of a means that the rate of star formation is low in the solar neighborhood. We can now see why it is so low in physical terms. Consider what would happen if even the smallest interstellar clouds were on the verge of instability, so that

$m_1 = 1$. It is readily shown from the basic equations that in this case, $a = 1 - \eta$, or 0.03, using our estimate for η. This is the rightmost line in Figure 5. This has the straightforward interpretation that $1 - \eta$ of the cloud mass goes into stars on each collision, because of the tendency for the bright stars to fragment the unstable clouds.

On the other hand, the actual value of a for $m_1 = 1300$ is 6×10^{-4}, only 0.02 times as large. This factor is to be explained as the inhibition factor due to the necessity for a unit cloud to make very many collisions before becoming unstable. This in turn is due to the large value of m_1. This interpretation is borne out by Figure 5, which indicates that if $\eta \rightarrow 1$, $a = f(m_1)(1 - \eta)$, where $f(m_1)$ is a decreasing function.

VI. RELATION TO OTHER OBSERVATIONAL DATA

The theoretical mass spectrum corresponds to $\beta < 1.001$ and is therefore very close to the spectrum given by equation (24), the lowest line of Figure 2. Since $a \ll 1$, equation (24) can be used to obtain $p(m)$ directly. We shall make some specific comparisons in what follows.

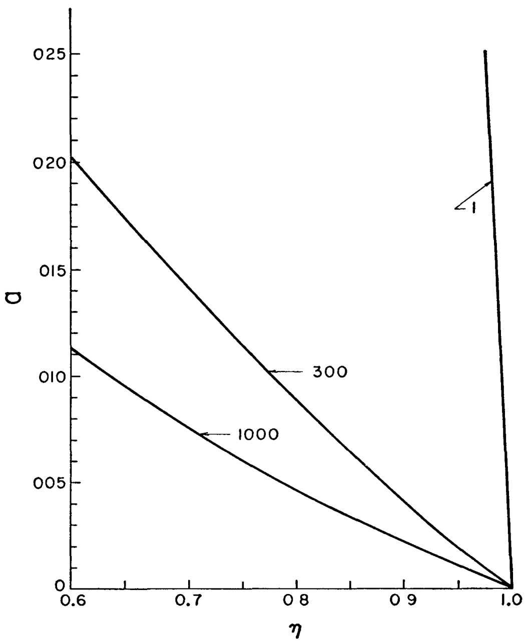

FIG 5 —The normalized rate of star formation, a, as a function of the ejection efficiency, η, for various values of the mass of an unstable cloud Note approximate proportionality to $1 - \eta$.

The term $p(1)$ should represent the fraction of all clouds which are unit clouds. Equation (24) predicts that $p(1) = 0.50$. These clouds are theoretically carrying the kinetic energy from H II regions needed to offset the dissipation in cloud collisions; they have not yet experienced collisions themselves. Therefore we may tentatively identify them with the high-velocity tails of histograms of observed cloud velocities (e.g., Spitzer 1965, Fig. 1). Indeed, negative radial velocities predominate in these tails as would be expected if the high-velocity clouds were seen projected against the stars that repel them. The observed fraction of such clouds ($|v| < 15$ km/sec) is about 15 per cent. This value should be corrected upward because the cross-section of a unit cloud is less than that of a typical cloud, and downward because unit clouds are not distributed uniformly, but tend to occur near hot stars where they are more readily seen. The correction for cross-section is a factor of 9 if all clouds have the same internal density (assuming $\langle M \rangle = 870\ M_\odot$, as below). The downward correction is uncertain, so that all that can be said is that the observational value of $p(1)$ is $\lesssim 1$.

From Figure 4 we obtain $\langle m \rangle = 29$, so that $\langle M \rangle = 870\, M_{\odot}$, in rough agreement with Spitzer's estimate of 400 $M_{\odot}$ for the "standard" interstellar cloud.

Schatzman (1950) and Münch (1952) have pointed out that some spread in cloud sizes is indicated by the statistics of color excesses. The same follows from calcium-line intensities according to Takakubo (1958). Both found it necessary to introduce clouds that were more effective than the average. Münch found that his data could be explained if 9 per cent of the clouds had five times the average color excess, while Takakubo required 3 per cent of the clouds to have nine times the average line intensity. If we assume that all clouds have the same density, these figures correspond to $m = 2500$ and 14600, respectively. The integral mass spectrum corresponding to equation (24) is

$$p(>m) = \frac{1}{\sqrt{\pi}}\, m^{-1/2}, \tag{33}$$

which gives 1.1 and 0.5 per cent as the predicted number density of the clouds considered above. The observed numbers will be greater because of the increased cross-section. If we assume a constant-density model, the observed numbers should be 28 and 40 per cent, respectively—three and ten times the numbers actually observed. If the constant-density assumption is correct we conclude that the theoretical mass spectrum disagrees with these observations. Perhaps more massive clouds have higher density because of the increased effectiveness of self-gravitation; this would tend to improve the agreement.

A final test concerns the fraction of all clouds that are massive objects in which star formation is presently under way. The mass per unit volume going into star formation is equivalent to $a\alpha A^2 \langle m \rangle$ unit masses per second, from equation (10). The mass going into unstable clouds is $(1 - \eta)^{-1}$ times larger, so that the number of unstable clouds formed per unit volume per unit time is

$$\frac{a\alpha A^2 \langle m \rangle}{m_1 (1 - \eta)} \tag{34}$$

and the fractional number of such clouds is

$$\frac{a \langle m \rangle}{m_1 (1 - \eta)}\, \alpha A \tau_F, \tag{35}$$

where τ_F is the time during which star formation and H II regions are conspicuous in the cloud. Taking $a = 6 \times 10^{-4}$, $\langle m \rangle = 29$, $m_1 = 1300$, $1 - \eta = 0.03$, $\alpha A = (6 \times 10^6$ years$)^{-1}$, and $\tau_F = 10^7$ years, the theoretically predicted fraction is 8×10^{-4}. The observed value may be found from Oort's estimate of five massive clouds undergoing star formation within a distance of 1.5 kpc, corresponding to about 3.5 such clouds per kpc^3. Spitzer (1965) adopts $A = 5 \times 10^4$ per kpc^3; estimates listed by van de Hulst (1958) indicate that this may easily be in error by a factor of 3. Adopting Oort's and Spitzer's estimates, we find that the observational estimate for the fraction of all clouds which are undergoing star formation is 7×10^{-5}, one-tenth of the theoretical estimate. In view of the uncertainties the agreement is perhaps acceptable.

VII. SUMMARY AND CONCLUSIONS

The statistical model of interstellar cloud collisions based on Oort's scheme presented here is in reasonable agreement with observation in several respects. First, it explains qualitatively why star formation varies as the square of the density. Second, it explains quantitatively why the rate of star formation is so low (a is small), if one adopts not unreasonable values for the parameters m and η. Third, it predicts an $m^{-3/2}$ mass spectrum which appears to agree roughly with observation as regards the number of young, rapidly moving clouds, the mean mass of all clouds, and the number of clouds considerably larger than the average. This degree of agreement suggests that the model should be developed further.

We may call attention to several deficiencies of the present model. First, the adopted expression for a is unrealistic, and should probably be replaced by one reflecting a positive correlation between mass and cross-section, and a negative correlation between mass and velocity. The discussion of Appendix III is a step in this direction. Second, our concept of cloud collisions is too idealized. In particular the effect of splintering into smaller clouds (Kahn 1955) was ignored. Third, the supposed hydrostatic equilibrium of individual clouds is open to grave doubt, as no one has explained satisfactorily why there should be a low-density high-temperature region uniformly pervading the intercloud regions and preventing clouds from expanding. An equally plausible model would be that "clouds" really represent compressed regions behind shocks which originate from regions containing hot stars and supernovae, while the "intercloud regions" represent regions recently affected by an expansion wave. It is conceivable that something like the present statistical model would be applicable to the interaction of such a collection of shocks and expansion waves. Finally, we may mention the problem of generating new quickly moving clouds at the boundaries of H II regions. While a start on this problem has been made by Oort and Spitzer (1955), it cannot be regarded as completely solved. In particular, the mass distribution of new clouds undoubtedly involves a considerable range of masses. All of these points call for further theoretical study.

On the observational side, the greatest need is for clear-cut data on the density distribution in interstellar space. It is well known that optical absorption-line data provide high resolution but inadequate sky coverage, while 21-cm data have suffered from the opposite defect. What is needed is an intensive 21-cm survey of a fairly large region, utilizing the highest possible angular and frequency resolution. Such a survey is planned with the 300-foot antenna of the National Radio Astronomy Observatory. It is possible that entirely new techniques could be applied to the problem. One that comes to mind is the observation of the infrared spectral lines and continuum in the 10–100-μ region, which should give evidence on dissipation and subsequent radiation in cloud collisions.

Finally, it is suggested that the model be developed into a true dynamical theory by inclusion of the velocity variable. Such a theory should provide the basis of understanding how energy flows from hot stars into the cloud motions, as well as how mass is exchanged between clouds of various masses. If successful, the rms velocity of clouds might be derived, instead of assumed as in the present paper.

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APPENDIX I

PROOF OF EQUATION (17)

First multiply the basic equation (15) by m, and sum over all values of m. The result is

$$\frac{1}{2}\sum_{m=1}^{m_1} m \sum_{m'=1}^{m-1} p_*(m')\,p_*(m-m') - \frac{\langle m\rangle}{1-a} + \frac{1}{2}\beta = 0\,. \tag{I.1}$$

The first term is found to be equal to

$$\sum_{m'=1}^{m_1} p_*(m') \sum_{m=m'}^{m_1} m p_*(m-m') \tag{I.2}$$

by interchanging the order of summation in the usual way (since $p_*(0) = 0$). Letting $m'' = m - m'$, we find that this can be written

$$\sum_{m'=1}^{m_1} \sum_{m''=1}^{m_1 - m'} (m' + m'') p_*(m') p_*(m''), \tag{I.3}$$

which equals

$$\sum_{m'=1}^{m_1} \sum_{m''=1}^{m_1} (m' + m'') p_*(m') p_*(m'') - \sum_{m'=1}^{m_1} \sum_{m''=m_1-m'+1}^{m_1} (m' + m'') p_*(m') p_*(m''). \tag{I.4}$$

The first term of equation (I.4) is readily shown to be equal to $2\langle m\rangle/(1-a)^2$, while the second is equal to β/η by equation (14). When these results are substituted into equation (I.1), we have

$$\frac{\langle m\rangle}{(1-a)^2} - \frac{1}{2}\frac{\beta}{\eta} - \frac{\langle m\rangle}{1-a} + \frac{1}{2}\beta = 0, \tag{I.5}$$

which is equivalent to equation (17).

APPENDIX II

APPROACH TO SEPARABLE SOLUTION AS $t \to \infty$

Suppose the input of unit clouds is F clouds per unit volume per unit time, fixed by external circumstances. Intuitively one expects this system to approach a steady state whatever the initial conditions. Indeed, in the special case $m_1 = \infty$ this steady state is the one described in Section III for the case $\beta = 1$, where F replaces ηU. Our aim here is to verify that this system in fact does approach the steady-state solution independent of the initial mass spectrum $N(m, 0)$.

The problem is best approached through the transform of equation (6):

$$\frac{\partial \bar{N}(z, t)}{\partial t} = \frac{1}{2}\alpha \bar{N}^2(z, t) - \alpha \bar{N}(z, t)\bar{N}(1, t) + Fz, \tag{II.1}$$

where we have used the fact that

$$\sum_{m=1}^{\infty} N(m, t) = A(t) = \bar{N}(1, t) \tag{II.2}$$

according to equation (18) Letting $z = 1$ in the above equation, we have

$$\frac{dA}{dt} = -\alpha A^2 + F, \tag{II.3}$$

the solution of which is

$$A(t) = \left(\frac{2F}{\alpha}\right)^{1/2} \frac{\gamma - \exp[-(2\alpha F)^{1/2} t]}{\gamma + \exp[-(2\alpha F)^{1/2} t]}, \tag{II.4}$$

where

$$\gamma = \frac{A_0 + (2F/\alpha)^{1/2}}{A_0 - (2F/\alpha)^{1/2}} \tag{II.5}$$

and A_0 is the initial number of clouds. Hence $A(t) \to (2F/\alpha)^{1/2}$ as $t \to \infty$, irrespective of the value of A_0.

We now formally define

$$\bar{p}(z, t) = \frac{\bar{N}(z, t)}{A(t)}, \tag{II.6}$$

which implies

$$\left(\frac{2}{\alpha F}\right)^{1/2}\left[A_*(t)\frac{\partial \bar{p}(z, t)}{\partial t} + \bar{p}(z, t)\frac{dA_*(t)}{dt}\right] = A_*^2(t)\bar{p}^2(z, t) - 2A_*^2(t)\bar{p}(z, t) + z, \tag{II.7}$$

where

$$A_*(t) = \left(\frac{\alpha}{2F}\right)^{1/2} A(t). \tag{II.8}$$

From equation (II.4)

$$A_* \to 1, \quad \frac{dA_*}{dt} \to 0 \quad \text{as} \quad t \to \infty. \tag{II.9}$$

In the limit of large time, then, equation (II.7) becomes

$$\left(\frac{2}{\alpha F}\right)^{1/2} \frac{\partial \bar{p}(z, t)}{\partial t} = \bar{p}^2 - 2\bar{p} + z = (\bar{p} - \bar{p}_+)(p - \bar{p}_-), \tag{II.10}$$

where

$$\bar{p}_\pm(z) = 1 \pm (1 - z)^{1/2} \tag{II.11}$$

are the roots of the separable problem. The general solution of equation (II.10) is

$$\bar{p}(z, t) = \bar{p}_-(z)\frac{\lambda - [\bar{p}_+(z)/\bar{p}_-(z)]\exp[-(2\alpha F)^{1/2}(1 - z)^{1/2}(t - t_1)]}{\lambda - \exp[-(2\alpha F)^{1/2}(1 - z)^{1/2}(t - t_1)]}, \tag{II.12}$$

where

$$\lambda = \frac{\bar{p}(z, t_1) - \bar{p}_+(z)}{\bar{p}(z, t_1) - \bar{p}_-(z)} \tag{II.13}$$

and t_1 is a time such that $(2\alpha F)^{1/2}t_1 \gg 1$ (so that eq. [II.9] is correct). From equation (II.12) it follows that

$$\bar{p}(z, t) \to \bar{p}_-(z) = 1 - (1 - z)^{1/2}, \qquad t \to \infty, \tag{II.14}$$

no matter what $\bar{p}(z, t_1)$ is. Correspondingly $p(m, t)$ approaches the previously derived solution for $\beta = 1$.

APPENDIX III

THE BASIC EQUATION FOR CONSTANT-DENSITY CLOUDS IN EQUILIBRIUM

A form of $\alpha(m, m')$ that is useful if all clouds have the same density (so $\sigma \propto m^{2/3}$) and are in a condition of equipartition of kinetic energy (so $v \propto m^{-1/2}$) is

$$\alpha(m, m') = 2\sqrt{(2)}\,\pi r_1^2 v_1 \frac{m + m'}{(m m')^{1/2}}, \tag{III.1}$$

where v_1 is the mean speed of a unit cloud and r_1 is its radius. About one-fourth of all collisions are between unit clouds (since they make up half of all clouds). For such collisions equation (III.1) gives

$$a(1, 1) = (4\pi r_1^2)\sqrt{(2)}\, v_1 , \tag{III.2}$$

which is precisely correct. On the other hand, when one mass (say m') is much larger than the other (m), it gives

$$a(m, m') = 2\pi\sqrt{(2)}\, r_1^2 v_1 \left(\frac{m'}{m}\right)^{1/2} . \tag{III.3}$$

The correct value in this limit is

$$a(m, m') = \pi r^2(m')\, v(m) = \pi r_1^2 v_1 \frac{m'^{(2/3)}}{m^{1/2}} . \tag{III.4}$$

The ratio of equation (III.3) to equation (III.4) is $2\sqrt{(2)}\, m'^{(-1/6)}$, which ranges from 1.9 for $m' = 10$ to 0.9 for $m' = 10^3$. Thus the factor of error is reduced very considerably from that obtained when only a mean value of a is used. We call $2\pi\sqrt{(2)}\, r_1^2 v_1 = \frac{1}{2}a(1, 1)$, a_0.

We now formulate the problem for the steady-state case, $m_1 = \infty$. We have

$$\frac{m}{2}\sum_{m'=1}^{m-1}\frac{N(m')}{m'^{1/2}}\,\frac{N(m-m')}{(m-m')^{1/2}} - \frac{N(m)}{m^{1/2}}\sum_{m'=1}^{\infty}(m+m')\frac{N(m')}{m'^{1/2}} + \frac{F}{a_0}\,\delta(m, 1) = 0 . \tag{III.5}$$

Introducing

$$N_*(m) = \frac{N(m)}{m^{1/2}} , \tag{III.6}$$

we may write this as

$$\begin{aligned} &\tfrac{1}{2}\sum_{m'=1}^{m-1} N_*(m')\, N_*(m-m') - N_*(m)\sum_{m'=1}^{\infty} N_*(m') \\ &\quad - \frac{N_*(m)}{m}\sum_{m'=1}^{\infty} m' N_*(m') + \frac{F}{a_0}\,\delta(m, 1) = 0 . \end{aligned} \tag{III.7}$$

If we define

$$B = \sum_{m=1}^{\infty} N_*(m) \tag{III.8}$$

and

$$p(m) = \frac{N_*(m)}{B} \tag{III.9}$$

so that $p(m)$ is normalized, equation (III.7) becomes

$$\tfrac{1}{2}\sum_{m'=1}^{m-1} p(m')\, p(m-m') - p(m) - \langle m\rangle \frac{p(m)}{m} + \tfrac{1}{2}\beta\,\delta(m\ \ 1) = 0 , \tag{III.10}$$

where

$$\langle m\rangle = \sum_{m=1}^{\infty} m\, p(m) = \frac{\sum_{m=1}^{\infty} m^{1/2} N(m)}{\sum_{m=1}^{\infty} m^{-1/2} N(m)} \tag{III.11}$$

and

$$\beta = \frac{2F}{a_0 B^2} . \qquad \text{(III.12)}$$

We note that equation (III.10) differs from equation (15) only by the addition of the term in $\langle m \rangle$. It follows from equation (III.11) that any solution $N(m)$ must decrease more rapidly than $m^{-3/2}$, otherwise $\langle m \rangle$ is infinite. This requirement is easily traced to the fact that large clouds have such low velocity that they may be considered stationary, but the rate of loss of small clouds by coalescence with large clouds is proportional to $\sigma(m)\, N(m)$, or $m^{1/2} N(m)$ with the present assumptions. If the integrated effect of such losses is to be finite, $N(m)$ must decrease faster than $m^{-3/2}$. (The precise value would be $m^{-5/3}$ if σ were proportional to $m^{2/3}$.) This means that the mass spectrum in this case is steeper than that calculated in the text for a = constant, which was just $m^{-3/2}$.

The transformed version of equation (III.10) is

$$\bar{p}^2(z) - 2\bar{p}(z) + \beta z - 2\langle m \rangle \left(\int_1^z \bar{p}(z) \frac{dz}{z} + \nu \right) = 0 , \qquad \text{(III.13)}$$

where $\nu = \langle m^{-1} \rangle$. This non-linear integral equation replaces the algebraic equation (19). Little progress has been made toward its solution.

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