THE STRUCTURE AND STABILITY OF ROTATING GAS MASSES

R. A. JAMES
Department of Mathematics, Faculty of Technology, and
Department of Astronomy, University of Manchester
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ABSTRACT

Structures have been determined for axially symmetric rotating gas masses, in the polytropic and white-dwarf cases. To solve the structure problem near the center of mass, the density and gravitational potential were expanded in power series in the radial variable. The coefficients in these expansions were themselves expanded in terms of Legendre polynomials in the cosine of the co-latitude. Analytic continuation, and finally a step-by-step integration, gave the structure elsewhere. The truncation error was about 0 002 in the worst case considered. Physical parameters for the rotating configurations were obtained for values of $n \leq 3$, and for a range of white-dwarf configurations.

The existence of forms of bifurcation of the axially symmetric series of equilibrium forms was also investigated. The white-dwarf series proved to lack such points of bifurcation, but they were found on the polytropic series for $n \leq 0.808$. The truncation error in this critical value of n is estimated at about 0.0004.

I. INTRODUCTION

We consider a mass of gas rotating as a rigid body about a fixed axis. We are concerned with two cases:

i) The polytropic case. The gas has equation of state

$$P = K\rho^{1+1/n}\,, \tag{1.1}$$

where P denotes the pressure, ρ the density, and K and n are constants (n is the polytropic index).

ii) The white-dwarf case. Following Chandrasekhar (1935) we take equation of state

$$P = af(x)\,, \qquad \rho = bx^3\,, \tag{1.2}$$

where

$$f(x) = x(2x^3 - 3)(x^2 + 1)^{1/2} + 3\sinh^{-1} x\,. \tag{1.3}$$

First, we consider the polytropic case. Emden (1907) and others have investigated the structure of non-rotating polytropes. Milne (1923) and Chandrasekhar (1933) have developed a theory for rotating configurations that is accurate to first order in the rotational distortion. For $n = 3$, Takeda (1934) has extended the theory to give a more accurate description of the geometry of the surface layers.

If $n = 0$, the density of the polytrope is constant throughout. This case has been investigated by many writers. A detailed account is given by Lyttleton (1953). If $n = 5$, the polytrope has finite mass and infinite radius. We may approximate to this case by the Roche model.

Comparing these two extreme cases, we observe important differences in dynamical behavior. For polytropes with $n \approx 5$, as for the Roche model, all equilibrium configurations are axially symmetric about the axis of rotation. These equilibrium configurations form a linear series, and are secularly stable. The corresponding series for the polytrope $n = 0$, however, has a point of bifurcation where it is crossed by a series of non-axially symmetric equilibrium configurations. If the angular momentum exceeds the value for this point of bifurcation, the axially symmetric equilibrium form is secularly unstable. Jeans (1919) has investigated the existence of such points of bifurcation for intermediate values of n.

Schatzman (1958) summarizes the white-dwarf problem in the non-rotating case. The simplest of the models he describes is that of Chandrasekhar (1935). This model was taken as the basis of the work on the white-dwarf part of this investigation.

The purpose of this paper is to investigate the properties of rotating polytropic and white-dwarf configurations. In particular, we determine the ranges of n, or the value of x for the center of mass, for which the series of axially symmetric equilibrium configurations have bifurcation points. This project requires much calculation. The calculations were performed on the Mercury computer in the University of Manchester, mostly in 1961 and 1962. Part of this work was included in a Ph.D. thesis submitted to Manchester University early in 1962.

II. FORMULATION OF THE STRUCTURE PROBLEM

We take the center of mass as the origin of coordinates, and denote the distance from the center of mass by r, the cosine of the co-latitude (measured from the axis of rotation) by μ, and the longitude by ϕ. We let Ψ denote the gravitational potential, ω the angular velocity.

The structure of the configuration is determined by Gauss's theorem:

$$r^{-2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+r^{-2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Psi}{\partial\mu}\right]+r^{-2}(1-\mu^2)^{-1}\frac{\partial^2\Psi}{\partial\phi^2}=-4\pi G\rho \tag{2.1}$$

together with the equations of hydrostatic equilibrium:

$$\frac{\partial P}{\partial r}=\rho\frac{\partial\Psi}{\partial r}+\rho\omega^2 r(1-\mu^2), \tag{2.2}$$

$$\frac{\partial P}{\partial\mu}=\rho\frac{\partial\Psi}{\partial\mu}-\rho\omega^2 r^2\mu, \tag{2.3}$$

$$\frac{\partial P}{\partial\phi}=\rho\frac{\partial\Psi}{\partial\phi}. \tag{2.4}$$

In the following, we attach the suffix c to a quantity to denote its value at $r = 0$.

a) *Polytropic Case*

In the region $\rho \neq 0$, we define the dimensionless variable θ by the equation

$$\rho = \rho_c\theta^n. \tag{2.5}$$

From equations (1.1), (2.2)–(2.5), we find

$$\theta=\psi+\frac{\omega^2}{2(n+1)K\rho_c^{1/n}}r^2(1-\mu^2)+C, \tag{2.6}$$

where

$$\psi=\Psi/(n+1)K\rho_c^{1/n}, \tag{2.7}$$

and C is a constant of integration. For convenience, we take $C = 0$. By equation (2.5), $\theta = 1$ at $r = 0$, so that

$$\psi = 1 \quad \text{at} \quad r = 0. \tag{2.8}$$

We adopt equation (2.6), with $C = 0$, as the definition of θ throughout space. We define

$$\rho' = \rho/\rho_c. \tag{2.9}$$

Then, if $\rho \neq 0$,

$$\rho' = \theta^n, \tag{2.10}$$

and otherwise

$$\rho' = 0. \tag{2.11}$$

We write

$$r = a\xi, \tag{2.12}$$

where

$$a^2 = (n+1)K/4\pi G\rho_c^{1-1/n}. \tag{2.13}$$

The substitutions (2.7), (2.9), and (2.12) reduce equation (2.1) to the form

$$\xi^{-2}\frac{\partial}{\partial\xi}\left(\xi^{2}\frac{\partial\psi}{\partial\xi}\right)+\xi^{-2}\frac{\partial}{\partial\mu}\left[(1-\mu^{2})\frac{\partial\psi}{\partial\mu}\right]+\xi^{-2}(1-\mu^{2})^{-1}\frac{\partial^{2}\psi}{\partial\phi^{2}}=-\rho', \tag{2.14}$$

and equation (2.6) to

$$\theta=\psi+A\xi^{2}(1-\mu^{2}), \tag{2.15}$$

where

$$A=\omega^{2}/8\pi G\rho_{c}. \tag{2.16}$$

b) White-Dwarf Case

From equation (1.3),

$$\frac{df}{dx}=8x^{4}(1+x^{2})^{-1/2}. \tag{2.17}$$

We substitute for x from

$$y^{2}=1+x^{2}, \qquad xdx=ydy. \tag{2.18}$$

Thus, we reduce equations (2.2)–(2.4) to the forms

$$\frac{8a}{b}\frac{\partial y}{\partial r}=\frac{\partial\Psi}{\partial r}+\omega^{2}r(1-\mu^{2}), \tag{2.19}$$

$$\frac{8a}{b}\frac{\partial y}{\partial\mu}=\frac{\partial\Psi}{\partial\mu}-\omega^{2}r^{2}\mu, \tag{2.20}$$

$$\frac{8a}{b}\frac{\partial y}{\partial\phi}=\frac{\partial\Psi}{\partial\phi}. \tag{2.21}$$

These equations integrate to

$$\theta=\psi+\frac{\omega^{2}b}{16a}r^{2}(1-\mu^{2}), \tag{2.22}$$

where, as before, the constant of integration is taken as zero. The symbols θ and ψ are defined by

$$y=y_{c}\theta, \tag{2.23}$$

$$=\Psi_{c}\psi=\frac{8a}{b}y_{c}\psi. \tag{2.24}$$

We now write

$$\alpha^{2}=2a/\pi Gb^{2}y_{c}^{2}, \tag{2.25}$$

$$A=\omega^{2}/8\pi Gy_{c}^{2}, \tag{2.26}$$

$$D=y_{c}^{-2}. \tag{2.27}$$

With these definitions, the substitutions (2.9) and (2.12) reduce the equations of the problem to the forms (2.14) and (2.15), together with

$$\rho'=(\theta^{2}-D)^{3/2} \qquad \text{if} \qquad \rho'\neq0, \tag{2.28}$$

$$=0 \qquad \text{otherwise}. \tag{2.29}$$

c) Boundary Conditions

At $\xi = 0$,

$$\psi = 1\,, \tag{2.30}$$

$$\partial\psi/\partial\xi = 0\,. \tag{2.31}$$

Outside the configuration, ψ is a free-space potential. Thus, if we expand

$$\psi = \sum_{j,\,k} \psi_{jk} P_j(\mu) \cos(k\phi + \delta_{jk})\,, \tag{2.32}$$

where the $P_j^k(\mu)$'s are the associated Legendre polynomials, then

$$\frac{d}{d\xi}\psi_{jk} + \frac{j+1}{\xi}\psi_{jk} = 0 \tag{2.33}$$

outside the body.

III. CALCULATION OF THE POTENTIAL FOR THE AXIALLY SYMMETRIC CASE

In the case of axial symmetry, equation (2.14) reduces to

$$\xi^{-2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\psi}{\partial\xi}\right) + \xi^{-2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right] = -\rho'\,. \tag{3.1}$$

Equations (2.10), (2.11), (2.15), (2.28), and (2.29) do not change. We denote the values of ξ corresponding to the polar and equatorial radii of the configuration by ξ_p, and ξ_e, respectively. In this section, we discuss the determination of the potential for $\xi \leq \xi_p$; ψ is clearly an analytic function of ξ in this region.

Near the center of mass, we obtain ψ by substituting into equation (3.1) a series in ascending powers of ξ. We extend this solution to $\xi = \xi_p$ by analytic continuation. We divide $\xi \leq \xi_p$ into region 1, where we use the expansion about $\xi = 0$, and region 2, the rest of the region $\xi \leq \xi_p$.

a) Solution in Region 1

We expand ψ, ρ' as

$$\psi = \sum_{i,\,j} A_{ij}\xi^i P_j(\mu)\,, \tag{3.2}$$

$$\rho' = \sum_{i,\,j} B_{ij}\xi^i P_j(\mu)\,. \tag{3.3}$$

The coefficient B_{pj} is a function of A and the coefficients A_{ij}, $i \leq p$. We discuss the practical determination of B_{pj} later.

The boundary conditions at $\xi = 0$, equations (2.30) and (2.31), give

$$A_{0j} = \delta_{0j}\,, \tag{3.4}$$

$$A_{1j} = 0\,, \tag{3.5}$$

where δ_{ij} is the Kronecker delta.

Lichtenstein (1933) has shown that all equilibrium forms for rotating fluid masses are symmetric about the equatorial plane. Thus A_{ij}, B_{ij} vanish for odd j. In addition, it is easy to show that they vanish when i is odd, and when $i < j$. Thus

$$A_{ij} = B_{ij} = 0\,, \qquad \text{odd } i \text{ or } j\,, \text{ or } i < j\,. \tag{3.6}$$

We substitute equations (3.2) and (3.3) into equation (3.1) to find

$$[i(i+1) - j(j+1)]A_{ij} = -B_{i-2,\,j}\,. \tag{3.7}$$

Equations (3.4)–(3.7) determine all coefficients A_{ij} for which $i \neq j$. If $i = j$, equation (3.7) is satisfied for any value of A_{ii}. Thus, any set of coefficients A_{ii}, $i = 2(2)\infty$, determines a solution of equation (3.1), satisfying the boundary conditions at $\xi = 0$. The coefficients A_{ii} are determined indirectly by the boundary conditions (2.33).

We terminate the μ-wise expansion for ψ with the term in $P_{10}(\mu)$. The truncation error depends on n (or D) and A. It is less than 2×10^{-3} in the cases which interest us. We terminate the ξ-wise expansion at $\xi = \xi_0$, where

$$\max_j (A_{20\ j}\xi_0^{20}) = 10^{-10} \tag{3.8}$$

defines ξ_0. This choice of the last term is a compromise between the need to maximize ξ_0, and the need to minimize the calculation to find the expansion (3.2).

b) Solution in Region 2

We assume that ψ, $\partial\psi/\partial\xi$ are known at $\xi = \xi_1$. We substitute

$$\xi = \xi_1 + \eta \tag{3.9}$$

in equation (3.1) and obtain

$$\eta^2 \frac{\partial^2\psi}{\partial\eta^2} + 2\eta\frac{\partial\psi}{\partial\eta} + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right] + \eta^2\rho' + 2\xi_1\left[\eta\frac{\partial^2\psi}{\partial\eta^2} + \frac{\partial\psi}{\partial\eta} + \eta\rho'\right] + \xi_1^2\left[\frac{\partial^2\psi}{\partial\eta^2} + \rho'\right] = 0\,. \tag{3.10}$$

In this equation, we substitute

$$\psi = \sum_{i=0}^{i=10}\sum_{j=0}^{j=10} \alpha_{ij}\eta^i P_j(\mu)\,, \tag{3.11}$$

$$\rho' = \sum_i \sum_{j=0}^{j=10} \beta_{ij}\eta^i P_j(\mu)\,. \tag{3.12}$$

We find

$$i(i-1)\xi_1^2\alpha_{ij} = -[(i-1)(i-2) - j(j+1)]\alpha_{i-2,\,j} - \beta_{i-4,\,j} - 2\xi_1[(i-1)^2\alpha_{i-1,\,j} + \beta_{i-3,\,j}] - \xi_1^2\beta_{i-2,\,j}\,. \tag{3.13}$$

We know

$$\alpha_{0j} = \psi_j(\xi_1)\,, \tag{3.14}$$

$$\alpha_{1j} = (d\psi_j/d\xi)_{\xi=\xi_1}\,. \tag{3.15}$$

Thus, we may find all coefficients α_{ij}, and determine ψ_j, $d\psi_j/d\xi$ at

$$\xi = \min(\xi_1', \xi_p)\,, \tag{3.16}$$

where

$$\xi_1' = \xi_1 + h \tag{3.17}$$

and

$$\max_j (\alpha_{10\ j}h^{10}) = 10^{-10}\,. \tag{3.18}$$

Initially, we take $\xi_1 = \xi_0$. We repeat the process until we obtain ψ_j, $d\psi_j/d\xi$ at $\xi = \xi_p$.

c) Determination of the Coefficients B_{pj}, β_{pj}

We work in terms of the expansions (3.11) and (3.12). Our arguments apply to region 1 if we interpret α_{ij} as $A_{2i,\,j}$ and β_{ij} as $B_{2i,\,j}$.

Consider the situation when we know a_{ij}, $i \leq p$. Let $\theta_p(\mu)$, $d_p(\mu)$ be the coefficients of η^p in the expansions of θ, ρ' about $\xi = \xi_1$. Then

$$d_p(\mu) = \sum_j \beta_{pj} P_j(\mu), \tag{3.19}$$

and by the orthogonality of the Legendre polynomials,

$$\beta_{pj} = (j + \tfrac{1}{2}) \int_{-1}^{1} d_p(\mu) P_j(\mu) d\mu . \tag{3.20}$$

We evaluate these integrals by an eleven-point Gauss-Legendre formula (Kopal 1961). Since our integrands are even functions of μ, equation (3.20) reduces to the form

$$\beta_{pj} = (j + \tfrac{1}{2}) \sum_{s=0}^{5} H_s d_p(\mu_s) P_j(\mu_s). \tag{3.21}$$

We find $d_p(\mu_s)$ as follows. The coefficient $\theta_i(\mu_s)$, $i \leq p$, may be found from the known part of the expansion for $\psi(\xi, \mu)$ about $\xi = \xi_1$. In the polytropic case, we raise this series to the nth power by means of the binomial theorem. In the white-dwarf case, we write

$$\theta^2 - D = -D + \sum_i \left(\sum_{r=0}^{i} \theta_r \theta_{i-r} \right) \eta^i , \tag{3.22}$$

and raise the series (3.22) to the power 1.5 by the binomial theorem.

IV. SOLUTION IN THE SURFACE REGION

We refer to the region $\xi_p < \xi < \xi_e$ as region 3. In region 3, ψ, ρ' are not analytic functions of ξ. If $r \geq n$ for the polytropic case, or $r \geq 2$ for the white-dwarf case, $\partial^r \rho' / \partial \xi^r$ is discontinuous across the surface, by equations (2.10), (2.11), (2.28), and (2.29). Clearly therefore, $\partial^r \psi / \partial \xi^r$ is discontinuous across the surface if $r \geq n + 2$ for the polytropic case, or $r \geq 4$ for the white-dwarf case.

We write

$$\psi = \sum_{j=0}^{j=10} \psi_j(\xi) P_j(\mu), \tag{4.1}$$

$$\rho' = \sum_{j=0}^{j=10} \rho_j'(\xi) P_j(\mu). \tag{4.2}$$

Substituting into equation (3.1), we find

$$\xi^2 \frac{d^2}{d\xi^2} \psi_j + 2\xi \frac{d}{d\xi} \psi_j - j(j+1)\psi_j = -\xi^2 \rho_j' . \tag{4.3}$$

The substitutions

$$\psi_j = a_j \xi^{-j-1} , \tag{4.4}$$

$$\frac{d}{d\xi} a_j = b_j \xi^{2j} , \tag{4.5}$$

lead to the equation

$$\frac{d}{d\xi} b_j = -\xi^{1-j} \rho_j' . \tag{4.6}$$

The set of equations (4.5) and (4.6) is integrated from ξ_p to ξ_e by the Runge-Kutta method. The initial values of a_j, b_j are determined from those of ψ_j, $d\psi_j/d\xi$ at $\xi = \xi_p$.

We determine the coefficients ρ_j' by inverting the expansion (4.2). We obtain

$$\rho_j'(\xi) = (j+\tfrac{1}{2})\int_{-1}^{1}\rho'(\xi, \mu)P_j(\mu)\,d\mu\,. \tag{4.7}$$

We suppose that

$$\mu = \mu'(\xi) \tag{4.8}$$

gives the boundary of the configuration. We substitute

$$\mu = \mu' t \tag{4.9}$$

into formula (4.7) to find

$$\rho_j'(\xi) = (j+\tfrac{1}{2})\mu'\int_{-1}^{1}\rho'(\xi, \mu' t)P_j(\mu' t)\,dt\,. \tag{4.10}$$

By the symmetry about the equatorial plane, $\theta(\xi, \mu)$ is an even function of μ. Thus, in the polytropic case,

$$\theta(\xi, \mu) = (1 - t^2)g(t^2)\,, \tag{4.11}$$

and in the white-dwarf case,

$$\theta^2(\xi, \mu) - D = (1 - t^2)g(t^2)\,, \tag{4.12}$$

where $g(1) \neq 0$ in general. Thus, the integrand in formula (4.10) behaves approximately as $(1 - t^2)^n$ in the polytropic case, and $(1 - t^2)^{3/2}$ in the white-dwarf case. We evaluate the integrals (4.10) by the appropriate Mehler-type quadrature formula. A general r-point Mehler formula gives

$$\int_{-1}^{1}(1-t)^u(1+t)^v g(t)\,dt$$

accurately for $g(t)$ a polynomial of degree $(2r - 1)$ in t, if $u, v > -1$. In this case, we could use $u = v = n$ for the polytropic case, and $u = v = 1.5$ for the white-dwarf case. However, we need to evaluate integrals of the form

$$\int_{-1}^{1}(1-t^2)^{n-1}G(t^2)\,dt \qquad \text{or} \qquad \int_{-1}^{1}(1-t^2)^{1/2}G(t^2)\,dt$$

simultaneously with the integrals (4.10), (see section V), so we use $u = v = n - 1$ for the polytropic case, and $u = v = 0.5$ for the white-dwarf case.

In the polytropic case, the computer reads the value of n, and immediately calculates the weights and abscissae for the Mehler formulae. The error is of order 10^{-6}.

It is not possible to integrate equations (4.5) and (4.6) from ξ_p to ξ_e as they stand. By the symmetry about the equatorial plane,

$$\frac{\partial\mu'}{\partial\xi}\rightarrow\infty \qquad \text{as} \qquad \xi\rightarrow\xi_e\,. \tag{4.13}$$

Thus, μ' is not a well-determined function of ξ near ξ_e. We change to μ' as independent variable, and solve the equations

$$\frac{d}{d\mu'}\,a_j = b_j\xi^{2j}\frac{d\xi}{d\mu'}\,, \tag{4.14}$$

$$\frac{d}{d\mu'}\,b_j = -\rho_j'\xi^{1-j}\frac{d\xi}{d\mu'}\,, \tag{4.15}$$

$$\frac{d\xi}{d\mu'} = \left\{\frac{\partial\theta}{\partial\mu}\bigg/\frac{\partial\theta}{\partial\xi}\right\}_{\mu'}\,. \tag{4.16}$$

The Runge-Kutta process has truncation error proportional to h^4 for integration with constant step length h over the interval (1, 0). The value of h required depends on n or D, and A. We describe the procedure for determining h later.

a) Stability of the Integration through Region 3

We are concerned with the stability of equations (4.14) and (4.15) for integration from $\mu' = 1$ to $\mu' = 0$. The term $d\xi/d\mu'$ is negative throughout this range, and thus the problem reduces to that of the stability of equations (4.3) for integration in the direction of ξ increasing. Suppose that, for a particular value of ξ, each $\psi_j(\xi)$ is subject to a small error $\delta_j(\xi)$. Let the consequent errors in μ', ρ', and ρ_j' be $\delta\mu'$, $\delta\rho'$, and $\delta\rho_j'$, respectively. We neglect terms in δ_j^2. In the region $\rho' \neq 0$,

$$\delta\rho' = \rho_*' \sum_k \delta_k P_k(\mu), \tag{4.17}$$

where

$$\rho_*' = \partial\rho'/\partial\theta. \tag{4.18}$$

Thus

$$\delta\rho_j' = (j+\tfrac{1}{2}) \sum_k \delta_k \int_{-\mu'}^{\mu'} \rho_*' P_j P_k d\mu + \text{contribution from the boundary}. \tag{4.19}$$

The last term in equation (4.19) arises from integrating $\delta\rho'$ over a range $\delta\mu'$, and is thus of second order in the δ_j's. Therefore, we obtain

$$\xi^2 \delta\rho_j' = \sum_k f_{jk}\delta_k, \tag{4.20}$$

where

$$f_{jk}(\xi) = (j+\tfrac{1}{2})\xi^2 \int_{-\mu'}^{\mu'} \rho_*' P_j P_k d\mu. \tag{4.21}$$

The functions ψ_j, $(\psi_j + \delta_j)$ both satisfy equation (4.3). Therefore, we obtain

$$\xi^2 \frac{d^2}{d\xi^2}\delta_j + 2\xi \frac{d}{d\xi}\delta_j - j(j+1)\delta_j + \sum_k f_{jk}\delta_k = 0. \tag{4.22}$$

These equations govern the stability of equation (4.3) for integration in the direction of ξ increasing.

One result is easily obtained. By their definition, the $f_{jk}(\xi)$'s are very sensitive to the magnitude of the distortion from spherical symmetry. If this distortion is small enough, we may neglect the f_{jk}'s, and reduce equation (4.22) to the form

$$\xi^2 \frac{d^2}{d\xi^2}\delta_j + 2\xi \frac{d}{d\xi}\delta_j - j(j+1)\delta_j = 0. \tag{4.23}$$

The δ_j's are now mutually independent, and the general expression for any one of them is

$$\delta_j = A_j\xi^j + B_j\xi^{-j-1}. \tag{4.24}$$

Thus the equation (4.3) is unstable for integration in either direction if the distortion is small.

We now consider the case of rapid rotation. The behavior of equation (4.3) depends critically on the functions $f_{jk}(\xi)$. As an example, we consider the polytropic case for $n = 1$. For the non-rotating polytrope, $\xi_p = \xi_e = 3.14$. If we take $\xi \approx 3.14$, we obtain $\xi^2 \approx 10$. For $n = 1$, $\rho_*' = n\theta^{n-1} = 1$. Thus, the $f_{jk}(\xi)$ calculated from equation (4.21) is comparable with the other coefficients in equation (4.22). This opens the possibility that, for rapid rotation, the equation (4.3) may be stable for integration in the direction of ξ increasing. This stability would not obtain throughout the entire range of integration, as the f_{jk}'s must always be small near $\mu' = 0$. However, for most of the range of integration, the equation (4.3) may well be stable for integration in the direction of ξ increasing.

It would be very difficult to establish the stability of the equation (4.3) analytically. However, it is not necessary to do this. In Section V, we describe the Newton-Raphson process used to determine the coefficients A_{ii}. An instability in the equation (4.3) would lead to failure of this process. Thus the stability of equation (4.3) may be verified a posteriori. However, we would expect that whatever happened in the case of fast rotation, the equation (4.3) would be unstable for slow rotation. We return to this matter in Section XI.

V. DETERMINATION OF THE COEFFICIENTS A_{ii}

We define

$$R_j = \left(\frac{d}{d\xi}\psi_j + \frac{j+1}{\xi}\psi_j\right)_{\xi=\xi_e}. \tag{5.1}$$

The boundary conditions (2.33) give

$$R_j = 0\,, \qquad j = 2(2)10\,, \tag{5.2}$$

a set of non-linear equations for A_{ii}, $i = 2(2)10$. We solve equation (5.1) by the multidimensional generalization of the Newton-Raphson process (Haselgrove 1961). The computer begins with a trial set of coefficients A_{ii}, and evaluates R_j, $\partial R_j/\partial A_{ii}$, for $i, j = 2(2)10$. The linear equations

$$\sum_{i=0}^{i=10} (\partial R_j/\partial A_{ii})\,\delta A_{ii} + R_j = 0, \qquad j = 2(2)10, \tag{5.3}$$

give first order corrections δA_{ii} to A_{ii}. We replace A_{ii} by $(A_{ii} + \delta A_{ii})$ and repeat the process. The machine iterates until

$$\sum_j R_j^2 < 2^{-36} \approx 10^{-11}. \tag{5.4}$$

We attach the superscripts q, 0 to a quantity to denote differentiation with respect to A_{qq}. The second superscript is needed in Section IX. We differentiate with respect to A_{qq} the equation (5.1) to obtain

$$R_j^{q0} = \left(\frac{d}{d\xi}\psi_j^{q0} + \frac{j+1}{\xi}\psi_j^{q0}\right)_{\xi=\xi_e}. \tag{5.5}$$

a) Determination of ψ^{q0} in Region 1

We differentiate with respect to A_{qq} the equations (3.2), (3.3), (3.6), and (3.7) to obtain

$$\psi^{q0} = \sum_{i=0}^{i=20} \sum_{j=0}^{j=10} A_{ij}^{q0}\xi^i P_j(\mu)\,, \tag{5.6}$$

$$\rho'^{q0} = \sum_i \sum_{j=0}^{j=10} B_{ij}^{q0}\xi^i P_j(\mu)\,, \tag{5.7}$$

$$A_{ij}^{q0} = 0 \text{ if } i \text{ or } j \text{ odd, or } i < j\,, \tag{5.8}$$

$$[i(i+1) - j(j+1)]\,A_{ij}^{q0} = -B^{q0}{}_{i-2\ j}\,. \tag{5.9}$$

We have also

$$\rho'^{q0} = \rho_*'^{q0}\psi^{q0} \qquad \text{if} \qquad \rho \neq 0\,, \tag{5.10}$$

$$= 0 \qquad \text{otherwise}\,, \tag{5.11}$$

and

$$A_{ii}^{q0} = \delta_{iq}\,. \tag{5.12}$$

Equations (5.5)–(5.12) determine ψ^{q0}, $q = 2(2)10$, throughout region 1. (The determination of B_{ij}^{q0} is discussed later.) The relative truncation error for the expansion (5.6) at $\xi = \xi_0$ may be larger than that for the expansion (3.2). However, this matters only if the resulting errors in R_j^{q0} prevent the convergence of the Newton-Raphson process. In practice the process does converge.

b) Determination of ψ^{q0} in Region 2

We use the expansion (5.6) to give ψ_j^{q0} at $\xi = \xi_0$. We calculate the analytic continuation of ψ^{q0} from the following equations:

$$\psi^{q0} = \sum_{i=0}^{i=10} \sum_{j=0}^{j=10} \alpha_{ij}^{q0} \eta^i P_j(\mu)\,, \tag{5.13}$$

$$\rho'^{q0} = \sum_{i} \sum_{j=0}^{j=10} \beta_{ij}^{q0} \eta^i P_j(\mu)\,, \tag{5.14}$$

$$\begin{aligned}\xi_1^2 i(i-1)\,\alpha_{ij}^{q0} = &-[(i-1)(i-2) - j(j+1)]\,\alpha^{q0}{}_{i-2\ j} - \beta^{q0}{}_{i-4\ j} \\ &- 2\xi_1[(i-1)^2\alpha^{q0}{}_{i-1,\ j} + \beta^{q0}{}_{i-3\ j}] - \xi_1^2\beta^{q0}{}_{i-2\ j}\,,\end{aligned} \tag{5.15}$$

$$\alpha_{0j}^{q0} = \psi_j^{q0}(\xi_1)\,, \tag{5.16}$$

$$\alpha_{1j}^{q0} = (d\psi_j^{q0}/d\xi)_{\xi=\xi_1}\,. \tag{5.17}$$

These equations are obtained by differentiating with respect to A_{qq} the equations (3.11)–(3.15).

c) Determination of B_{ij}^{q0}, β_{ij}^{q0}

We work with expansions (3.11) and (5.13). Formula (3.21) becomes

$$\beta_{pj}^{q0} = (j + \tfrac{1}{2}) \sum_{s=0}^{5} H_s \rho_p'^{q0}(\mu_s) P_j(\mu_s)\,. \tag{5.18}$$

We use the binomial theorem and the rules for the multiplication of power series to construct the series expansion for ρ_*' about $\xi = \xi_1$. We multiply this series by the known part of the series for $\psi^{q0}(\xi, \mu_s)$, to obtain the series for ρ'^{q0}.

d) Determination of ψ^{q0} in Region 3

We differentiate with respect to A_{qq} the equations (4.1), (4.2), (4.4), (4.7), (4.14), and (4.15):

$$\psi^{q0} = \sum_{j=0}^{j=10} \psi_j^{q0}(\xi) P_j(\mu)\,, \tag{5.19}$$

$$\rho'^{q0} = \sum_{j=0}^{j=10} \rho_j'^{q0}(\xi) P_j(\mu), \tag{5.20}$$

$$\psi_j^{q0} = a_j^{q0} \xi^{-j-1}, \tag{5.21}$$

$$\rho_j'^{q0}(\xi) = (j+\tfrac{1}{2}) \mu' \int_{-1}^{1} \rho'^{q0}(\xi, \mu' t) P_j(\mu' t)\, dt, \tag{5.22}$$

$$\frac{d}{d\mu'} a_j^{q0} = b_j^{q0} \xi^{2j} \frac{d\xi}{d\mu'}, \tag{5.23}$$

$$\frac{d}{d\mu'} b_j^{q0} = -\rho_j'^{q0} \xi^{1-j} \frac{d\xi}{d\mu'}. \tag{5.24}$$

We integrate equations (5.23) and (5.24) from $\mu' = 1$ to $\mu' = 0$ simultaneously with equations (4.14)–(4.16). Equations (5.10) and (5.19–(5.22) determine $\rho_j'^{q0}$. The Mehler formulae used give accurate results for integrals of the form (5.22).

VI. MISCELLANEOUS FUNCTIONS OF THE SOLUTION

For each pair of values of n, A or D, A, the computer prints the polar and equatorial radii, the equatorial gravity g_e, the mass M and volume V of the configuration, and also the moments of inertia, C about the axis of rotation, and B about a perpendicular axis through the center of mass. The units are shown in Table 1. The computer also prints A_{ii}, R_j, $\psi_j(\xi_e)$, and $(d\psi_j/d\xi)_{\xi_e}$.

TABLE 1

UNITS FOR PHYSICAL QUANTITIES

Quantity	Units in Polytropic Case	Units in White-Dwarf Case
Polar radius, Equatorial radius	$[(n+1)K/4\pi G\rho_c^{1-1/n}]^{1/2}$	$(2a/\pi G b^2 y_c^2)^{1/2}$
Equatorial gravity	$[4\pi G(n+1)K\rho_c^{1+1/n}]^{1/2}$	$4y_c^2\sqrt{(2a\pi G)}$
Mass	$4\pi[(n+1)K/4\pi G]^{3/2}\rho_c^{(3-n)/2n}$	$4\pi(2a/\pi G)^{3/2}b^{-2}$
Volume	$[(n+1)K/4\pi G\rho_c^{1-1/n}]^{3/2}$	$(2a/\pi G b^2 y_c^2)^{3/2}$
Moments of inertia	$[(n+1)K/4\pi G]^{5/2}\rho_c^{(5-3n)/2n}$	$\frac{8}{3}\pi(2a/\pi G)^{5/2} b^{-4} y_c^{-3}$

Of these quantities, ξ_p, ξ_e, ψ_j, $d\psi_j/d\xi$, A_{ii}, and R_j are immediately available. The calculation of g_e is trivial. We find the mass and volume from

$$M = -\xi_e^2 (d\psi_0/d\xi)_{\xi_e}, \tag{6.1}$$

$$V = \frac{2\pi}{3} \int_{-1}^{1} \xi^3(\mu')\, d\mu. \tag{6.2}$$

The moments of inertia are

$$B = \tfrac{1}{2} \int \rho' \xi^4 (1+\mu^2)\, d\xi\, d\mu\, d\phi \tag{6.3}$$

and

$$C = \int \rho' \xi^4 (1-\mu^2)\, d\xi\, d\mu\, d\phi, \tag{6.4}$$

where the integrations are carried out over all the volume occupied by the material.

These expressions reduce to

$$B=\frac{4\pi}{3}\left[\int_0^{\xi_e}\rho_0'\xi^4 d\xi+\tfrac{1}{10}\int_0^{\xi_e}\rho_2'\xi^4 d\xi\right], \tag{6.5}$$

$$C=\frac{4\pi}{3}\left[\int_0^{\xi_e}\rho_0'\xi^4 d\xi-\tfrac{1}{5}\int_0^{\xi_e}\rho_2'\xi^4 d\xi\right]. \tag{6.6}$$

Clearly

$$\int_0^{\xi_e}\rho_0'\xi^4 d\xi=2\xi_e^3\psi_0(\xi_e)-\xi_e^4\left(\frac{d}{d\xi}\psi_0\right)_{\xi_e}-6I, \tag{6.7}$$

$$\int_0^{\xi_e}\rho_2'\xi^4 d\xi=2\xi_e^3\psi_2(\xi_e)-\xi_e^4\left(\frac{d}{d\xi}\psi_2\right)_{\xi_e}, \tag{6.8}$$

where

$$I=\int_0^{\xi_e}\xi^2\psi_0 d\xi. \tag{6.9}$$

I is calculated directly from the solution for ψ.

VII. GENERAL STRATEGY

Within certain ranges, each pair of values of n, A defines a unique axially symmetric equilibrium configuration. The determination of this configuration is effectively the determination of A_{ii}, $i = 2(2)10$. The computing time needed for this is greatly reduced by starting with a good guess for the A_{ii}'s.

For each value of n, or D, we construct solutions in order of increasing A. The machine estimates the A_{ii}'s for each new value of A by extrapolating from the known solutions. It refines the estimates to obtain a new solution. If more than five solutions are known, only those for the five largest values of A are used. To start the process, we take $A_{ii} = A = 0$ as one solution, and guess the A_{ii}'s for a small value of A. These values are refined as in Section V, and the process continues as above. The values of A for which solutions are obtained are read from a data tape. The program includes facilities for reading a set of solutions for subsequent use in extrapolation—this permits of restarting the calculation of a sequence if it is interrupted. The construction of a sequence is stopped manually near the end. The end is defined by the equation

$$g_e=0. \tag{7.1}$$

In general, a coarse tabulation is used for most of each series, together with a fine tabulation near the end. The fine tabulation permits extrapolation for terminal values of physical parameters.

For any particular sequence of solutions, the number of Runge-Kutta steps for the integration through region 3 is an increasing function of A. We denote the number required by m. Initially, we set $m = 10$. After finding each solution, we check the adequacy of m by repeating the integration through region 3 with $(m + 10)$ steps. If condition (5.4) ceases to hold, m is increased by 10, and the approximate solution is refined again. However, m is not reset to 10 when the A_{ii}'s are extrapolated for the next value of A. The number of Runge-Kutta steps used is printed at the end of the calculation. Facilities are provided for setting m when restarting in the middle of a sequence.

VIII. SPECIFICATION OF NON-AXIALLY SYMMETRIC FORMS

The structure of a general equilibrium form is governed by the equations (2.10), (2.11), or (2.28), (2.29) and (2.14), (2.15). Near the origin, we assume the expansions

$$\psi=\sum_{i,j,k}\xi^i P_j^k(\mu)\left[A_{ijk}\cos k\phi+C_{ijk}\sin k\phi\right], \tag{8.1}$$

$$\rho' = \sum_{i,j,k} \xi^i P_j^k(\mu)\,[B_{ijk}\cos k\phi + D_{ijk}\sin k\phi]\,, \tag{8.2}$$

where the $P_j^k(\mu)$'s are the associated Legendre functions. Substitution into equation (2.14) yields

$$[i(i+1) - j(j+1)]\,A_{ijk} = -B_{i-2\ jk}\,, \tag{8.3}$$

$$[i(i+1) - j(j+1)]\,C_{ijk} = -D_{i-2\ jk}\,. \tag{8.4}$$

It is easy to prove that A_{ijk} and C_{ijk} vanish if $i < j$. If $j < k$, $P_j^k(\mu) \equiv 0$. By the symmetry about the equatorial plane, A_{ijk} and C_{ijk} vanish if $(j - k)$ is odd. Thus, we can write

$$A_{ijk} = C_{ijk} = 0\,, \qquad i < j\,, j < k\,, \text{ or } (j-k) \text{ odd}\,. \tag{8.5}$$

In region 1, the potential is determined by equations (8.3) and (8.4) and the coefficients A_{iik}, C_{iik} for which $(i - k)$ is even and positive.

The theory of preceding sections is easily generalized to cover non-axially symmetric equilibrium forms. However, the Mercury computer is not powerful enough to permit the investigation of the general case. We are restricted to the case where the non-axially symmetric term in the potential is small, and may be treated as a small perturbation.

IX. PERTURBATION THEORY FOR NON-AXIALLY SYMMETRIC FORMS

We consider a pair of values of n, A or D, A such that an axially symmetric equilibrium form exists, together with an adjacent non-symmetric equilibrium form. We attach a suffix a to any quantity to show that it describes the axially symmetric form. We write the potential ψ for the non-symmetric form as

$$\psi = \psi_a + \delta\psi \tag{9.1}$$

and assume that we may neglect $(\delta\psi)^2$. For the non-symmetric form,

$$\theta = \theta_a + \delta\psi\,, \tag{9.2}$$

so that

$$\rho' = \rho_a' + \rho_*'\delta\psi \tag{9.3}$$

to first order in $\delta\psi$.

If we substitute expressions (9.1) and (9.3) into equation (2.14), the terms in ψ_a, ρ_a' cancel, and we are left with a linear homogeneous equation for $\delta\psi$. We expand

$$\delta\psi = \sum_{k=0} (\psi_{.k1}\cos k\phi + \psi_{.k2}\sin k\phi) \tag{9.4}$$

and obtain the system of equations

$$\xi^{-2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial}{\partial\xi}\psi_{.k}\right) + \xi^{-2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}\psi_{.k}\right] + k^2\xi^{-2}(1-\mu^2)^{-1}\psi_{.k} = -\rho_*'\psi_{.k}\,, \tag{9.5}$$

where $\psi_{.k}$ stands for either $\psi_{.k1}$ or $\psi_{.k2}$. From the linearity of equation (9.5), we see that $\psi_{.k}$ is a linear function of the A_{iik}'s. The equations (9.5) are mutually independent, and we may consider each value of k separately.

The boundary conditions (2.30) and (2.31) show that

$$\psi_{.k} = \partial\psi_{.k}/\partial\xi = 0 \qquad \text{at} \qquad \xi = 0\,. \tag{9.6}$$

We expand

$$\psi_{.k} = \sum_j \psi_{jk} P^k(\mu) \tag{9.7}$$

to obtain

$$\xi^{-2} \frac{d}{d\xi}\left(\xi^{2} \frac{d}{d\xi} \psi_{jk}\right) - \frac{j(j+1)}{\xi^{2}} \psi_{jk} = -\rho_{*}'\psi_{jk} . \tag{9.8}$$

The functions ψ_{jk} satisfy boundary conditions (2.33) at $\xi = \xi_e$. We define

$$R_{jk} = \left(\frac{d}{d\xi} \psi_{jk} + \frac{j+1}{\xi} \psi_{jk}\right)_{\xi=\xi_e} . \tag{9.9}$$

The residuals R_{jk} are linear functions of the coefficients A_{iik} for $i = k(2)$. Thus,

$$R_{jk} = \sum_{q} (\partial R_{jk}/\partial A_{qqk}) A_{qqk} \tag{9.10}$$

$$= \sum_{q} R_{jk}{}^{qk} A_{qqk} , \tag{9.11}$$

where the superscripts q, k denote differentiation with respect to A_{qqk}. This notation is a generalization of the notation introduced in Section V. The necessary and sufficient condition for the existence of a non-axially symmetric equilibrium form, adjacent to the symmetric form, is that there exists a set of coefficients A_{qqk}, not all zero, such that

$$R_{jk} = 0 , \qquad \text{all } j,\ k . \tag{9.12}$$

Thus, we require the singularity of one or more of the matrices R^k with $R_{jk}{}^{qk}$ as the element in row $(q - k + 2)/2$, column $(j - k + 2)/2$, when $k \neq 0$, and row $q/2$, column $j/2$ when $k = 0$.

The matrix elements $R_{j0}{}^{q0}$ are the $R_j{}^{q0}$'s of Section V. Essentially the same computer programs are used to compute $R_{jk}{}^{qk}$, $k \neq 0$, and $R_{j0}{}^{q0}$. The main modification required is the addition of certain subroutines to the program. These replace $P_j(\mu_s)$ by $P_j{}^k(\mu_s)$ before computing expansions for $\psi_{.k}{}^{qk}$ in regions 1 and 2, or finding ρ'^{qk} in region 3. The $P_j{}^k(\mu_s)$'s are renormalized so that

$$\int_{-1}^{1} P_u^k(\mu) P_v^k(\mu) d\mu = \frac{2}{2\mu + 1} \delta_{uv} . \tag{9.13}$$

This simplifies the integration subroutines. With this convention, we may obtain the equations governing the calculation of $R_{jk}{}^{qk}$ from equations (5.6)–(5.24) by the obvious changes in notation, with the exception of equation (5.12), which becomes

$$A_{iik}{}^{qk'} = \delta_{iq}\delta_{kk}' . \tag{9.14}$$

The rank of the matrix R^k could be determined by finding its eigenvalues. However, elements of R^k can be of order 10^{10}, which introduces difficulties in the calculation of the eigenvalues. Therefore, the rows of R^k are first normalized to make the diagonal element equal to unity. We denote the normalized matrix by N^k. Thus, to locate a point of bifurcation, we need only find a configuration for which a matrix N^k has a zero eigenvalue.

The methods described above cannot be extended to include terms of the second order in $\delta\psi$ for the polytropic case. This would involve computing integrals of the form

$$\int_{-\mu'}^{\mu'} \theta_a^{n-2} \psi_{.k}{}^{ik} d\mu \tag{9.15}$$

in region 3. If $n < 1$, these integrals do not exist. In fact, bifurcation forms are not found on the axially symmetric series for $n > 1$. Thus, we cannot investigate the variation of physical parameters along the branch series of non-symmetric forms to the second order in $\delta\psi$. The investigation of the secular stability of these forms requires a knowledge of this variation. Thus, we cannot investigate the secular stability of non-axially symmetric equilibrium polytropes with the facilities available.

X. NUMERICAL CHECKS

The program includes an optional facility for checking the solutions obtained. The checks verify the adequacy of the integration formulae used in regions 1 and 2, and also give some security against program errors.

In regions 1 and 2, we check the expansions for ψ, $\psi._k{}^{qk}$ by back substitution for all $\xi = 0.1r$, r a positive integer, such that $0 < \xi < \xi_p$. When the computer tests ψ, it calculates ψ_j, $\xi^{-2}\, d(\xi^2 d\psi_j/d\xi)/d\xi$ directly from the coefficients in the expansion. From ψ_j' through equations (2.15) and (2.10) or (2.28), it calculates $\rho'(\xi, \mu_s)$ and evaluates ρ_j, from the formula

$$\rho_j' = (j + \tfrac{1}{2}) \sum_{s=0}^{5} H_s \rho'(\xi, \mu_s) P_j(\mu_s)\,. \tag{10.1}$$

The integration formula is the same as is used for computing the expansion for ψ. The computer stores for each j the maximum value of the residual on substituting for ψ_j, $\xi^{-2}\, d(\xi^2 d\psi_j/d\xi)/d\xi$, ρ_j' in equation (3.1). The adequacy of the integration formula is checked indirectly. For $\mu = 0.2(0.2)1$, the computer determines $\rho'(\xi, \mu)$ from the coefficients ρ_j', and also from the coefficients ψ_j. The machine records for each value of μ the maximum deviation for all the values of ξ tested. Similar checks were performed when required on the expansions for $\psi._k{}^{qk}$.

In region 3, the computer checks that certain functions of ξ are constant. Consider the integral

$$I_j = \int_0^{\xi} \xi'^{j+2} \rho_j' d\xi'\,. \tag{10.2}$$

We substitute for ρ_j' from equation (4.3):

$$I_j = j\xi^{j+1}\psi_j - \xi^{j+2} \frac{d}{d\xi} \psi_j\,. \tag{10.3}$$

The computer calculates $I_j(\xi_p)$ by substituting $\xi = \xi_p$ in equation (10.3). It integrates from $\mu' = 1$ to $\mu' = 0$ the differential equations

$$\frac{d}{d\mu'} I_j = \rho_j' \xi^{j+2} \frac{d\xi}{d\mu'} \tag{10.4}$$

simultaneously with equations (4.14)–(4.16). At $\mu' = 0$, $\xi = \xi_e$, the computer substitutes I_j, ψ_j, $d\psi_j/d\xi$ into equations (10.3), and stores the residuals. No attempt is made to check the accuracy of the integration formulae in region 3.

In general, the machine takes as long to check a solution as it takes to obtain it in the first place. Thus, only two or three solutions for ψ are checked for each value of n, or D. The functions $\psi._k{}^{qk}$ are checked only in cases of particular interest.

XI. GENERAL PROPERTIES OF ROTATING CONFIGURATIONS

For a non-rotating polytrope, the mass (in units of $\alpha[n+1]K\rho_c{}^{1/n}$) is a decreasing function, and the dimensionless radius an increasing function of n. Thus, for fixed A, the importance of the rotational part of the total potential θ increases with n. The maximum value of A on the series decreases with n. The central condensation of a polytrope increases with n, and the distortion of the central regions is always small. Thus, for fixed n, the non-spherically symmetric terms in the gravitational potential ψ diminish as n increases. The μ-wise expansion of ψ near the end of a series converges most rapidly for large n.

For values of $n \leq 3$, the Newton-Raphson process of Section V proved satisfactory. For $n > 3$, however, the process failed to produce solutions. This phenomenon is explained by the analysis of Section IV. It is clear that the last term on the left-hand side of equation (4.22) serves to stabilize the integration for $n \leq 3$, but that for larger values of n, these terms are not sufficiently large for the purpose. Fortunately, however, bifurcation forms are not found when n is as large as 3, and so the region $n > 3$ is not of primary interest in this investigation. Series of equilibrium forms were constructed for $n = 1(0.5)3$, and also for some values of n in the range (0.5, 1). To facilitate extrapolation of the terminal values of physical quantities, a fine tabulation was made near the end of each series. The proximity of the end of the series led to some difficulty in obtaining solutions, as $d\theta/d\xi$ is small near $\xi = \xi_e$, and small errors in the potential led to larger errors in $d\xi/d\mu'$. However, an adequate tabulation was obtained in all cases except $n = 3$. In this case, the instability due to the integration through region 3, combined with proximity to the end of the series, rendered a satisfactory tabulation impractical. The numerical checks on ψ described in Section X were applied to two configurations on each polytropic series. The results proved satisfactory in all cases.

Series of white-dwarf configurations were obtained for $D = 0.025$, 0.25, 0.1, and 0.2(0.2)0.8. The Newton-Raphson process of Section V was satisfactory in all these cases. Numerical checks were applied to a few configurations only, as computing time for more thorough checking was not available. The results of the checks performed were satisfactory, and served to verify the accuracy of the program for the white-dwarf case.

Appendix Table 2 gives the variation of physical properties for the axially symmetric equilibrium configuration. For $A = 0$ in the polytropic case, Chandrasekhar (1933) gives the derivatives with respect to A of several physical quantities. The same derivatives are obtained by numerical differentiation of the values in Appendix Table 2, and a comparison is made in Appendix Table 3. In Appendix Table 4 we give the physical properties for the last configuration of each series. These values were obtained by numerical extrapolation of the values given in Appendix Table 2. The number of figures given in each case reflected the number considered meaningful. The missing values for the polytrope $n = 3$ are those considered untrustworthy. Figures 1 and 2 show the variation with n and D, respectively, of the terminal values of A, the oblateness $\sigma = (\xi_e - \xi_p)/\xi_e$, and of $(C - B)/C$. It will be noted that the terminal value of σ is not sensitive to variation in n or D.

a) *Estimation of the Truncation Error*

When each solution of equation (2.14) is obtained, the computer prints $\psi_j(\xi_e)$, $(d\psi_j/d\xi)_{\xi_e}$ as part of the information for this case. Inspection of these coefficients gives an estimate of the truncation error due to the truncation of the μ-wise expansion of ψ. Where possible, this estimate was confirmed by the checking facilities of Section X. In Figure 3, we summarize the behavior of the truncation error near the end of a series, considered as a function of n or D. The truncation error in g_e is approximately that in ψ. In all cases, it exceeds the random error in g_e, which is of order 10^{-6}. The truncation error in other physical quantities is easily obtained from that in g_e. The number of significant figures given in Appendix Table 2 reflects the magnitude of the truncation error—the last figure given is subject to error but is never without meaning.

The figures given in Appendix Table 2 were checked by differencing wherever possible. This check proved satisfactory in the polytropic case. The tabulation was much coarser in the white-dwarf case, and differencing did not give so reliable a check on the values of Appendix Table 2. Computing time was not available to rectify this situation.

XII. THE EXISTENCE OF BIFURCATION FORMS

For the Roche model, the series of axially symmetric equilibrium forms has no point of bifurcation. The series terminates when the surface gravity becomes zero at the equator. The angular velocity increases monotonically along the series. We expect a similar behavior for a polytrope with $n \approx 5$, with A increasing monotonically along the series. For the homogeneous case, corresponding to the polytrope $n = 0$, a bifurcation point exists. The series does not terminate as above. The angular momentum H increases monotonically along the series, and can be arbitrarily large. A has a maximum on the series, and $\rightarrow 0$ as $H \rightarrow 0$ or $H \rightarrow \infty$. For other values of n, we do not know a priori if a series of polytropes terminates. However, all the polytropic series constructed for

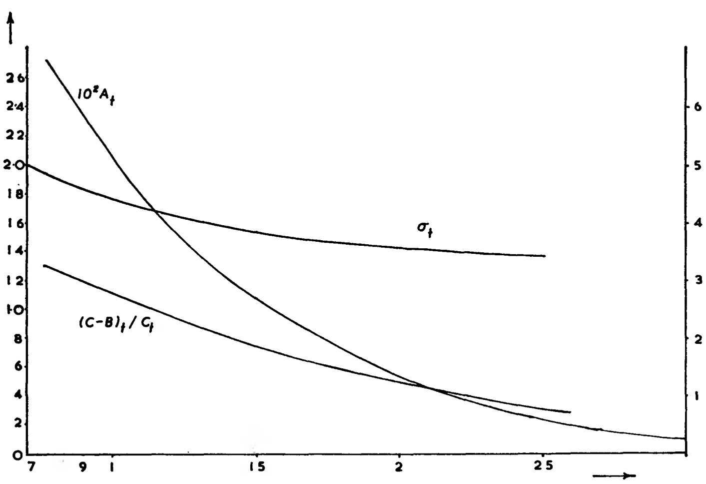

FIG. 1.—Variation of A_t, σ_t, $([C - B]/C)_t$ with n in the polytropic case. Abscissa: n; ordinate (left-hand): $10^2 A_t$; ordinate (right-hand): σ_t and $([C - B]/C)_t$.

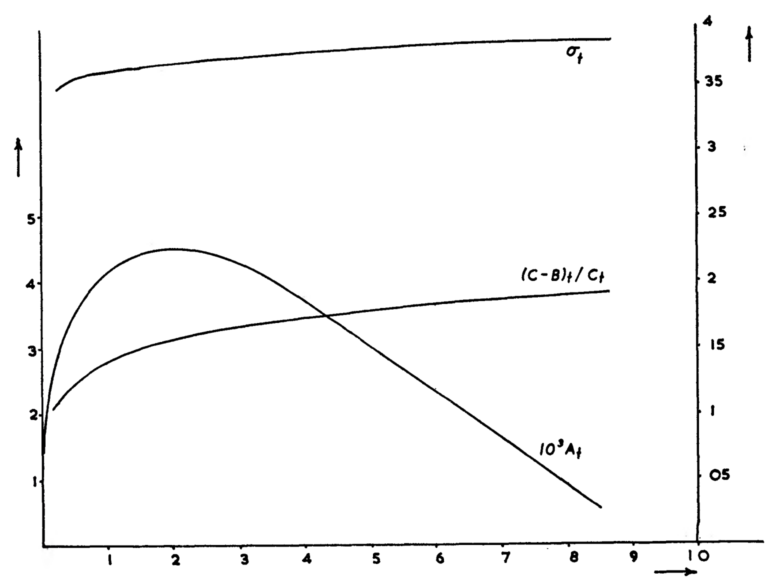

FIG. 2.—Variation of A_t, σ_t, $([C - B]/C)_t$ with D in the white-dwarf case. Abscissa: D; ordinate (left-hand): $10^3 A_t$; ordinate (right-hand): σ_t and $([C - B]/C)_t$.

$n \geq 0.7$ were found to terminate with $g_e = 0$ for the last configuration, as were all the white-dwarf series. This made possible the strategy of Section VII and greatly simplified the search for points of bifurcation. The series constructed for $n < 0.7$ were not taken far enough to show whether or not they terminated in the same way.

Comparing the cases $n = 0$ and $n \approx 5$, it seems likely that there exists a value n_c of n, such that bifurcation forms exist if and only if $0 \leq n \leq n_c$. If $n_c > 1.5$, we would expect a corresponding value D_c of D, such that bifurcation forms exist for $1 \geq D \geq D_c$.

We consider first the polytropic case. The matrices N^k were obtained in the cases $k = 0$, $k = 2$ only. These cases give perturbing potentials containing second-order spherical harmonics. For other cases, the spherical harmonics in the potentials are all of order ≥ 3. (In the case $k = 1$, the term $\psi_{11}P_1^1[\mu] \cos \phi$ is ruled out by the boundary

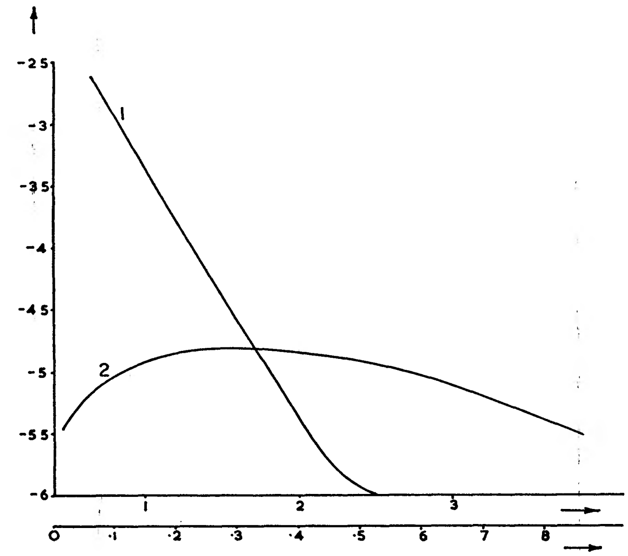

FIG. 3 —Truncation errors near A_t for the polytropic and white-dwarf cases. Curve 1: polytropic case; Curve 2: white-dwarf case. Abscissa (upper): n; abscissa (lower): D; ordinate: $\log_{10}$ (*error*).

conditions [9.6].) In the homogeneous case, the first bifurcation point of the axially symmetric series corresponds to the second-order harmonic. The higher the order of a given harmonic, the farther along the series is the corresponding bifurcation point. It seems likely that a similar behavior occurs in the polytropic case. It would have been desirable to verify this conjecture, but the computing time available was not adequate for this purpose.

At $A = 0$, the eigenvalues of N^k are all equal to unity. This follows from equations (9.8). If $A = 0$, ρ_*' depends on ξ only, and the equations are mutually independent. Thus R^k is a diagonal matrix and N^k a unit matrix. As A increases, the eigenvalues of N^k spread out. If n is small enough, the smallest eigenvalue reaches zero before the end of the series. It is found that the eigenvalues of N^2 spread out more rapidly than those of N^0. In Figure 4, we show the variation of the eigenvalues of N^2 in the case $n = 0.7$. This behavior is broadly similar to that for other values of n. We take $E_p(n, A)$ as being numerically the smallest of the eigenvalues of the matrices N_k. For a bifurcation point,

$$E_p(n, A) = 0 . \tag{12.1}$$

If this point is at the end of a series, $A = A_t$ say,

$$g_e = 0 . \tag{12.2}$$

We solve equations (12.1) and (12.2) for A and n to find the critical bifurcation form.

Obviously, we cannot construct equilibrium forms at or beyond the end of a series. Therefore, we cannot use the Newton-Raphson process to solve equations (12.1) and (12.2) for n and A. The equations were solved as follows. First the series of equilibrium configurations for $n = 1$ was examined. It proved that $E_p(1, A)$ is positive everywhere on the series. Series of equilibrium forms were constructed for $n = 0.6$, 0.7, and 0.8. These series proved to have bifurcation forms. Thus

$$0.8 < n_c < 1 . \tag{12.3}$$

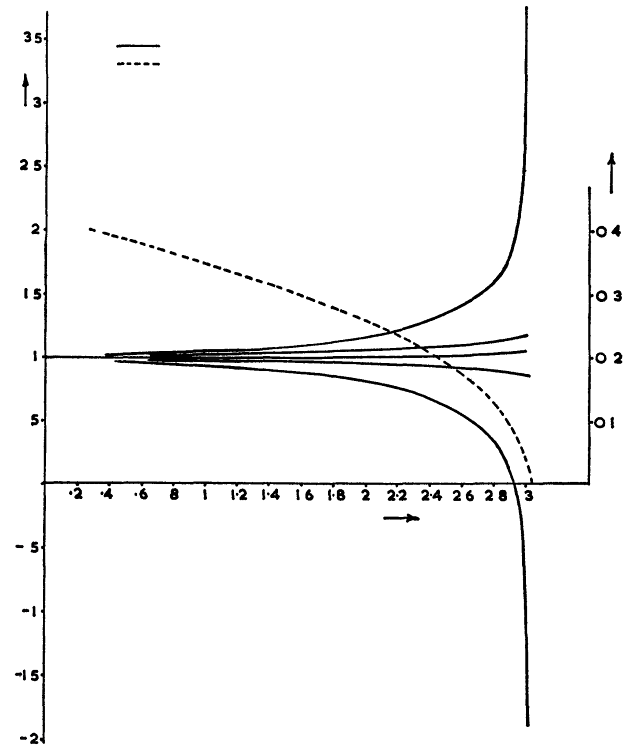

FIG. 4.—Eigenvalues of N^2 for $n = 0.7$. Abscissa: $10^2 A$; ordinate (left-hand): eigenvalue; ordinate (right-hand): g_e; solid line: eigenvalues; broken line: g_e.

Here n was taken as a function of g_e for the bifurcation forms, and Aitken's (1932) process used to extrapolate n for $g_e = 0$. For values of n near this estimated n_c, series of equilibrium configurations were constructed, and $E_p(n, A_t)$ obtained by numerical extrapolation. Inverse interpolation between these values led to the result

$$n_c = 0.808 . \tag{12.4}$$

This result was verified by constructing series for $n = 0.808$ and 0.8085. These series showed that

$$0.808 < n_c < 0.8085 . \tag{12.5}$$

Figure 5 shows the variation of g_e with n for bifurcation forms, where these exist. Figure 6 shows the variation of $E_p(n, A)$ as a function of A/A_t for other values of n. Clearly the series for $n = 1(0.5)2.5$ lacks points of bifurcation. Also, if A/A_t is constant and ≤ 1, $E_p(n, A)$ is an increasing function of n in the range (0.8, 2.5). We conclude that no bifurcation forms exist on the axially symmetric series if $n > 0.808$.

As $n_c < 1.5$, we do not expect to find points of bifurcation on axially symmetric equilibrium series for the white-dwarf case. It was not possible to explore this case as thoroughly as the polytropic case, because of shortage of computing time. However, very rough estimates for the minimum eigenvalues, denoted by $E_d(D, A_t)$ in this case, were obtained for some series. In Figure 7, we see the variation of n as a function of $E_p(n, A_t)$, and also of D as a function of $E_d(D, A_t)$. The scale in D has been chosen so that $D = 0$ corresponds to $n = 3$, and $D = 1$ to $n = 1.5$. We can easily show that $E_d(D, A_t) \rightarrow E_p(1.5, A_t)$ as $D \rightarrow 1$. Thus the two curves intersect at $n = 1.5$, $D = 1$. They appear to diverge as n increases, but this is a spurious effect. If the effective polytropic index were used as the ordinate in the white-dwarf curve, the scale of D would be compressed toward $D = 1$, and the two curves would be in better agreement.

In the homogeneous case, the instability after the point of bifurcation enters through a term in $P_2^2(\mu) \cos 2\phi$. Jeans (1919) found this term dominant in the polytropic case also. Near a point of bifurcation, the coefficient R_{22}^{22} becomes very small—this is essen-

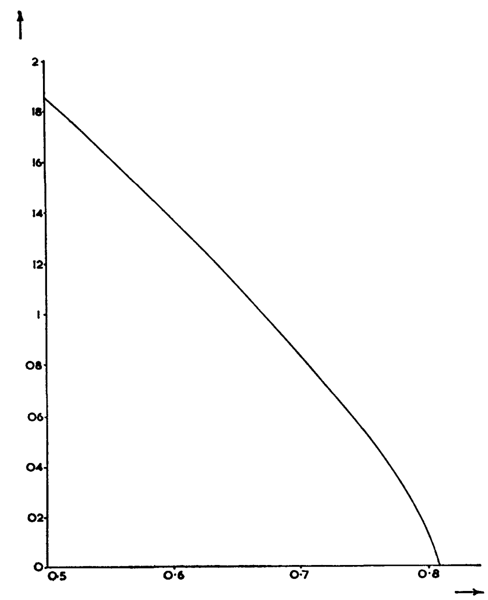

FIG. 5.—Variation of g_e with n for bifurcation points in the polytropic case. Abscissa: n; ordinate: g_e

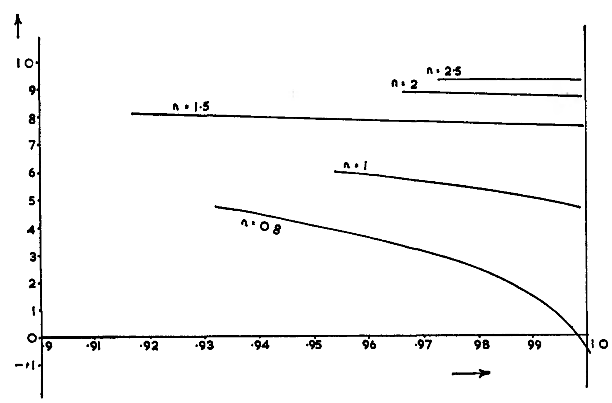

FIG. 6.—Variation of $E(n, A)$ with A/A_t for 5 values of n. Abscissa: A/A_t; ordinate: $E(n, A)$

tially the reason for E_p vanishing. Thus, the term in $P_2^2(\mu)\cos 2\phi$ in the potential is the dominant term in the perturbation giving rise to instability of the axially symmetric form. This result is consistent with Jeans's work, and with the corresponding results for the homogeneous case, and the generalized Roche model.

a) Estimation of the Truncation Error in n_c

The truncation error in $E_p(n, A_t)$ arises from three sources:

i) The truncation error in ψ. This contributes a relative error of ± 0.001 in $R_{jk}{}^{ik}$.

ii) The truncation error in the μ-wise expansion for $\psi_{.2}$. The relative error in $R_{jk}{}^{ik}$ from this source is about ± 0.005, so we may neglect the truncation error from source (i). Random errors are negligible in comparison with these truncation errors.

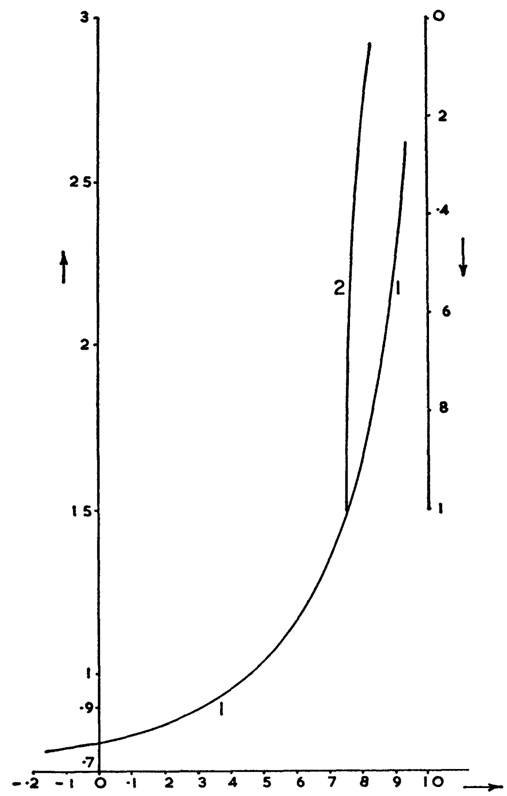

FIG. 7.—Variation of n with $E_p(n, A_t)$ and D with $D_d(D, A_t)$. Curve 1: polytropic case; Curve 2: white-dwarf case. Abscissa: $E_p(n, A_t)$ and $E_d(n, A_t)$; ordinate (left-hand): n; ordinate (right-hand): D.

iii) The truncation of the ξ-wise expansion for $\psi_{.2}$ in regions 1 and 2. This was examined by repeating one case with 7×10^{-9} replacing the right-hand side of equation (3.18). The truncation error proved to be negligible. Confirmation was obtained by testing $\psi_{.2}{}^{i2}$ by the methods of Section X, in the case $n = 0.808$, $A = 0.0264$.

To estimate the effect of the μ-wise truncation error on $E_p(n, A_t)$, three matrices N^2 were examined. The corresponding equilibrium forms were all near the critical bifurcation form. The smallest eigenvalue of the matrix formed by the first i rows and columns of N^2 was calculated for $i = 2(1)5$. These four successive approximations to E_p gave an estimate of about ± 0.002 for the truncation error in all three cases. From Figure 4,

$\partial n/\partial E_p \approx 0.2$ in the critical region. Thus, we estimate the truncation error in n_c as ± 0.0004.

It is of interest to compare these results with those of Roberts (1962, 1963). Roberts' work is based on the assumption that surfaces of constant density are ellipsoids of revolution with a constant ellipticity throughout the star. In the later paper, he shows that for $n = 1$, the value of A at the end of the series lies in the range (0.02035, 0.02590). This is consistent with the value 0.020930 obtained in the current investigation. The ellipticity of a form of bifurcation is given as 0.81267. However, no purpose would be served by comparing this with the "ellipticity" of the critical configuration for $n = 0.808$. Near the end of a series, an equatorial bulge develops. Its extent may be estimated by computing the equatorial radius of an ellipsoid with the same polar radius and volume as the actual equilibrium form. Reference to Appendix Table 4 shows that an ellipsoidal model underestimates the equatorial radius by 8.9 per cent at the end of the series for $n = 1$, and 8.8 per cent at the end of that for $n = 0.808$. Roberts is unable to rule out the possibility of a bifurcation form on the sequence for $n = 1$. This possibility is excluded by the results presented above.

b) Conclusions

Bifurcation points exist on the series of axially symmetric equilibrium forms provided that $n \leq 0.808$ in the polytropic case. The error in this critical value of n is of order ± 0.0004—thus all the decimals given are meaningful. This result is in very good agreement with the limiting value of 0.83 given by Jeans (1919). There are no such bifurcation forms in the white-dwarf case. For a slightly non-symmetric equilibrium form, the dominant term in the non-symmetric part of the potential is a term in $P_2^2(\mu)\cos(\mathit{or}\ \sin)2\phi$. This is also in agreement with Jeans. The critical bifurcation form is that for $n = 0.808$, $A = 0.0265074$. The central condensation for this critical form is 3.117. This compares with the value ≈ 1.3 for the critical case for the generalized Roche model (Jeans 1919). The numerical agreement is not close, but both figures are small compared with the central condensations for most stars. Therefore, the existence of non-symmetric equilibrium forms for rotating stars would require large deviations from rigid body rotation inside these stars.

I would like to express my gratitude to Professor Z. Kopal, for suggesting this problem to me in the first place, and encouraging and advising me with it ever since. My thanks go also to Dr. C. B. Haselgrove and Dr. D. J. Evans, both of Manchester University, for much valuable advice. Dr. Evans also permitted me the use of a complete program for the evaluation of the eigenvalues of a non-symmetric matrix. Mr. R. A. Brooker, director of the Manchester University Computing Machine Laboratory, very kindly permitted me the use of the Mercury computer, without which this work would have been impossible.

APPENDIX

NOTATION

The following conventions are used throughout:

Suffix c attached to a variable quantity indicates the value at the center of mass.

Suffix c attached to n or D indicates a critical value (see Sec. XII).

Suffix t attached to a parameter indicates its value for the last of a series of equilibrium forms.

Suffix a attached to a quantity indicates that it describes an axially symmetric equilibrium form (Sec. IX).

Superscripts q and k attached to a quantity denote its derivative with respect to A_{qqk}.

a) Coordinates and Physical Variables

r = distance from center of mass of the configuration

μ = cosine of the co-latitude, measured from the axis of rotation

ϕ = longitude

P = pressure

ρ = density

ψ = gravitational potential.

b) Physical Constants

G = universal constant of gravitation

K = constant factor in equation (1.1)

$a = 6.01 \times 10^{22}$

$b = 9.82 \times 10^{5}\mu_e$

μ_e = molecular weight per electron.

c) Dimensionless Variables

$\rho' = \rho/\rho_c$

x = parameter in equation of state for a degenerate gas

$y = (1 + x^2)^{1/2}$

$\theta = \rho'^{1/n}$ (polytropic case)

$= y/y_c$ (white-dwarf case)

$\psi = \psi/(n + 1)K\rho_c^{1/n}$ (polytropic case)

$= b/8ay_c$ (white-dwarf case)

$\rho_*' = \partial\rho'/\partial\theta$

$\xi = r/\alpha$

$\alpha = [(n + 1)K/4\pi G\rho_c^{1-1/n}]^{1/2}$ (polytropic case)

$= [2a/\pi G b^2 y_c^2]^{1/2}$ (white-dwarf case)

$\mu = \mu'(\xi)$ is the boundary of the configuration

ξ_0 = boundary of region 1

ξ_1 = origin for an expansion in region 2

ξ_1' = end of range for an expansion in region 2

$\eta = \xi - \xi_1$

$h = \xi_0$ (region 1)

$= \xi_1' - \xi_1$ (region 2)

= Runge-Kutta step-length in region 3

$$f_{jk}(\xi) = (j + \tfrac{1}{2})\xi^2 \int_{-\mu'}^{\mu'} \rho^* P_j P_k d\mu \,.$$

d) Coefficients in Expansions

i) *Axially symmetric cases*

A_{ij} = coefficient of $\xi^i P_j(\mu)$ in ψ (region 1)

B_{ij} = coefficient of $\xi^i P_j(\mu)$ in ρ' (region 1)

α_{ij} = coefficient of $\eta^i P_j(\mu)$ in ψ (region 2)

β_{ij} = coefficient of $\eta^i P_j(\mu)$ in ρ' (region 2)

$\psi_j(\xi)$ = coefficient of $P_j(\mu)$ in ψ

$\rho_j'(\xi)$ = coefficient of $P_j(\mu)$ in ρ'

$\theta_p(\mu)$ = coefficient of η^p in θ (region 2)

$d_p(\mu)$ = coefficient of η^p in ρ' (region 2).

ii) *Non-axially symmetric cases*

A_{ijk} = coefficient of $\xi^i P_j^k(\mu) \cos k\phi$ in ψ

B_{ijk} = coefficient of $\xi^i P_j^k(\mu) \cos k\phi$ in ρ'

C_{ijk} = coefficient of $\xi^i P_j^k(\mu) \sin k\phi$ in ψ

D_{ijk} = coefficient of $\xi^i P_j^k(\mu) \sin k\phi$ in ρ'

$\psi_{jk}(\xi)$ = coefficient of $P_j^k(\mu) \cos(k\phi + \delta_{jk})$ in ψ

δ_{jk} = phase constant.

e) Physical Parameters

ω = angular velocity

n = polytropic index

$D = y_c^{-2}$

H = angular momentum of configuration

$A = \omega^2/8\pi G\rho_c$ (polytropic case)

$= \omega^2/8\pi G y_c^2$ (white-dwarf case)

σ = oblateness of configuration

R^k = matrix formed by the derivatives of the R_{jk}'s with respect to the A_{qqk}'s

N^k = normalized matrix R^k

$E_p(n, A)$ = minimum of eigenvalues of N^0, N^2 (polytropic case)

$E_d(D, A)$ = minimum of eigenvalues of N^0, N^2 (white-dwarf case).

The units for the following are given in Table 1:

ξ_p = polar radius

ξ_e = equatorial radius

M = mass

V = volume

g_e = equatorial gravity

B = moment of inertia about principal axis, perpendicular to axis of rotation

C = moment of inertia about axis of rotation.

f) Miscellaneous

$$f(x) = x(2x^3 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x$$

$$\delta_j(\xi) = \text{error in } \psi_j(\xi) \text{ (Sec. IV)}$$

$$R_j = \left(\frac{d}{d\xi}\psi_j + \frac{j+1}{\xi}\psi_j\right)_{\xi=\xi_e} \text{ (axially symmetric case)}$$

$$R_{jk} = \left(\frac{d}{d\xi}\psi_{jk} + \frac{j+1}{\xi}\psi_{jk}\right)_{\xi=\xi_e} \text{ (general case)}$$

$$I_j(\xi) = \int_0^{\xi} \xi'^{j+2}\rho_j' d\xi$$

$$a_j = \xi^{j+1}\psi_j$$

$$b_j = \xi^{1-j}\left(\frac{d}{d\xi}\psi_j + \frac{j+1}{\xi}\psi_j\right)$$

$$t_s = \text{Gaussian abscissa}$$

$$H_s = \text{Gaussian weight.}$$

APPENDIX TABLE 1

THE PHYSICAL PROPERTIES OF POLYTROPES FOR INCREASING ANGULAR VELOCITY OF ROTATION ($A = \omega^2/8\pi G\rho_c$)

$n = 1$

10^2A	ξ_p	ξ_e	$10g_e$	M	$10^{-2}V$	$10^{-2}B$	$10^{-2}C$
0 0	3 1416	3 1416	3 1831	3 142	1 299	1 0184	1.0184
0 1	3 1227	3 1704	3 1075	3.171	1 315	1 0308	1 0444
0 2	3 1037	3 2005	3 0299	3 201	1 332	1 0440	1 0719
0 3	3 0845	3 2320	2 9501	3 233	1 349	1 0580	1 1009
0 4	3 0653	3 2649	2 8680	3 267	1 368	1 0729	1 1316
0 5	3 0459	3 2994	2 7833	3 301	1 387	1 0888	1 1643
0 6	3 0263	3 3357	2 6958	3 338	1 408	1.1059	1 1990
0 7	3 0066	3 3739	2 6052	3 376	1 430	1 1243	1 2360
0 8	2 9866	3 4144	2 5111	3.416	1 454	1 1440	1 2756
0 9	2 9665	3 4574	2 4132	3 458	1 479	1 1652	1.3181
1 0	2 9461	3 5032	2 3110	3.502	1 506	1 1882	1 3638
1 1	2 9255	3 5523	2 2037	3 549	1 535	1 2131	1 4133
1 2	2 9046	3 6051	2 0907	3 599	1 566	1 2403	1 4669
1 3	2 8834	3 6625	1 9710	3 652	1 600	1 2700	1 5254
1 4	2 8617	3 7253	1.8432	3 709	1 638	1 3027	1 5896
1 5	2 8397	3 7948	1 7055	3 770	1 680	1 3390	1.6605
1 6	2 8172	3 8727	1 5553	3 837	1 727	1 3795	1 7394
1 7	2 7940	3 9620	1 3886	3.909	1 781	1 4252	1 8284
1 8.	2 7702	4 0674	1 1985	3 988	1 844	1 4774	1 9300
1 9	2 7454	4 1984	0 9717	4 076	1 920	1 5384	2 0483
2 0.	2 7194	4 3797	0 6732	4 177	2 020	1 6117	2 1908
2 02	2 7139	4.4282	0 5963	4 199	2 045	1 6284	2 2233
2 04	2 7084	4 4847	0 5082	4 222	2 073	1 6460	2 2576
2 06	2 7028	4.5542	0 4019	4 246	2 105	1 6648	2 2941
2 08	2 6971	4 6519	0 2569	4 272	2 144	1 6848	2 3332
2 09	2 6942	4 7349	0 1369	4 285	2 169	1 6955	2 3540

APPENDIX TABLE 1—*Continued*

$n = 1\ 5$

10^2A	ξ_p	ξ_e	$10g_e$	M	$10^{-2}V$	$10^{-2}B$	$10^{-2}C$
0 00	3 6538	3 6538	2 0330	2 7141	2 0432	0 9316	0 9316
0 05	3 6383	3 6811	1 9829	2 7297	2 0650	0 9391	0 9470
0 10	3 6228	3 7095	1 9315	2 7457	2 0878	0 9469	0 9631
0 15	3 6073	3 7390	1 8790	2 7622	2 1116	0 9551	0 9797
0 20	3 5917	3 7698	1 8251	2 7791	2 1366	0 9636	0 9971
0 25	3 5760	3 8020	1 7697	2 7966	2 1629	0 9725	1 0151
0 30	3 5603	3 8358	1 7128	2 8145	2 1905	0 9819	1 0340
0 35	3 5446	3 8712	1 6542	2 8331	2 2196	0 9917	1 0537
0 40	3 5287	3 9085	1 5936	2 8522	2 2505	1 0020	1 0742
0 45	3 5128	3 9478	1 5309	2 8719	2 2832	1 0128	1 0958
0 50	3 4968	3 9896	1 4659	2 8923	2 3180	1 0242	1 1184
0 55	3 4807	4 0341	1 3983	2 9134	2 3553	1 0362	1.1422
0 60	3.4645	4 0818	1 3277	2 9354	2 3953	1 0489	1.1672
0 65	3 4481	4 1331	1 2537	2 9581	2 4384	1 0623	1 1937
0 70	3 4317	4 1888	1 1756	2 9818	2 4854	1 0766	1 2217
0 75	3 4151	4 2498	1 0928	3 0065	2 5368	1 0918	1 2515
0 80	3 3983	4 3174	1 0040	3 0323	2 5937	1 1080	1 2833
0 85	3 3814	4 3937	0 9080	3 0593	2 6576	1 1255	1 3174
0 90	3 3642	4 4815	0 8021	3 0877	2 7304	1 1444	1 3541
0 95	3 3468	4 5863	0 6825	3 1177	2 8157	1 1649	1 3939
1 00	3 3292	4 7190	0 5410	3 1496	2 9199	1 1874	1 4376
1 04	3 3148	4 8641	0 3983	3 1767	3 0265	1 2072	1 4761
1 06	3 3075	4 9650	0 3063	3 1910	3 0940	1 2179	1 4968
1 08	3 3001	5 1187	0 1769	3 2057	3 1820	1 2292	1 5187
1 09	3 2964	5 3007	0 0400	3 2133	3 2491	1 2351	1 5302

$n = 2$

10^3A	ξ_p	ξ_e	$10g_e$	M	$10^{-2}V$	$10^{-2}B$	$10^{-2}C$
0 00	4 35288	4 35288	1 27249	2 4110	3 4548	0 88895	0 88895
0 25	4 33945	4 38274	1 23940	2 4197	3 4914	0 89397	0 89887
0 50	4 32595	4 41383	1 20558	2 4286	3 5296	0 89915	0 90908
0 75	4 31251	4 44620	1 17103	2 4376	3 5697	0 90450	0 91958
1 00	4 29905	4 48001	1 13565	2 4468	3 6117	0 91002	0 93040
1 25	4 28553	4 51543	1 09934	2 4562	3 6558	0 91572	0 94155
1 50	4 27203	4 55256	1 06209	2 4657	3 7024	0 92162	0 95305
1 75	4 25845	4 59168	1 02374	2 4755	3 7515	0 92772	0 96493
2 00	4 24486	4 63295	0 98426	2 4855	3 8035	0 93404	0 97721
2 25	4 23127	4 67665	0 94347	2 4957	3 8588	0 94059	0 98991
2 50	4 21763	4 72315	0 90126	2 5062	3 9178	0 94740	1.00307
2 75	4 20395	4 77287	0 85740	2 5169	3 9810	0 95448	1 01673
3 00	4 19024	4 82630	0 81172	2 5278	4 0491	0 96184	1 03091
3 25	4 17648	4 88416	0 76388	2 5391	4 1228	0 96952	1 04567
3 50	4 16267	4 94728	0 71359	2 5507	4 2033	0 97754	1 06106
3 75	4 14880	5 01693	0 66029	2 5625	4 2918	0 98594	1 07714
4 00	4 13488	5 09479	0 60333	2 5748	4 3903	0 99475	1 09398
4 25	4 12088	5 18346	0 54171	2 5874	4 5015	1 00402	1 11168
4 50	4 10680	5 28708	0 47384	2 6004	4 6296	1 01381	1 13034
4 75	4 09263	5 41319	0 39694	2 6139	4 7816	1 02419	1 15011
5 00	4 07836	5 57816	0 30516	2 6279	4 9714	1 03528	1 17118
5 25	4 06396	5 83670	0 17943	2 6425	5 2366	1 04721	1 19386
5 30.	4 06107	5 91885	0 14369	2 6456	5 3092	1 04973	1 19863
5 325	4 05962	5 96985	0 12246	2 6471	5 3505	1 05100	1 20105
5 350	4 05816	6 03318	0 09704	2 6486	5 3972	1 05230	1 20351
5 375	4 05671	6 12214	0 06312	2 6501	5 4525	1 05364	1 20602

APPENDIX TABLE 1—*Continued*

$n = 2\ 5$

10^3A	ξ_p	ξ_e	10^2g_e	M	$10^{-3}V$	$10^{-2}B$	$10^{-2}C$
0 0	5 35527	5 35527	7 6265	2 18720	0 64333	0 88130	0 88130
0 1	5 34505	5 38585	7.4499	2 19124	0 64943	88427	0 88687
0 2	5 33483	5 41744	7 2708	2 19532	0 65577	88731	0 89254
0 3	5 32466	5 45028	7 0879	2 19945	0 66237	.89041	0 89833
0 4	5 31447	5 48441	6 9016	2 20363	0 66924	89357	0 90422
0 5	5 30426	5 51992	6 7115	2 20785	0 67642	89680	0 91023
0 6	5 29404	5 55694	6 5175	2 21213	0 68393	90010	0 91637
0 7	5 28383	5 59563	6 3191	2 21647	0 69180	90347	0 92263
0 8	5 27361	5 63614	6 1161	2 22085	0 70005	90691	0 92902
0 9	5 26338	5 67866	5 9080	2 22530	0 70874	91044	0 93555
1 0	5 25315	5 72343	5 6943	2 22980	0 71790	91404	0 94223
1 1	5 24290	5 77068	5 4747	2 23437	0 72759	91774	0 94906
1 2.	5 23264	5 82079	5 2483	2 23900	0 73788	92152	0 95606
1 3	5 22242	5 87408	5 0145	2 24369	0 74883	92540	0 96323
1 4	5 21212	5 93109	4 7721	2 24846	0 76055	92939	0 97058
1 5	5 20189	5 99234	4 5204	2 25330	0 77313	93348	0 97812
1 6	5 19160	6.05865	4 2576	2 25821	0.78673	93769	0 98586
1 7	5 18126	6 13101	3 9820	2 26320	0 80152	94202	0 99383
1 8	5 17095	6 21072	3 6912	2 26828	0 81775	94648	1 00204
1 9	5.16062	6 29969	3 3819	2 27344	0 83572	95108	1 01049
2 0	5 15026	6 40074	3 0490	2 27870	0 85590	95584	1 01923
2 1	5 13990	6 51818	2 6854	2 28406	0 87897	96077	1 02827
2 2	5 12949	6 65982	2 2777	2 28952	0 90606	96588	1 03764
2 3	5 11907	6 84138	1 8005	2.29510	0 93922	97121	1 04739
2 4.	5 10862	7 10819	1 1817	2 30081	0 98349	.97677	1 05758
2 42	5 10652	7 18413	1 0219	2 30197	0 99482	97791	1 05968
2 44	5 10442	7 27657	0 8362	2 30314	1 00765	97908	1 06180
2 46	5 10233	7 40012	0 6023	2 30431	1 02286	0 98026	1 06396

$n = 3$

10^4A	ξ_p	ξ_e	10^2g_e	M	$10^{-3}V$	$10^{-2}B$	$10^{-2}C$
0 0 .	6 89685	6 89685	4 2430	2 01824	1 37417	0 90910	0 90910
0 5	6 88591	6 94462	4 1215	2 02072	1 39101	91162	0 91352
1 0	6 87504	6 99456	3 9975	2 02323	1 40867	91418	0 91802
1 5	6 86411	7 04690	3 8708	2 02576	1 42724	91679	0 92260
2 0	6 85317	7 10189	3 7411	2 02832	1 44681	91945	0 92726
2 5	6 84227	7.15980	3 6083	2 03090	1 46748	.92216	0 93201
3 0	6 83136	7 22101	3 4719	2 03351	1 48937	92493	0 93686
3 5	6 82048	7 28587	3 3318	2 03615	1 51263	92776	0 94180
4 0	6 80949	7 35492	3 1874	2 03882	1 53745	93064	0 94685
4 5	6 79857	7 42879	3 0380	2 04152	1 56401	93359	0 95200
5 0	6 78772	7 50803	2 8837	2 04425	1 59258	93660	0 95726
5 5	6 77682	7 59388	2 7228	2 04701	1 62348	93969	0 96265
6 0	6 76591	7 68735	2 5550	2 04980	1 65712	94284	0 96816
6 5	6 75500	7 79007	2 3790	2 05263	1 69401	94608	0 97381
7 0	6 74406	7 90435	2 1928	2 05550	1 73488	94940	0 97961
7 5	6 73301	8 03338	1 9943	2 05841	1 78070	95281	0 98556
8 0	6 72208	8 18197	1 7797	2 06136	1 83289	95632	0 99168
8 5	6 71111	8 35797	1 5439	2 06435	1 89366	95994	0 99799
9 0.	6 70013	8 57635	1.2763	2 06738	1 96685	96367	1 00450
9 2	6 69575	8 68182	1 1562	2 06861	2 00111	.96520	1 00716
9 4	6 69138	8 80295	1 0249	2 06985	2 03935	96675	1 00986
9 5	6 68918	8 87126	0 9540	2 07047	2 06035	96754	1 01123
9 6	6 68698	8 94619	0 8786	2 07110	2 08289	96833	1 01261
9.7.	6 68477	9 02964	0 7976	2 07172	2 10730	0 96912	1.01400

APPENDIX TABLE 2

THE PHYSICAL PROPERTIES OF WHITE-DWARF CONFIGURATIONS FOR INCREASING ANGULAR VELOCITY OF ROTATION ($A = \omega^2/8\pi G y_c^2$)

$D = 0\ 025$

10^3A	ξ_p	ξ_e	$10^2 g_e$	M	$10^{-2}V$	$10^{-1}B$	$10^{-1}C$
0 0	4 85982	4 85982	7 7718	1 8355	4 8078	7 123	7 123
0 3	4 83197	4 93601	7 3000	1 8472	4 9303	7 207	7 280
0 6	4 80406	5 02005	6 8041	1 8593	5 0666	7 298	7 447
0 9	4 77604	5 11387	6 2790	1 8719	5 2198	7 395	7 627
1 2	4 74792	5 22018	5 7186	1 8851	5 3947	7 500	7.820
1 5	4 71966	5 34317	5 1129	1 8990	5 5979	7 615	8 031
1 8	4 69122	5 48971	4 4467	1 9137	5 8403	7 740	8 261
2 1	4 66252	5 67259	3 6923	1 9293	6 1409	7 880	8 516
2 4	4 63349	5 92064	2 7911	1 9460	6 5393	8 038	8 805
2 7	4 60402	6 33812	1 5425	1 9642	7 1566	8 222	9 142

$D = 0\ 05$

10^3A	ξ_p	ξ_e	$10^2 g_e$	M	$10^{-2}V$	$10^{-1}B$	$10^{-1}C$
0 0	4 4601	4 4601	8 594	1 7097	3 7165	6 165	6.165
0 4	4 4292	4 5375	8 030	1 7235	3 8190	6 252	6.331
0 8	4 3982	4 6236	7.443	1 7379	3 9341	6 348	6.511
1 2	4 3670	4 7208	6 797	1 7532	4 0651	6 451	6 705
1 6	4 3357	4 8326	6 111	1 7693	4 2170	6 565	6 918
2 0	4 3041	4 9646	5 358	1 7864	4 3974	6 691	7.153
2 4	4 2723	5 1272	4 513	1 8047	4 6192	6 832	7.415
2 8	4 2400	5 3422	3 518	1 8245	4 9086	6 993	7 714
3 2	4 2071	5 6765	2 210	1 8462	5 3356	7 182	8 064
3 352384	4 1943	5 8937	1 495	1 8552	5 5846	7 265	8 218
3 430255	4 1878	6 0721	0 977	1 8600	5 7614	7 310	8 303

$D = 0\ 1$

10^3A	ξ_p	ξ_e	$10^2 g_e$	M	$10^{-2}V$	$10^{-1}B$	$10^{-1}C$
0 0	4 0690	4 0690	9 172	1 5186	2 8220	5 014	5 014
0 5	4 0361	4 1446	8 545	1 5336	2 9033	5 096	5 174
1 0	4 0029	4 2291	7 880	1 5495	2 9951	5 187	5 347
1 5	3 9696	4 3250	7 166	1 5664	3 1004	5 286	5 538
2 0	3 9361	4 4363	6 390	1 5843	3 2236	5 396	5 748
2 5	3 9022	4 5695	5 531	1 6036	3 3719	5 520	5 983
3 0	3 8678	4 7373	4.549	1.6245	3 5582	5 662	6 250
3 5	3 8329	4 9692	3 353	1 6474	3 8108	5 827	6 561
3 8	3 8115	5 1766	2 426	1 6625	4 0254	5 942	6 777
3 911354	3 8035	5 2834	1 996	1 6684	4 1291	5 989	6 865
4 095762	3 7900	5 5655	0 995	1 6786	4 3674	6 073	7 024

APPENDIX TABLE 2—*Continued*

$D = 0\ 2$

10^3A	ξ_p	ξ_e	10^2g_e	M	$10^{-2}V$	$10^{-1}B$	$10^{-1}C$
0 0	3 7271	3 7271	8 948	1 2430	2 1687	3 762	3 762
0 5 .	3 6962	3 7920	8 392	1 2561	2 2258	3 824	3 885
1 0	3 6652	3 8638	7 804	1 2700	2 2896	3 892	4 019
1 5	3 6339	3 9442	7 177	1 2847	2 3617	3 967	4 165
2 0	3 6024	4 0357	6 501	1 3003	2 4446	4 050	4 325
2 5	3 5706	4 1424	5 762	1 3171	2 5419	4 142	4 503
3 0 ..	3 5384	4 2712	4 936	1 3352	2 6596	4 246	4 703
3 5...	3 5056	4 4358	3 976	1 3549	2 8091	4 366	4 932
3 756649	3 4885	4 5438	3 400	1 3658	2 9055	4 435	5 064
4 0. .	3 4721	4 6723	2 766	1 3767	3 0172	4 507	5 201
4 236581	3 4560	4 8441	2 001	1 3881	3 1582	4 584	5 349
4 356443	3 4476	4 9699	1 494	1 3941	3 2526	4 627	5 430

$D = 0\ 4$

10^3A	ξ_p	ξ_e	10^2g_e	$10M$	$10^{-2}V$	$10^{-1}B$	$10^{-1}C$
0 0 ..	3 5244	3 5244	6 922	8 598	1 8338	2 518	2 518
0 5 ..	3 4858	3 6008	6 408	8 723	1 8925	2 573	2 629
1 0....	3 4468	3.6871	5 857	8 858	1 9597	2 635	2 752
1 5 . .	3 4074	3 7866	5 258	9.005	2 0381	2 704	2 890
2 0 . .	3 3675	3 9045	4 595	9.164	2 1321	2 785	3 047
2 5 ...	3 3270	4 0505	3 839	9 340	2 2491	2 878	3 229
3 0...	3 2855	4 2464	2 929	9.536	2.4050	2 989	3 444
3 25 ..	3 2643	4 3803	2.370	9 645	2 5089	3 054	3 569
3.5 ...	3 2426	4 5679	1 665	9.762	2 6473	3 128	3 711
3 665359	3 2280	4.7714	0 996	9 846	2 7803	3 183	3 817
3 7	3 2249	4 8380	0 797	9.864	2 8180	3 196	3 841

$D = 0\ 6$

10^3A	ξ_p	ξ_e	10^2g_e	$10M$	$10^{-2}V$	$10^{-1}B$	$10^{-1}C$
0 0.	3 6038	3 6038	4 373	5 680	1 960	1 811	1 811
0 25	3 5704	3 6666	4 117	5 750	2 010	1 844	1.878
0 5. .	3 5368	3 7356	3 846	5 826	2 065	1 880	1 951
0 75 .	3 5030	3 8124	3 557	5 906	2 128	1 920	2 031
1 00	3 4688	3 8991	3 247	5 992	2 198	1 965	2 119
1 25	3 4342	3 9988	2 908	6 085	2 281	2 015	2 218
1 50	3 3991	4 1170	2 532	6 185	2 379	2 072	2 329
1 65	3 3778	4 2005	2 282	6 250	2 448	2 110	2 403
1 75.	3 3634	4 2635	2 101	6 296	2 500	2 137	2 456
1 85..	3 3489	4 3342	1 906	6 343	2 557	2 166	2 513
1.95....	3 3342	4 4154	1 692	6 392	2 624	2 197	2 573
2 203855	3 2961	4 7129	0 993	6 530	2 853	2 289	2 750
2 320633	3 2780	5 0009	0 432	6 601	3 035	2 340	2 848

APPENDIX TABLE 2—*Continued*

$D = 0\ 8$

10^4A	ξ_p	ξ_e	$10^2 g_e$	$10M$	$10^{-2}V$	$10^{-1}B$	$10^{-1}C$
0	4 0446	4 0446	1 890	3 0913	2 7714	1 275	1 275
1	4 0046	4 1183	1 776	3 1332	2 8443	1 300	1 326
2	3 9642	4 1996	1 655	3 1778	2 9256	1 327	1 381
3	3 9236	4 2904	1 526	3 2256	3 0175	1 358	1 442
4	3 8825	4 3933	1 387	3 2771	3 1227	1 392	1 510
5	3 8409	4 5126	1 234	3 3329	3 2459	1 431	1 587
6	3 7987	4 6555	1 063	3 3939	3 3941	1 475	1 674
7	3 7554	4 8358	0 864	3 4612	3 5806	1 527	1 774
7.5	3 7334	4 9492	0 748	3 4978	3 6964	1 556	1 831
8	3 7110	5 0892	0 614	3 5368	3 8362	1 588	1 894
8 5	3 6882	5 2787	0 448	3 5787	4 0164	1 624	1 964
9	3 6648	5 6274	0 184	3 6245	4 2950	1 666	2 045
9 080420	3 6609	5 7636	0 095	3 6324	4 3709	1 673	2 059

APPENDIX TABLE 3

COMPARISON WITH CHANDRASEKHAR*

	$\partial\xi_p/\partial A$	$\partial\xi_e/\partial A$	$\partial M/\partial A$	$\partial V/\partial A$
$n=1$:				
Current ..	− 18 87	28 27	28 78	1559
Chandrasekhar	− 18 85	28 27	28 78	1558
$n=1\ 5$:				
Current ...	− 30 84	53 63	30 83	4272
Chandrasekhar	− 30 98	53 68	31 05	4272
$n=2$:				
Current ..	− 53 40	117 0	34 41	14330
Chandrasekhar	− 53 77	117.2	34 48	14330
$n=3$:				
Current ..	−219 3	935.2	49 44	328900
Chandrasekhar	−218 6	934 8	49 43	329000

* The deviations between the results of the current investigation and those of Chandrasekhar are due to the poor convergence of the differentiation formulae used.

APPENDIX TABLE 4

TERMINAL VALUES OF PHYSICAL QUANTITIES

n	10^2A	ξ_p	ξ_e	M	$10^{-2}V$	$10^{-2}B$	$10^{-2}C$
	Polytropic Case						
0 808	2 65074	2 4852	4 7652	5 0248	1 968	2 0890	3 0695
1 0	2 0930	2 6933	4 8265	4 289	2 1825	1 6955	2 3540
1 5	1 0906	3 2962	5 3585	3 2137	3 259	1 2355	1 5309
2 0	0 5401	4 0553	6 307	2 6518	5 527	1 0553	1 209
2 5	0 24825	5 0999	7 7623	2 30563	10 503	0 9818	1 0666
3 0	0 0983	6 58		2 089	.		
D	White-Dwarf Case						
0 025	0 2855	4 5883	7 033	1 9747	7 981	0 834	0 935
05	34920	4 1825	6 4723	1 8638	6 036	7348	8373
1	41650	3 7849	5 908	1 6826	4 569	.6107	7089
2	452065	3 4359	5 4216	1 4026	3 4959	4689	5550
4	37650	3 2191	5 1418	0 9899	2 938	3220	3888
6	23518	3 2730	5 2659	0 6621	3 153	2354	2877
0 8	0 91127	3 6594	5 9233	0 36355	4 430	0 1676	0 2064

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