THE ASTRONOMICAL JOURNAL VOLUME 70, NUMBER 4 MAY 1965

Universal Variables

SAMUEL HERRICK
University of California, Los Angeles, California
(Received 13 January 1965)

Universal variables—which may be used with elliptic, hyperbolic, and parabolic orbits, including circular and rectilinear (or "collision") orbits—are reviewed and generalized to show their relationship to the proposals of Stumpff and others. A basis of comparison is established with simplified universal formulas for representation in the two-body problem, for differential correction, and for perturbations by variation of parameters, and with alternative sets of universal parameters.

1. INTRODUCTION

"UNIVERSAL" variables, parameters, and formulas are those that can be used with any of the two-body conic-section orbits, including the circular and rectilinear extremes of the ellipse, and the rectilinear extremes of the hyperbola and the parabola. Their primary contribution is nonsingular transition between reference orbits of differing type—e.g., between an elliptic reference orbit and a hyperbolic one—for use in the development of perturbations by the method of variation of parameters, or in either observational or thrust correction of an orbit, whether or not there are perturbations present. They also reduce the programming necessary, perhaps for an "on-board" computer, when the orbits to be encountered in a given mission may range from nearly circular satellite orbits to nearly rectilinear transition orbits.

A subgroup that may be referred to as "unified variables" is somewhat simplified by reference to the line of apsides, and can be used effectively with any of the conics except the circle and the nearly circular ellipse.

The use of such variables as "universal" or "unified" ones for ephemeris integration and orbit correction is a fairly recent development (Herrick 1945, 1953a, 1960a, 1960b, 1961; Stumpff 1947, 1959). But the classical series expansions associated with the Lambert–Euler equation (e.g., Bauschinger 1928, p. 172) are an earlier manifestation of the same principles. For his integration of the two-body problem in universal variables, P. Wong (in an unpublished manuscript) starts with the transformation $d\tau = r\,d\hat{X}$, developed hereafter as Eq. (24), which may be recognized as a "regularizing" transformation such as is used in the three-body problem (Leimanis 1958; Pitkin 1964).

In comparing the several proposals for "universal" variables, parameters, and formulas, we must first of all reject those that involve the denominator q of the early "nearly parabolic" developments, or that cannot include rectilinear orbits for some other reason (Garafalo 1960; Newton 1961). Functions of the true anomaly v, for example, become constant (and some of them infinite) because $v = 180°$ for all positions in a rectilinear orbit.

Then, in a common notation, we must explore the behavior of the proposed functions in the simple determination of position and velocity in the two-body problem, in the calculation of partial differential coefficients for differential corrections and related statistical and optimization problems, and in the method of variation of parameters.

As a basis for these explorations and comparisons we have selected the formulas by which one proceeds from position and velocity at an arbitrary "epoch" $(\mathbf{r}_0, \dot{\mathbf{r}}_0, t_0)$, adopted as parameters or "elements," to position and velocity at any other time $(\mathbf{r}, \dot{\mathbf{r}}, t)$. To simplify the necessary formulations we absorb the gravitation-mass constant, $\mu^{\frac{1}{2}} = k(\Sigma m)^{\frac{1}{2}}$, into the unit of time by

$$d\tau = \mu^{\frac{1}{2}} dt, \quad \dot{\mathbf{r}} = d\mathbf{r}/d\tau, \quad \ddot{\mathbf{r}} = d^2\mathbf{r}/d\tau^2, \text{ etc.,} \tag{1}$$

and, in particular, by defining the mean angular motion n, which is the rate of change of M (in elliptic orbits the mean anomaly), by

$$n = \dot{M} = \frac{dM}{d\tau} = \frac{1}{\mu^{\frac{1}{2}}} \frac{dM}{dt}. \tag{2}$$

When it is desirable to do so, we introduce derivatives with respect to t by

$$\dot{\mathbf{r}}_t = \frac{d\mathbf{r}}{dt} = \mu^{\frac{1}{2}}\dot{\mathbf{r}}, \quad \ddot{\mathbf{r}}_t = \frac{d^2\mathbf{r}}{dt^2} = \mu\ddot{\mathbf{r}}, \quad n_t = \frac{dM}{dt} = n\mu^{\frac{1}{2}}. \tag{3}$$

2. BASIC ELLIPTIC FORMULATION

The equations by which one proceeds from $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$, t_0 to $\mathbf{r}$, $\dot{\mathbf{r}}$, t, defined in accordance with the end of Sec. 1, may be subdivided into three groups, the first and third of which are "universal," and so may be used directly in subsequent sections. The middle group we show, in this section, only in its minimum elliptic formulation, but with Kepler's equation, $M = E - e\sin E$, and its companion equation, $r/a = 1 - e\cos E$, each expanded into two forms that may be used with circular and nearly circular orbits, one of which is easily developed into a "universal" form in Sec. 3.

a. Universal Formulas:

$$\tau=\mu^{\frac{1}{2}}(t-t_0), \quad (4)$$

$$r_0^2=\mathbf{r}_0\cdot\mathbf{r}_0, \qquad \dot{s}_0^2=\dot{\mathbf{r}}_0\cdot\dot{\mathbf{r}}_0, \quad D_0=r_0\dot{r}_0=\mathbf{r}_0\cdot\dot{\mathbf{r}}_0, \quad (5)$$

$$\alpha=-\frac{1}{a}=\dot{s}_0^2-\frac{2}{r_0}, \qquad c_0=1-\frac{r_0}{a}. \quad (6)$$

We do not use α in the early stages of the discussion, in part because of the greater familiarity of a, the semimajor axis in elliptic orbits, but α proves to be more convenient than a in differential correction and in variation of parameters.

b. Elliptic Formulas:

$$n=1/a^{\frac{3}{2}}, \quad e\cos E_0=c_0, \quad e\sin E_0=D_0/a^{\frac{1}{2}}. \quad (7)$$

Then with the notation

$$X=E-E_0 \quad (8)$$

Kepler's equation becomes [cf. Eq. (2)]

$$n\tau=M-M_0=X+e\sin E_0(1-\cos X)-e\cos E_0\sin X$$
$$=(1-e\cos E_0)X+e\sin E_0(1-\cos X)+e\cos E_0(X-\sin X) \quad (9)$$

and

$$r/a=1+e\sin E_0\sin X-e\cos E_0\cos X$$
$$=1-e\cos E_0+e\sin E_0\sin X+e\cos E_0(1-\cos X). \quad (10)$$

If, after the solution of Kepler's equation (9) for X, one next forms the functions

$$\hat{S}=a^{\frac{1}{2}}\sin X, \quad \hat{C}=a(1-\cos X), \quad \hat{U}=a^{\frac{3}{2}}(X-\sin X), \quad (11)$$

he will find that they fit naturally into well-known elliptic formulas for f, g, $\dot{f}$, $\dot{g}$—and in such a way as to make them actually become:

c. Universal Formulas:

$$f=1-\hat{C}/r_0, \quad \dot{f}=-\hat{S}/rr_0,$$
$$g=\tau-\hat{U}, \qquad \dot{g}=1-\hat{C}/r, \quad (12)$$

$$\mathbf{r}=f\mathbf{r}_0+g\dot{\mathbf{r}}_0, \quad \dot{\mathbf{r}}=\dot{f}\mathbf{r}_0+\dot{g}\dot{\mathbf{r}}_0. \quad (13)$$

3. UNIVERSAL FORMULATION

If we replace the $X=E-E_0$ of Eq. (8) by the universal variable

$$\hat{X}=a^{\frac{1}{2}}X=a^{\frac{1}{2}}(E-E_0) \quad (14)$$

the elliptic formulas of Sec. 2b are easily developed into their universal forms. We note also that the equivalent hyperbolic and parabolic formulas are satisfied by exactly the same universal forms and the relationships

$$\hat{X}=(-a)^{\frac{1}{2}}(F-F_0), \quad (15)$$

where F is the hyperbolic anomaly equivalent to the eccentric anomaly E, and

$$\hat{X}=D-D_0 \quad (16)$$

for the parabola, in which

$$D=r\dot{r}=(2q)^{\frac{1}{2}}\tan\tfrac{1}{2}v. \quad (17)$$

Then the equations by which one proceeds from t_0, $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$ t_0 t, $\mathbf{r}$, $\dot{\mathbf{r}}$, subdivided as in Sec. 2, are:

a. Group (a):

Equations (4), (5), (6) are used to obtain τ, r_0, $\dot{s}_0^2$, D_0, $\alpha=-1/a$, and c_0.

b. Group (b):

Kepler's equation (9), when its second form is multiplied through by $a^{\frac{3}{2}}$, becomes

$$\tau=r_0\hat{X}+D_0\hat{C}+c_0\hat{U} \quad (18)$$

to be solved in conjunction with the following expressions for $\hat{U}$ and $\hat{C}$ [easily derivable from Eqs. (11)], in which $\hat{U}^*$ and $\hat{C}^*$ are segregated because of the later requirements of the formulas for the partial differential coefficients, and for variation of parameters:

$$\hat{U}=\frac{\hat{X}^3}{3!}-\frac{1}{a}{}^*\hat{U}, \quad \hat{U}^*=\frac{\hat{X}^5}{5!}-\frac{1}{a}\frac{\hat{X}^7}{7!}+\left(\frac{1}{a}\right)^2\frac{\hat{X}^9}{9!}-\cdots, \quad (19)$$

$$\hat{C}=\frac{\hat{X}^2}{2!}-\frac{1}{a}\hat{C}^*, \quad \hat{C}^*=\frac{\hat{X}^4}{4!}-\frac{1}{a}\frac{\hat{X}^6}{6!}+\left(\frac{1}{a}\right)^2\frac{\mathfrak{X}^8}{8!}-\cdots. \quad (20)$$

When Eqs. (18), (19), (20) have been solved by successive approximations for $\hat{X}$, $\hat{U}$, $\hat{C}$, one proceeds, if $\mathcal{S}$ and r are needed for $\dot{f}$, $\dot{g}$, $\dot{\mathbf{r}}$, with

$$\hat{S}=\hat{X}-(1/a)\hat{U}, \quad (21)$$

$$r=r_0+D_0\hat{S}c_0\hat{C}, \quad (22)$$

which follows directly from the second form of Eq. (10).

c. Group (c):

Equations (12), (13) are used to calculate f, g, $\mathbf{r}$ and $\dot{f}$, $\dot{g}$, $\dot{\mathbf{r}}$. (The latter group is often dispensed with, of course, when there are no velocity-component observations or velocity perturbations.)

To the foregoing minimum formulas we may add, for alternatives, for differential formulations, or for perturbation developments, the following:

d. Group (d), Auxiliary formulas:

$$\partial\hat{S}/\partial\hat{X}=\hat{K}=1-(1/a)\hat{C}, \quad (23)$$

which is $\cos(E-E_0)$ for the ellipse,

$$\frac{\partial\hat{C}}{\partial\hat{X}}=\hat{S},\quad \frac{\partial\hat{U}}{\partial\hat{X}}=\hat{C},\quad \frac{\partial\tau}{\partial\hat{X}}=r, \tag{24}$$

$$\frac{\partial r}{\partial\hat{X}}=D=r\dot{r}=D_0\hat{K}+c_0\hat{S},\qquad \frac{\partial D}{\partial\hat{X}}=c=1-\frac{r}{a}, \tag{25}$$

$$\tau=r_0\hat{S}+D_0\hat{C}+\hat{U}=r\hat{S}-D\hat{C}+\hat{U}=r\hat{X}-D\hat{C}+c\hat{U}, \tag{26}$$

$$\begin{aligned} r&=r_0\hat{K}+D_0\hat{S}+\hat{C};\\ r_0&=r\hat{K}-D\hat{S}+\hat{C}=r-D\hat{S}+c\hat{C}, \end{aligned} \tag{27}$$

and from Eqs. (12), (26), (27)

$$\begin{aligned} g&=r_0\hat{S}+D_0\hat{C}=r\hat{S}-D\hat{C},\\ \dot{g}&=\frac{1}{r}(r_0\hat{K}+D_0\hat{S}),\qquad f=\frac{1}{r_0}(r\hat{K}-D\hat{S}),\\ D_0&=D\hat{K}-c\hat{S}=D-\hat{S}-\alpha g. \end{aligned} \tag{28}$$

4. ALTERNATIVE FORMS OF THE UNIVERSAL VARIABLES

In inspecting the equations of the preceding sections we find two "degrees of freedom"—i.e., two places in which we may introduce arbitrary constants. These we shall designate β and $\bar{n}$. The former we introduce by setting

$$\bar{X}=\beta^{-\frac{1}{2}}\hat{X}=\left(\frac{a}{\beta}\right)^{\frac{1}{2}}X=\left(\frac{a}{\beta}\right)^{\frac{1}{2}}(E-E_0), \tag{29}$$

$$\begin{aligned} \bar{U}&=\beta^{-\frac{3}{2}}\hat{U}=\frac{\bar{X}^3}{3!}-\frac{\beta}{a}\frac{\bar{X}^5}{5!}+\left(\frac{\beta}{a}\right)^2\frac{\bar{X}^7}{7!}-\cdots,\\ \bar{C}&=\beta^{-1}\hat{C}=\frac{\bar{X}}{2!}-\frac{^2\beta}{a}\frac{\bar{X}^4}{4!}+\left(\frac{\beta}{a}\right)^2\frac{\bar{X}^6}{6!}-\cdots, \end{aligned} \tag{30}$$

$$\bar{S}=\beta^{-\frac{1}{2}}\hat{S}=\bar{X}-\frac{\beta}{a}\bar{U},\quad \bar{K}=\hat{K}=1-\frac{\beta}{a}\bar{C}. \tag{31}$$

The latter, $\bar{n}$, we introduce by setting [cf. Eqs. (18), (26)]

$$\bar{M}=\bar{n}\tau=\bar{r}_0\bar{X}+\bar{D}_0\bar{C}+\bar{c}_0\bar{U}=\bar{r}_0\bar{S}+\bar{D}_0\bar{C}+\bar{n}\beta^{\frac{3}{2}}\bar{U} \tag{32}$$

so that, evidently,

$$\bar{r}_0=\bar{n}\beta^{\frac{1}{2}}r_0,\quad \bar{D}_0=\bar{n}\beta D_0,\quad \bar{c}_0=\bar{n}\beta^{\frac{3}{2}}c_0 \tag{33}$$

and

$$\bar{r}=\bar{n}\beta^{\frac{1}{2}}r=\bar{r}_0+\bar{D}_0\bar{S}+\bar{c}_0\bar{C}, \tag{34}$$

$$\bar{D}=\bar{n}\beta D=\bar{n}\beta r\dot{r}=\bar{D}_0\bar{K}+\bar{c}_0\bar{S}. \tag{35}$$

Finally the expressions for f, g, $\dot{f}$, $\dot{g}$ [Eqs. (11), (13)] must be rewritten

$$\begin{aligned} f&=1-\beta\bar{C}/r_0,\quad \dot{f}=\beta^{\frac{1}{2}}\bar{S}/rr_0,\\ g&=\tau-\beta^{\frac{3}{2}}\bar{U},\quad \dot{g}=1-\beta\bar{C}/r. \end{aligned} \tag{36}$$

The following choices of the arbitraries β and $\bar{n}$ are of interest:

	β	$\bar{n}$
No. 1	1	1
No. 2	1	$1/r_0$
No. 3	r_0	$1/r_0^{\frac{3}{2}}$
No. 4	$1/r_0^2$	1
No. 5	τ^2/r_0^2	$1/\tau$
No. 6	μ	$1/\mu^{\frac{1}{2}}$
No. 7	μ/r_0^2	$1/\mu^{\frac{1}{2}}$

Choice No. 1, of course, reduces Eqs. (29)–(36) to the simpler forms of Sec. 3. Choices Nos. 2, 3, and 4 have the property that $\bar{r}_0=1$; these were developed to explore their effect on the question of single versus double integration in the method of variation of parameters (Herrick 1964). Choice No. 4, in which $\bar{n}=1$ also, reduces Eq. (32) to the relatively simple form

$$\tau=\bar{X}+\bar{D}_0\bar{C}+\bar{c}_0\bar{U}=\bar{S}+\bar{D}_0\bar{C}+r_0^{-3}\bar{U}. \tag{33}$$

Choice No. 5, which leads to the equations and "universal variables" developed by Stumpff (1947), may be looked upon as an ingenious variation of choice No. 4; it makes $\bar{M}=1$ as well as $\bar{r}_0=1$. The consequences of choice No. 5 we find interesting to develop below.

Choice No. 6 was proposed in a manuscript titled, "A Modification of Herrick's Solution of the Two-Body Problem for All Cases", sent to me early in 1964 by W. H. Goodyear, and later revised and retitled. It is designed to replace the independent variable $\tau=\mu^{\frac{1}{2}}(t-t_0)$ by $t-t_0$ itself, by absorbing $\mu^{\frac{1}{2}}$ into certain other quantities. It has the interesting effect that μ enters into the actual computing formulas only in the first power and not in $\mu^{\frac{1}{2}}$, so that transitions between attractive and repulsive forces can be added to the transitions discussed in Sec. 1. Choice No. 7 has the properties of choice No. 6 and, in addition, those of choice No. 4.

Choice No. 5—Stumpff's form:

Equations (32)–(34) become

$$1=\bar{X}+\bar{D}_0\bar{C}+\bar{c}_0\bar{U}=\bar{S}+\bar{D}_0\bar{C}+\bar{n}\beta^{\frac{3}{2}}\bar{U}, \tag{37}$$

where

$$\begin{aligned} \bar{D}_0&=\frac{\tau D_0}{r_0^2}=\frac{\tau\dot{r}_0}{r_0},\quad \bar{n}\beta^{\frac{3}{2}}=\frac{\tau^2}{r_0^3},\\ \bar{c}_0&=\frac{\tau^2c_0}{r_0^3}=\frac{\tau^2}{r_0^3}-\frac{\tau^2}{r_0^2a}=\frac{\tau^2\dot{s}_0^2}{r_0^2}-\frac{\tau^2}{r_0^3}, \end{aligned} \tag{38}$$

$$r=r_0\bar{r}=r_0(1+\bar{D}_0\bar{S}+\bar{c}_0\bar{C}). \tag{39}$$

Interestingly the arguments for his series that are the equivalent of $\bar{U}$, $\bar{C}$, $\bar{S}$, $\bar{K}$ become

$$\frac{\beta}{a}=\frac{\tau^2}{r_0^2a},\quad \bar{X}=\frac{r_0a^{\frac{1}{2}}}{\tau}(E-E_0)=\frac{E-E_0}{M-M_0}\left(\frac{dM}{dE}\right)_0. \tag{40}$$

Stumpff's notation (double in some cases) is as follows:

$$
\begin{aligned}
&\bar{X}=z, \quad \bar{D}_0=\eta=\sigma\tau, \quad \bar{c}_0=\zeta=\epsilon\tau^2,\\
&\frac{\beta}{a}=\chi=\rho\tau^2, \quad \left(\frac{a}{\beta}\right)^{\frac{1}{2}}\bar{X}=E-E_0=\lambda=\chi^{\frac{1}{2}}z,\\
&\bar{K}=c_0(\lambda^2)=c_0(\chi z^2), \quad \frac{\bar{C}}{\bar{X}^2}=c_2(\lambda^2)=c_2(\chi z^2),\\
&\frac{\bar{S}}{\bar{X}}=c_1(\lambda^2)=c_1(\chi z^2), \quad \frac{\bar{U}}{\bar{X}^3}=c_3(\lambda^2)=c_3(\chi z^2).
\end{aligned}
\tag{41}
$$

His expressions for f, g, $\dot{f}$, $\dot{g}$ are, in effect, as follows:

$$
\begin{aligned}
&f=1-\frac{\tau^2\bar{C}}{r_0^3}, \quad \dot{f}=\frac{\tau\bar{S}}{rr_0^2},\\
&g=\tau-\frac{\tau^3\bar{U}}{r_0^3}, \quad \dot{g}=1-\frac{\tau^2\bar{C}}{rr_0^2}.
\end{aligned}
\tag{42}
$$

Perhaps the chief drawback of Stumpff's ingenious formulation is that the coefficients $\bar{D}_0$, $\bar{c}_0$, and β/a, as they appear in Kepler's Eq. (37) and in Eqs. (30) for $\bar{U}$ and $\bar{C}$, are functions of τ and so must be computed anew for each value of t. There appears to be only disadvantage in making $\bar{M}=1$ in Eq. (37) as compared with $\bar{M}=\tau$ in Eqs. (18), (33).

5. VARIATION OF PARAMETERS

Formulas for the method of variation of parameters may be developed somewhat more simply than those for the differential correction, and they have been very well developed by Wong (1962), using the "perturbative differentiation" technique developed in my first paper (Herrick 1948, cf. also 1953b, 1960c). He adheres rather too strictly, however, to earlier usage on $\mu^{\frac{1}{2}}$, etc. and his equations accordingly are not quite in the simplest form.

In order to develop formulas for the perturbative variations, $\mathbf{r}_0^{\backprime}$ and $\dot{\mathbf{r}}_0^{\backprime}$, of the parameters selected for the illustrative problem of this paper (cf. Sec. 1), one must first invert the equations of Secs. 2 and 3, so that $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ are expressed as functions of $\mathbf{r}$ and $\dot{\mathbf{r}}$. To this end one may simply drop the subscript zero from Eqs. (5) and (6), but in order to continue using the values of τ, $\hat{X}$, $\hat{U}$, $\hat{S}$, f, g, $\dot{f}$, $\dot{g}$ in the same sense as in Eqs. (12), (13), (18)–(22), one must modify Eqs. (18), (22), (13) to [cf. Eqs. (25), (26), (27)]

$$\tau=r\hat{X}-D\hat{C}+c\hat{U}, \tag{43}$$

$$r_0=r-D\hat{S}+c\hat{C}, \tag{44}$$

$$\mathbf{r}_0=\dot{g}\mathbf{r}-g\dot{\mathbf{r}}, \quad \dot{\mathbf{r}}_0=-\dot{f}\mathbf{r}+f\dot{\mathbf{r}}, \tag{45}$$

which last equations follow from the fact that

$$f\dot{g}-g\dot{f}=1.$$

Then, by the simple process of perturbative differentiation, using $\alpha=-1/a$ rather than a itself, designating the perturbative acceleration by $\dot{\mathbf{r}}^{\backprime}$, whatever its character or mathematical expression (and whether or not the perturbing forces are conservative or dissipative), and remembering that the perturbative variation of $\mathbf{r}$ is zero, we obtain

$$D^{\backprime}=\mathbf{r}\cdot\dot{\mathbf{r}}^{\backprime}, \quad \alpha^{\backprime}=2\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}^{\backprime}, \quad c^{\backprime}=r\alpha^{\backprime}, \tag{46}$$

and, with

$$
\begin{aligned}
&\hat{U}_\alpha=\frac{\partial\hat{U}}{\partial\alpha}=\tfrac{1}{2}(\hat{X}\hat{C}^*-3\hat{U}^*),\\
&\hat{C}_\alpha=\frac{\partial\hat{C}}{\partial\alpha}=\tfrac{1}{2}(\hat{X}\hat{U}-2\hat{C}^*),
\end{aligned}
\tag{47}
$$

$$\hat{X}^{\backprime}=\frac{1}{r_0}[(D\hat{C}_\alpha-c\hat{U}_\alpha-r\hat{U})\alpha^{\backprime}+\hat{C}D^{\backprime}], \tag{48}$$

$$\hat{U}^{\backprime}=\hat{C}\hat{X}^{\backprime}+\hat{U}_\alpha\alpha^{\backprime}, \quad \hat{C}^{\backprime}=\hat{S}\hat{X}^{\backprime}+\hat{C}_\alpha\alpha^{\backprime}, \tag{49}$$

$$\hat{S}^{\backprime}=\hat{X}^{\backprime}+\alpha\hat{U}^{\backprime}+\hat{U}\alpha^{\backprime}, \tag{50}$$

$$r_0^{\backprime}=c\hat{C}^{\backprime}+r\hat{C}\alpha^{\backprime}-D\hat{S}^{\backprime}-\hat{S}D^{\backprime}, \tag{51}$$

$$
\begin{aligned}
&f^{\backprime}=\frac{1}{r_0^2}(\hat{C}r_0^{\backprime}-r_0\hat{C}^{\backprime}),\\
&\dot{f}^{\backprime}=\frac{1}{rr_0^2}(\hat{S}r_0^{\backprime}-r_0\hat{S}^{\backprime}),\\
&g^{\backprime}=-\hat{U}^{\backprime}, \quad \dot{g}^{\backprime}=\hat{C}^{\backprime},
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
&\mathbf{r}_0^{\backprime}=\mathbf{r}\dot{g}^{\backprime}-\dot{\mathbf{r}}g^{\backprime}-g\dot{\mathbf{r}}^{\backprime},\\
&\dot{\mathbf{r}}_0^{\backprime}=-\mathbf{r}\dot{f}^{\backprime}+\dot{\mathbf{r}}f^{\backprime}+f\dot{\mathbf{r}}^{\backprime}.
\end{aligned}
\tag{53}
$$

It is these last perturbative variations, $\mathbf{r}_0^{\backprime}$ and $\dot{\mathbf{r}}_0^{\backprime}$, that are to be integrated, whether numerically (special perturbations) or from series approximations (general perturbations, and by whatever integration formula (e.g., Gaussian or Runge–Kutta).

Upon considering the effect of choice No. of Sec. 4—the Stumpff form—on the foregoing formulation for variation of parameters, we note that the introduction of r_0 into β/a will complicate rather than simplify the perturbative variations $\bar{U}^{\backprime}$, $\bar{C}^{\backprime}$, $\bar{S}^{\backprime}$, and so $r_0^{\backprime}$ itself, and the whole of the formulation.

6. PARTIAL DIFFERENTIAL COEFFICIENTS

The universal-variable analytic forms of the partial differential coefficients of $\mathbf{r}$ with respect to $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$, which are useful in differential correction, in thrust guidance, in error analyses, in optimization, and in similar problems, were first developed in the author's study of the two-body precision aspects of rendezvous (Herrick 1961). They were extended to $\dot{\mathbf{r}}$ by Pierce in the second edition (1962) of the same study.

The complexities in the formulation of these partial differential coefficients can be reduced greatly by a thoughtful selection of equations, sets of parameters, and notations. In order to orient ourselves in the problem we may start effectively at the end of the "representation" outlined by Secs. 2 and 3, with Eqs. (13). Let the notation $u, v=x, y, z$ indicate every possible pairing of x, y, z: namely, $xx, xy, xz, yx, yy, yz, zx, zy, zz$. Then, from Eqs. (13),

$$\begin{aligned}\frac{\partial v}{\partial u_0}&=f\delta_{uv}+v_0\frac{\partial f}{\partial u_0}+\dot v_0\frac{\partial g}{\partial u_0},\\ \frac{\partial v}{\partial \dot u_0}&=g\delta_{uv}+v_0\frac{\partial f}{\partial \dot u_0}+\dot v_0\frac{\partial g}{\partial \dot u_0},\\ \frac{\partial \dot v}{\partial u_0}&=\dot f\delta_{uv}+v_0\frac{\partial \dot f}{\partial u_0}+\dot v_0\frac{\partial \dot g}{\partial u_0},\\ \frac{\partial \dot v}{\partial \dot u_0}&=\dot g\delta_{uv}+v_0\frac{\partial \dot f}{\partial \dot u_0}+\dot v_0\frac{\partial \dot g}{\partial \dot u_0},\end{aligned}\qquad u, v=x, y, z \quad (54)$$

where δ_{uv} is the Kronecker delta,

$$\begin{aligned}\delta_{uv}&=1, \quad u=v,\\ &=0, \quad u\neq v.\end{aligned}\quad (55)$$

For the development of the partial differential coefficients of $f, g, \dot f, \dot g$ with respect to $u_0, \dot u_0$, we note from Eqs. (4)–(6), (12), (18)–(22) that r_0, D_0, α will serve effectively as an intermediate set of parameters, and that [from Eqs. (5), (6)]

$$\begin{aligned}&r_0\frac{\partial r_0}{\partial u_0}=u_0, &&\frac{\partial r_0}{\partial \dot u_0}=0,\\ &\frac{\partial D_0}{\partial u_0}=\dot u_0, &&\frac{\partial D_0}{\partial \dot u_0}=u_0, \qquad u=x, y, z\\ &r_0^3\frac{\partial \alpha}{\partial u_0}=2u_0, &&\frac{\partial \alpha}{\partial \dot u_0}=2\dot u_0,\end{aligned}\quad (56)$$

and

$$\frac{\partial c_0}{\partial r_0}=\alpha, \quad \frac{\partial c_0}{\partial D_0}=0, \quad \frac{\partial c_0}{\partial \alpha}=r_0. \quad (57)$$

Then, from Eqs. (19), (20), (21),

$$\frac{\partial \hat U}{\partial r_0}=\hat C\frac{\partial \hat X}{\partial r_0}, \quad \frac{\partial \hat U}{\partial D_0}=\hat C\frac{\partial \hat X}{\partial D_0}, \quad \frac{\partial \hat U}{\partial \alpha}=\hat C\frac{\partial \hat X}{\partial \alpha}+\hat U_\alpha, \quad (58)$$

$$\frac{\partial \hat C}{\partial r_0}=\hat S\frac{\partial \hat X}{\partial r_0}, \quad \frac{\partial \hat C}{\partial D_0}=\hat S\frac{\partial \hat X}{\partial D_0}, \quad \frac{\partial \hat C}{\partial \alpha}=\hat S\frac{\partial \hat X}{\partial \alpha}+\hat C_\alpha, \quad (59)$$

$$\frac{\partial \hat S}{\partial r_0}=\hat K\frac{\partial \hat X}{\partial r_0}, \quad \frac{\partial \hat S}{\partial D_0}=\hat K\frac{\partial \hat X}{\partial D_0}, \quad \frac{\partial \hat S}{\partial \alpha}=\hat K\frac{\partial \hat X}{\partial \alpha}+\hat S_\alpha, \quad (60)$$

where

$$\hat U_\alpha=\tfrac{1}{2}(\hat X\hat C^*-3\hat U^*), \quad \hat C_\alpha=\tfrac{1}{2}(\hat X\hat U-2\hat C^*) \quad (61)$$

are the partial derivatives of $\hat U$ and $\hat C$ with respect to α [cf. Eq. (47)] for the set of parameters $\hat X, \alpha$, but not for the set r_0, D_0, α; and where [cf. Eqs. (23), (50)]

$$\hat K=1+\alpha\hat C, \quad \hat S_\alpha=\hat U+\alpha\hat U_\alpha. \quad (62)$$

The partial derivatives of $\hat X$ are obtained from Kepler's equation (18), in which τ is a constant in each partial differentiation, with the aid of Eqs. (21), (22) for $\hat S$ and r, and of Eqs. (57), (58), (59):

$$r\frac{\partial \hat X}{\partial r_0}=-\hat S, \quad r\frac{\partial \hat X}{\partial D_0}=-\hat C, \quad r\frac{\partial \hat X}{\partial \alpha}=-X_\alpha, \quad (63)$$

where

$$X_\alpha=r_0\hat U+D_0\hat C_\alpha+c_0\hat U_\alpha. \quad (64)$$

For the partial derivatives of f and g with respect to r_0, D_0, α we differentiate Eqs. (12) and obtain, with the aid of Eqs. (58), (59), (63), (64),

$$\begin{aligned}&\frac{1}{r_0}\frac{\partial f}{\partial r_0}=\frac{\hat C}{r_0^3}+\frac{\hat S^2}{rr_0^2}, \quad \frac{\partial f}{\partial D_0}=\frac{\hat S\hat C}{rr_0}, \quad \frac{\partial f}{\partial \alpha}=\frac{\hat SX_\alpha-r\hat C_\alpha}{rr_0},\\ &\frac{1}{r_0}\frac{\partial g}{\partial r_0}=\frac{\hat S\hat C}{rr_0}=\frac{\partial f}{\partial D_0}, \quad \frac{\partial g}{\partial D_0}=\frac{\hat C^2}{r}, \quad \frac{\partial g}{\partial \alpha}=\frac{\hat CX_\alpha-r\hat U_\alpha}{r}.\end{aligned}\quad (65)$$

For the partial derivatives of $\dot f$ and $\dot g$ we must first have those of r, from Eqs. (22), (25), (28), (59)–(64)

$$\begin{aligned}&r\frac{\partial r}{\partial r_0}=r\hat K-D\hat S=r_0 f, \quad r\frac{\partial r}{\partial D_0}=r\hat S-D\hat C=g,\\ &r\frac{\partial r}{\partial \alpha}=rr_\alpha-DX_\alpha, \qquad r_\alpha=r_0\hat C+D_0\hat S_\alpha+c_0\hat C_\alpha.\end{aligned}\quad (66)$$

Then, again from Eqs. (12),

$$\begin{aligned}\frac{1}{r_0}\frac{\partial \dot f}{\partial r_0}&=\frac{\hat S}{r^3r_0^3}[r^2+r_0(r\hat K+r_0 f)],\\ \frac{1}{r_0}\frac{\partial \dot g}{\partial r_0}&=\frac{1}{r^3r_0}(r_0 f\hat C+r\hat S^2)=\frac{\hat C}{r^3r_0}(r+r\hat K+r_0 f),\\ \frac{\partial \dot f}{\partial D_0}&=\frac{\hat Sg+r\hat K\hat C}{r^3r_0}=\frac{(\hat Sg+r\hat S)-r\hat C}{r^3r_0},\\ \frac{\partial \dot g}{\partial D_0}&=\frac{\hat C}{r^3}(g+r\hat S),\\ \frac{\partial \dot f}{\partial \alpha}&=\frac{r_0 fX_\alpha+r(\hat Sr_\alpha-r\hat S_\alpha)}{r^3r_0},\\ \frac{\partial \dot g}{\partial \alpha}&=\frac{gX_\alpha+r(\hat Cr_\alpha-r\hat S_\alpha)}{r^3}.\end{aligned}\quad (67)$$

If we now set

$$f_d=\frac{\partial f}{\partial D_0},\quad f_s=2\frac{\partial f}{\partial \alpha},\quad f_r=\frac{f_s}{r_0^3}+\frac{1}{r_0}\frac{\partial f}{\partial r_0},\quad f\rightarrow g,\, f,\, \dot{g},\tag{68}$$

we may write, from Eqs. (56), (65), (67), (68),

$$\frac{\partial f}{\partial u_0}=f_r u_0+f_d\dot{u}_0,\quad \frac{\partial f}{\partial \dot{u}_0}=f_d u_0+f_s\dot{u}_0,\quad f\rightarrow g,\, f,\, \dot{g}.\tag{69}$$

(We may recognize in f_r, f_d, f_s partial derivatives of the set r_0, D_0, $\dot{s}_0$ in place of the set r_0, D_0, α.)

The order of calculations, if r_0, D_0, c_0, $\hat{X}$, $\hat{U}^*$, $\hat{C}^*$, $\hat{U}$, $\hat{C}$, $\hat{S}$, r, are given by the representation calculation of Secs. 2 and 3, is as follows: For $\mathbf{r}$ alone, $\hat{U}_\alpha$, $\hat{C}_\alpha$ by Eqs. (61), X_α by (64), the partial derivatives of f and g by (65), (68), the partial derivatives of $\mathbf{r}$ (or $v=x, y, z$) by (54), (55); additionally for $\dot{\mathbf{r}}$, r_α by (66), the partial derivatives of $\dot{f}$ and $\dot{g}$ by (67), (68), and the partial derivatives of $\dot{\mathbf{r}}$ (or $\dot{v}=\dot{x}, \dot{y}, \dot{z}$) by (54), (55).

7. PARAMETERS ALTERNATIVE TO $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$

In Sec. 4 we explored alternative forms to the universal variables; it is appropriate also to consider the effect of replacing $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ by other parameters. Again the possibilities are infinite and we can consider only a few. They should be judged by the following criteria:

(1) Simplification or complication of the formulas for representation, variation of parameters, and differential correction.

(2) Removal or introduction of indeterminacies, whether mathematical or computational, in particular circumstances.

First consideration should be given to parameters associated with the orthogonal unit vectors

$$\mathbf{U}_0=\mathbf{r}_0/r_0,\quad \mathbf{V}_0=(\dot{\mathbf{r}}_0-\dot{r}_0\mathbf{U}_0)/r_0\dot{v}_0\tag{70}$$

and with r_0 itself, $\dot{r}_0$ or $D_0=r_0\dot{r}_0$, and $\dot{v}_0$ or $p^{\frac{1}{2}}=r_0^2\dot{v}_0$. These parameters are suggested by the problem of determining an orbit satisfying two positions, $\mathbf{r}_0$ and $\mathbf{r}$, separated by 180° (i.e., $v-v_0=180°$). In such circumstances $\dot{\mathbf{r}}_0$ is indeterminate; so also is $\mathbf{V}_0$, but $\dot{r}_0$, $\dot{v}_0$, D_0, and $p^{\frac{1}{2}}$ are all determinate. They will give useful information about the orbit even with $\mathbf{V}_0$ lacking, e.g., in interception problems.

Because $\mathbf{V}_0$ is indeterminate also for rectilinear orbits, though for a different reason ($\dot{\mathbf{r}}_0-\dot{r}_0\mathbf{U}_0=0$, $p^{\frac{1}{2}}=r_0^2\dot{v}_0=0$), and for other reasons that are developed later, the author has come to prefer the following associated parameters:

$$\mathbf{r}_0=r_0\mathbf{U}_0,\quad \hat{\mathbf{V}}_0=r_0p^{\frac{1}{2}}\mathbf{V}_0=r_0^2\dot{\mathbf{r}}_0-D_0\mathbf{r}_0,\quad D_0=r_0\dot{r}_0=\mathbf{r}_0\cdot\dot{\mathbf{r}}_0.\tag{71}$$

We note that $\hat{\mathbf{V}}_0=0$ for the rectilinear orbit. The rectilinear indeterminacy of $\mathbf{V}_0$ accordingly cannot affect our calculations. We note also that the three conditions on $\mathbf{U}_0$ and $\mathbf{V}_0$, namely $\mathbf{U}_0\cdot\mathbf{U}_0=1$, $\mathbf{V}_0\cdot\mathbf{V}_0=1$, $\mathbf{U}_0\cdot\hat{\mathbf{V}}_0=0$, have been reduced to one, $\mathbf{r}_0\cdot\hat{\mathbf{V}}_0=0$.

With the adopted parameters the representation proceeds as follows (cf. Secs. 2 and 3): Equations (5) are replaced by

$$r_0^2=\mathbf{r}_0\cdot\mathbf{r}_0,\quad p=\frac{\hat{\mathbf{V}}_0\cdot\hat{\mathbf{V}}_0}{r_0^2},\quad \dot{s}_0^2=\frac{D_0^2+p}{r_0^2}.\tag{72}$$

Equations (6), (18), (19), (20), and (21) are then used to obtain $\alpha=-1/a$, c_0, $\hat{X}$, $\hat{U}$, $\hat{C}$, $\hat{S}$. Then, for $K=rr_0\cos(v-v_0)$, $g=rr_0\sin(v-v_0)/p^{\frac{1}{2}}$ [cf. Eqs. (12), (13)],

$$f=1-\frac{\hat{C}}{r_0},\quad g=\tau-\hat{U},\qquad \frac{K}{r_0^2}=f+\frac{g}{r_0^2}D_0,\tag{73}$$

$$\mathbf{r}=\frac{K}{r_0^2}\mathbf{r}_0+\frac{g}{r_0^2}\hat{\mathbf{V}}_0,\qquad \hat{\mathbf{V}}=-p\frac{g}{r_0^2}\mathbf{r}_0+\frac{K}{r_0^2}\hat{\mathbf{V}}_0,\tag{74}$$

$$D=D_0\hat{K}+c_0\hat{S}=D_0+\hat{S}+\alpha g,\tag{75}$$

$$\dot{\mathbf{r}}=(D\mathbf{r}+\hat{\mathbf{V}})/r^2,\qquad r^2=\mathbf{r}\cdot\mathbf{r}.\tag{76}$$

To obtain the variations of the adopted parameters one proceeds as in Sec. 5 with Eqs. (46) through (50). Then with the aid of Eqs. (12) or (52), (28) or (75), (76) and the inverted equations

$$K=\dot{g}r^2-gD,\qquad p=r^2\dot{s}^2-D^2=(\hat{\mathbf{V}}\cdot\hat{\mathbf{V}})/r^2,\tag{77}$$
$$r^2\mathbf{r}_0=K\mathbf{r}-g\hat{\mathbf{V}},\quad r^2\hat{\mathbf{V}}_0=pg\mathbf{r}+K\hat{\mathbf{V}},$$

we obtain

$$g^{`}=-\hat{U}^{`},\quad \dot{g}^{`}=-\hat{C}^{`}/r,\tag{78}$$

$$K^{`}=r^2\dot{g}^{`}-(Dg^{`}+gD^{`}),\tag{79}$$

$$\hat{\mathbf{V}}^{`}=r^2\dot{\mathbf{r}}^{`}-\mathbf{r}D^{`},\tag{80}$$

$$p^{`}=r^2\alpha^{`}-2DD^{`}=2(\hat{\mathbf{V}}\cdot\hat{\mathbf{V}}^{`})/r^2,\tag{81}$$

$$r^2\mathbf{r}_0^{`}=\mathbf{r}K^{`}-(\hat{\mathbf{V}}g^{`}+g\hat{\mathbf{V}}^{`}),\tag{82}$$

$$r^2\hat{\mathbf{V}}_0^{`}=\mathbf{r}(pg^{`}+gp^{`})+(\hat{\mathbf{V}}K^{`}+K\hat{\mathbf{V}}^{`}),\tag{83}$$

$$D_0^{`}=D^{`}-\hat{S}^{`}-(\alpha g^{`}+g\alpha^{`}).\tag{84}$$

We note that $r_0^{`}$ is not necessary to the determination of the parameter variations to be integrated, as given by Eqs. (82)–(84), but that it would have been had we not included r_0 in $\hat{\mathbf{V}}_0=r_0p^{\frac{1}{2}}\mathbf{V}_0$, as well as in $\mathbf{r}_0=r_0\mathbf{U}_0$. This is a fourth and final reason for the selection of the set of parameters specified by Eqs. (71), (82)–(84).

It is evident that Eqs. (78)–(84) cannot compete in brevity and simplicity with Eqs. (51)–(53) in the method of variation of parameters, nor Eqs. (73)–(76) with Eqs. (12), (13), (22) in the representation, although of course circumstances may be found in which the longer sets are preferable.

Pitkin (1964) has studied another suggestion—that $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ be replaced by the "scaled and regularized" set $\mathbf{r}_0/r_0^{\frac{1}{2}}$ and $r_0\dot{\mathbf{r}}_0$. He also uses $r\dot{\mathbf{r}}$, postponing the division by r, but not $\mathbf{r}/r^{\frac{1}{2}}$. The necessary accompanying replacement of f, g, $\dot{f}$, $\dot{g}$ by $r_0^{\frac{1}{2}}f$, g/r_0, $rr_0^{\frac{1}{2}}\dot{f}$, $\dot{g}r/r_0$ only complicates the formulation, as may be judged from Eqs. (12), (13), (51)–(53). It would be more effective to use $\mathbf{r}_0$ and $r_0^{\frac{3}{2}}\dot{\mathbf{r}}_0$ in conjunction with $\mathbf{r}$ and $r_0^{\frac{1}{2}}r\dot{\mathbf{r}}$, especially when choice No. 3 of Sec. 4 is adopted; but the resulting theories would still be more complicated than those adopted here in Secs. 2, 3, 5, 6.

This work was supported in part by NASA, Goddard Space Flight Center, USAF Office of Scientific Research and North American Aviation Space and Information Systems Division.

REFERENCES

Bauschinger, J. 1906, 1928, *Die Bahnbestimmung der Himmelskorper* (Engelmann, Leipzig).

Garafalo, A. M. 1960, *Astron. J.* **65**, 117–121.

Herrick, S. 1945, *ibid.* **51**, 123.

——. 1948, *Publ. Astron. Soc. Pacific* **60**, 321–323; *UCLA Astron. Papers* **1**, 60–62.

——. 1953a, *Tables for Rocket and Comet Orbits, Natl. Bur. Stds. Appl. Math. Ser.*, **20**, xxiv+100.

——. 1953b, *Astron. J.*, **58**, 156–164; *UCLA Astron. Papers* **2**, 6–14.

——. 1960a, *Aeronutronic Publ.* U-908.

——. 1960b, *UCLA Astrodynamical Report* No. 7.

——. 1960c, *UCLA Astrodynamical Report* No. 9.

——. 1961, *Lockheed Contract Report*; 2nd ed., 1962: *Lockheed Astrodynamics Research Report* No. 11; *UCLA Astrodynamics Report* No. 15. (The expressions for the partial differential coefficients in $\Delta\dot{f}$ and $\Delta\dot{g}$ contain some errors and should be replaced by the simpler expressions in the present paper.)

——. 1964, *Proc. Intern. Astron. Union Symp.* No. 25, Salonike, Greece (to be published).

Leimanis, E. 1958, *Dynamics and Non-Linear Mechanics* (John Wiley & Sons, Inc., New York), pp. 1–108.

Newton, R. R. 1961, *ARS J.* **31**, 364–366.

Pitkin, E. T. 1964, UCLA thesis (privately printed).

Stumpff, K. 1947, *Astron. Nachr.* **275**, 108–128.

——. 1959, *Himmelsmechanik* **1**, 200–207. (VEB Deutscher Verlag der Wissenschaften, Berlin).

Wong, P. 1962, *ARS J.*, **32**, 264–265.

NEW BOOKS RECEIVED

Journey to Alpha Centauri, by John W. Macvey. Published by the MacMillan Company, New York, January 1965. Price $5.95.

Reference Catalogue of Bright Galaxies, by Gerard de Vaucouleurs and Antoinette de Vaucouleurs. Published by the University of Texas Press, Austin, Texas, January 1965. Price $12.50.

Space Exploration and the Solar System—Course 24, edited by B. Rossi. Proceedings of the International School of Physics, "Enrico Fermi," Italian Physics Society, G. Polvani, Series Editor, October 1964. Price $13.00.

The Physics and Astronomy of Meteors, Comets and Meteorites, by Gerald S. Hawkins. Published by McGraw-Hill Book Company, Inc., New York, 1964.

Astronautics and Aeronautics 1963. Published by Superintendent of Documents, U. S. Government Printing Office, Washington D. C. 20402. Price $2.00.

Whistlers and Related Ionospheric Phenomena, by Robert A. Helliwell. Published by Stanford University Press, Stanford, California, January 1965. Price $12.50.

Kepler's Dream, by John Lear. Published by the University of California Press, Berkeley, California (translated by Patricia L. Kirkwood), February 1965. Price $5.00.

Communications of the Lunar and Planetary Laboratory, Vol. 2, Nos. 31–35; Vol. 3. Published by the University of Arizona Press, Tucson, Arizona. Price $4.50 each.

Annuaire Pour L'An 1965. Published by Le Bureau des Longitudes, Gauthier-Villars, Paris.

Ground-Based Astronomy—A Ten Year Program. Published by the National Academy of Sciences–National Research Council, Washington, D. C. 1964.

Astronomy Highlights. Contributions of the American Museum–Hayden Planetarium, Series 1, Publication 12–19., Editor Thomas D. Nicholson: *Apollo and the Moon*, by Franklin M. Branley; *Captives of the Sun*, by James S. Pickering; *Design of the Universe*, by S. I. Gale; *Man in Space*, by Fred C. Hess; *Space Age Astronomy*, by Kenneth L. Franklin; *The Sun in Action*, by Thomas D. Nicholson; *Time and the Stars*, by Joseph M. Chamberlain. Published by the Natural History Press. Price 50¢ each.

Satellite Environment Handbook, edited by Francis S. Johnson. Published by Stanford University Press, Stanford, California, February 1965. Price $7.50.

Galileo Galilei, by Ludovico Geymonat. Published by McGraw-Hill Book Company, Inc., New York, February 1965 (available in paperback in the fall). Price $6.50.

Astrophysics and Space Science—An Integration of Science, by Allen McMahan. Published by Prentice-Hall Inc., Englewood Cliffs, New Jersey, February 1965. Price $15.00.

Quasi-Stellar Sources and Gravitational Collapse, edited by Ivor Robinson, Alfred Schild, and E. L. Schucking. Published by the University of Chicago Press, Chicago, Illinois. February 1965. Price $10.00.

Fundamental Astrometry by B. B. Podobed (English edition edited by A. N. Vyssotsky). Published by the University of Chicago Press, Chicago, Illinois. March 1965. Price $7.50.