GENERAL RELATIVISTIC POLYTROPIC FLUID SPHERES*

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ABSTRACT

This paper discusses solutions of the field equations of general relativity for a compressible fluid sphere in gravitational equilibrium under the assumption that the fluid obeys a polytropic equation of state. With suitable transformations the equations of static equilibrium are equivalent to two coupled first-order non-linear differential equations analogous to the Lane-Emden equation in the Newtonian theory of polytropes. Solutions of the equilibrium equations are obtained in terms of the polytropic index n and a parameter σ whose physical interpretation is the ratio of pressure to energy density at the center of the sphere. The quantity σ measures the deviation from Newtonian gravitational theory. A solution is obtained in closed form for $n = 0$; this corresponds to the Schwarzschild interior solution for a fluid sphere of uniform density. Solutions for $n = 1.0(0.5)3.0$ are obtained by numerical integration. The ratio of total mass to invariant radius of a polytropic sphere is found in terms of boundary values of the relativistic Lane-Emden functions. Integrals for the gravitational potential energy and rest energy are obtained and evaluated numerically. Properties of the solutions are tabulated for each n and a range of values of σ. The distributions of density, pressure, mass, and metric tensor components are shown graphically for some typical cases Plots of the mass-radius relation are given in a form suitable for determination of the internal structure of a polytrope of given mass, radius, and polytropic index. The existence of multiple solutions for some values of mass and radius is a general-relativistic feature. The maximum ratio of half the gravitational radius to the geometrical radius is 0.214 for $n = 1.0$ and 0.0631 for $n = 3.0$. These values are smaller than the limiting ratio 0.340 previously known for the Schwarzschild interior solution ($n = 0.0$). It appears that models with $n = 3.0$ and $\sigma > 0\ 5$ are energetically unstable. The gravitational collapse of massive ($\sim 10^8 M_\odot$) starlike objects which may exist near the centers of galaxies is discussed as a possible general-relativistic mechanism for producing the large amounts of energy ($\sim 10^{60}$ ergs) associated with strong radio sources.

I. INTRODUCTION

Only a small number of solutions describing the gravitational field inside a spherical mass distribution are currently available in the general theory of relativity. This paper describes some new solutions of the field equations of general relativity which give the internal structure of a compressible fluid sphere in static equilibrium under its own gravitation. The pressure and energy density within the sphere are supposed to be related by a polytropic equation of state. Solution of the equations of equilibrium leads to the conclusion that some of the general-relativistic configurations may be unstable. It is shown that transition from an unstable state to one of stable equilibrium may be an important mechanism for the sudden generation of energy in massive stars. The large amounts of energy associated with strong radio sources can perhaps be explained in terms of this general-relativistic process.

The present investigation represents an attempt to provide a basis for a relativistic theory of the stellar interior by solving the field equations for a simple relation between pressure and energy density. It is possible to determine the general-relativistic features of the results by comparing them with the corresponding Newtonian solutions. The properties of polytropic fluid spheres treated according to Newtonian gravitational theory are well known, and a discussion of these properties is fundamental to the theory of stellar structure (Chandrasekhar 1939). We find the relativistic generalization of the basic equations in the Newtonian theory of polytropes and exhibit some features of their solutions. Because the characteristics of the non-relativistic polytropic spheres are known in

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detail, it is possible to make a direct comparison with the relativistic solutions and separate out in an unambiguous way the effects predicted by the relativistic theory of gravitation as opposed to the Newtonian.

The structure of polytropic fluid spheres is treated here under the assumption of quasi-static equilibrium. All time-dependent terms in the gravitational field equations are set equal to zero. During its lifetime a star is continuously evolving, depending on the availability of thermonuclear energy sources, and there is a continuous irreversible output of energy due to radiation. But ordinarily this evolution proceeds so slowly that the variation of physical quantities with time can be ignored. It will be shown that the assumption of equilibrium can imply much more in the relativistic theory than in the Newtonian case. In the polytropic models about to be discussed, the configurations are usually states of stable equilibrium, for which the quasi-static description is a good one. In some cases the configurations are not states of lowest energy corresponding to a given rest mass, and the possibility arises that the equilibrium is unstable. Detailed consideration of the dynamics of sudden transitions between states of different energy, although of great interest, is beyond the scope of the present treatment.

In a relativistic theory mass and energy must be equivalent. The density which appears in the relativistic equations of gravitation represents the density of internal energy from all sources instead of simply the rest-energy density of matter. The pressure consists of gas pressure plus radiation pressure. In a massive star the radiation pressure can be comparable with or exceed the gas pressure. In choosing the polytropic equation of state, we assume that the total density is related to the total pressure by a simple power law without going into detail concerning the processes by which such a relation might arise. Adoption of the "standard model," in which the ratio of radiation pressure to gas pressure is constant throughout the star, gives an equation of state which to first approximation is polytropic with index 3. The deviation from the polytropic equation of state in the relativistic standard model lies in the difference between total energy density and rest-mass energy density. For values of the parameters such that both radiation pressure and general relativity are important, the mass density of internal energy (due to both microscopic kinetic energy of the gas particles and the energy of radiation) is comparable with the gas density near the center of the star.

The body of this paper is divided into three major sections as follows. First, the relativistic equations for a fluid sphere in gravitational equilibrium are derived and put in suitable dimensionless form. Second, a general discussion of the properties of the solutions is given. This includes the closed-form solution for zero polytropic index, the mass-radius relation imposed by the boundary conditions, the means of finding the internal structure of a given polytrope, and the gravitational potential energy. Finally, the results obtained by numerical integration of the equations of equilibrium are presented. These include illustration of the internal structure for some sample cases, a method of determining the structure of a polytrope with given mass and radius, the limitations imposed by general relativity on the mass-radius ratio, and a discussion of stability of the models based on energy considerations.

II. THE EQUATIONS OF EQUILIBRIUM

We derive the general-relativistic equations of equilibrium for a spherically symmetric distribution of fluid obeying a polytropic equation of state. For a fluid capable of supporting only isotropic stresses, the mixed components of the energy-momentum tensor in a co-moving coordinate system are given by

$$T_1{}^1 = T_2{}^2 = T_3{}^3 = P, \qquad T_0{}^0 = -\epsilon = -\rho c^2, \tag{2.1}$$

where P is the pressure and $\epsilon = \rho c^2$ is the energy density (written as an equivalent mass density ρ times the square of the speed of light). In a spherical coordinate system (r, ϑ, φ)

at rest with respect to the fluid and chosen such that the metric reduces to the standard form

$$ds^2 = e^{\nu(r)}c^2dt^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) - e^{\lambda(r)}dr^2\,, \tag{2.2}$$

the time-independent gravitational equations reduce to (Tolman 1939, p. 244; Landau and Lifshitz 1962, p. 325)

$$e^{-\lambda}\left(\frac{1}{r}\frac{d\nu}{dr}+\frac{1}{r^2}\right)-\frac{1}{r^2}=\frac{8\pi G}{c^4}P\,, \tag{2.3}$$

$$e^{-\lambda}\left(\frac{1}{r}\frac{d\lambda}{dr}-\frac{1}{r^2}\right)+\frac{1}{r^2}=\frac{8\pi G}{c^4}\rho c^2\,, \tag{2.4}$$

$$\tfrac{1}{2}e^{-\lambda}\left[\frac{d^2\nu}{dr^2}+\tfrac{1}{2}\left(\frac{d\nu}{dr}-\frac{d\lambda}{dr}\right)\left(\frac{d\nu}{dr}+\frac{2}{r}\right)\right]=\frac{8\pi G}{c^4}P\,. \tag{2.5}$$

(Here $G = 6.670 \times 10^{-8}$ gm^{-1} cm^3 sec^{-2} is the Newtonian gravitational constant.) As a consequence of the Bianchi identities, the gravitational equations admit the covariant conservation law $T_{\mu}{}^{\nu}{}_{;\,\nu} = 0$. The r-component (which is the only non-zero component in the static case) of this conservation law is

$$\frac{dP}{dr}+\tfrac{1}{2}(P+\rho c^2)\frac{d\nu}{dr}=0\,. \tag{2.6}$$

This equation follows directly from the field equations by setting equations (2.3) and (2.5) equal to each other and using equation (2.4).

In the sequel we drop equation (2.5) and take equations (2.3), (2.4), and (2.6) plus the equation of state $\rho = \rho(P)$ as the full set for determination of λ, ν, P, and ρ as functions of r. We want to find a solution of this third-order differential system which is non-singular at the origin and which goes over to the Schwarzschild exterior solution

$$\left.\begin{aligned} &P=\rho c^2=0\\ &e^{\nu}=e^{-\lambda}=1-\frac{2GM}{c^2r}\end{aligned}\right\}\ (r\geq R)\,, \tag{2.7}$$

giving the field of a spherically symmetric distribution of total mass M and coordinate radius R, at the boundary. We require λ, ν, P, and ρ to be continuous everywhere; the field equations then insure the continuity of $d\nu/dr$ and dP/dr, but not necessarily $d\lambda/dr$ and $d\rho/dr$, at all points inside and outside the mass distribution.

Consistent with its role as a conservation law, equation (2.6) permits a first integration, and this gives a relation between the pressure P and metric component e^{ν} when the dependence of density on pressure is known explicitly.

Equation (2.4) can also be integrated, although only formally, since the density distribution $\rho(r)$ is unknown (Oppenheimer and Volkoff 1939; Buchdahl 1959). Define a new function u by

$$u=\frac{c^2r}{2GM}(1-e^{-\lambda})\,, \tag{2.8}$$

where M is the total mass of the sphere as measured by an external observer. Then

$$e^{-\lambda}=1-\frac{2GMu}{c^2r}\,. \tag{2.9}$$

In terms of u, equation (2.4) becomes

$$M\frac{du}{dr}=4\pi r^2\rho\,. \tag{2.10}$$

We see from equation (2.10) that $Mu(r) = M(r)$ can be interpreted as the total mass, including contributions from the internal energy as well as from matter, within a sphere of coordinate radius r. We put $u(0) = 0$ to avoid a singularity in mass at the origin. Equation (2.9) is a solution for $e^{-\lambda}$ which is continuous with the exterior solution (2.7) at $r = R$ provided $u(R) = 1$. The simple form of this solution is a consequence of the choice of coordinate system (2.2).

At this point in order to make further progress we must introduce the explicit form of the equation of state. We shall assume that the pressure and density are connected by a power law of the form

$$P = K\rho^{1+(1/n)} , \tag{2.11}$$

where K and n are constants. The constant n, which need not be an integer, will be called the "polytropic index." We regard n as a known physical constant, while K is a constant to be determined by the thermal characteristics of a given fluid sphere. The constant of proportionality K may be established for a particular body by specifying the density and pressure at a single point, say the center. Since for a given pressure the density is a function of temperature, the constant K in equation (2.11) contains the temperature at a given point in the gas implicitly. Later we shall see that K can be determined from the total mass, radius, and ratio of pressure to energy density at the center.

An equation of the type (2.11) is obtained as the limiting form of the parametric equations of state for a completely degenerate gas at zero temperature. In this case both K and n are universal physical constants and the density at a given point is determined by the pressure independently of the temperature. The polytropic form of the limiting pressure-density relation for a highly degenerate gas will be made the basis of a series of general-relativistic models for white-dwarf stars in a later paper.

The explicit relation (2.11) between pressure and density permits an integration of equation (2.6) independently of the radial coordinate r. It is most convenient to perform this integration in terms of a new variable θ related to the density ρ at a given point and the central density ρ_c by

$$\rho = \rho_c\theta^n . \tag{2.12}$$

(Here n is the same n that appears in equation [2.11].) From equation (2.12) we note that θ must be "normalized" to unity when $r = 0$. With the change of variable (2.12) the equation of state appears as

$$P = K\rho_c^{1+(1/n)}\theta^{n+1} . \tag{2.13}$$

Equation (2.6) becomes

$$2\sigma(n+1)d\theta + (1+\sigma\theta)d\nu = 0 , \tag{2.14}$$

with

$$\sigma = \frac{K\rho_c^{1/n}}{c^2} . \tag{2.15}$$

The parameter σ defined by equation (2.15) is interpreted as the ratio of pressure to energy density at the center of the sphere:

$$\sigma = \frac{P_c}{\rho_c c^2} . \tag{2.16}$$

Now equation (2.6) is the general-relativistic analogue of the Newtonian expression

$$\frac{dP}{dr} + \rho\frac{d\phi}{dr} = 0 \tag{2.17}$$

for the dependence of pressure on gravitational potential ϕ. In the non-relativistic case (2.17) only the density and not the pressure enters into the expression for the pressure gradient. Thus the parameter σ measures the deviation from the Newtonian barometric law (2.17). The non-relativistic limit is obtained by letting $c \to \infty$, which implies $\sigma \to 0$.

The variables are separable in equation (2.14), and we may integrate to get e^{ν} in terms of θ. Letting e^{ν} have the value e^{ν_c} at the center (where $\theta = 1$), we obtain

$$e^{\nu} = e^{\nu_c} \left(\frac{1+\sigma}{1+\sigma\theta} \right)^{2(n+1)} . \tag{2.18}$$

The integration constant e^{ν_c} is determined by requiring e^{ν} to be continuous at the boundary. When $r = R$, $\rho = 0$, which implies $\theta = 0$; comparing equation (2.18) with the exterior solution (2.7) for $r = R$, we see that

$$e^{\nu_c}(1+\sigma)^{2(n+1)} = e^{\nu(R)} = 1 - \frac{2GM}{c^2 R},$$

and equation (2.18) for e^{ν} in terms of θ becomes

$$e^{\nu} = (1+\sigma\theta)^{-2(n+1)} \left(1 - \frac{2GM}{c^2 R} \right) . \tag{2.19}$$

Up to this point we have used only the equation of state (2.13) and the relativistic barometric law (2.6) to get a relation between the unknown function θ and the metric component $g_{00} = -e^{\nu}$. Also θ^n gives the mass-density ρ in terms of the central density ρ_c and θ^{n+1} gives the pressure in units of the central pressure $P_c = \sigma \rho_c c^2$. It remains to determine the metric component $g_{rr} = e^{\lambda}$, or equivalently the function u defined by equation (2.8), together with θ. To accomplish this we have the field equations (2.10) and (2.3). Rewriting (2.14) as

$$\frac{d\nu}{dr} = -\frac{2\sigma(n+1)}{1+\sigma\theta} \frac{d\theta}{dr},$$

and substituting this as well as the expression (2.9) for $e^{-\lambda}$ in terms of u into the $d\nu/dr$ field equation (2.3), we obtain a first-order differential equation connecting u and θ:

$$\frac{\sigma(n+1)}{1+\sigma\theta} r \frac{d\theta}{dr} \left(1 - \frac{2GMu}{c^2 r} \right) + \frac{GMu}{c^2 r} + \frac{GM}{c^2} \sigma\theta \frac{du}{dr} = 0 . \tag{2.20}$$

Equation (2.10) expressed in terms of θ is a second equation connecting u and θ:

$$M \frac{du}{dr} = 4\pi r^2 \rho_c \theta^n . \tag{2.21}$$

The system (2.20) and (2.21) is to be solved with the initial conditions $\theta(0) = 1$, $u(0) = 0$. We make a change of variables which puts equations (2.20) and (2.21) in dimensionless form. Let

$$r = \xi / A \tag{2.22}$$

$$v(\xi) = \frac{A^3 M}{4\pi\rho_c} u(r), \tag{2.23}$$

where

$$A = \left[\frac{4\pi G \rho_c}{(n+1) K \rho_c^{1/n}} \right]^{1/2} \tag{2.24}$$

has dimensions of inverse length.

In terms of the new variables v and ξ, equations (2.20) and (2.21) become

$$\xi^2 \frac{d\theta}{d\xi} \frac{1-2\sigma(n+1)v/\xi}{1+\sigma\theta} + v + \sigma\xi\theta \frac{dv}{d\xi} = 0\,, \tag{2.25}$$

$$\frac{dv}{d\xi} = \xi^2 \theta^n\,. \tag{2.26}$$

In the non-relativistic limit $\sigma \rightarrow 0$ the variable v can be eliminated and the system (2.25) (2.26) reduces to the Lane-Emden equation (Chandrasekhar 1939, p. 88)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0\,. \tag{2.27}$$

Equations (2.25) and (2.26) are to be solved for given n and σ subject to the initial conditions

$$\theta(0) = 1\,, \qquad v(0) = 0\,. \tag{2.28}$$

Since $v \rightarrow 0$ like ξ^3 as $\xi \rightarrow 0$ it is clear from equation (2.25) that $d\theta/d\xi \rightarrow 0$ as $\xi \rightarrow 0$. The boundary of the sphere is represented by the first zero ξ_1 of θ:

$$\theta(\xi_1) = 0\,. \tag{2.29}$$

For a given K and n the total mass M is proportional to the value $v(\xi_1)$ of the transformed mass function at this point.

III. GENERAL PROPERTIES OF THE SOLUTIONS

a) The Relativistic Lane-Emden Functions

The equations of equilibrium can be solved exactly for $n = 0$, which corresponds to uniform density. In this case the θ-dependence disappears from equation (2.26), and we may integrate directly to obtain v as a function of ξ. Since $v = 0$ when $\xi = 0$,

$$v = \tfrac{1}{3}\xi^3\,. \tag{3.1}$$

Then equation (2.26) becomes

$$\xi(1+\sigma\theta)(1+3\sigma\theta)d\xi + (3-2\sigma\xi^2)d\theta = 0\,. \tag{3.2}$$

The variables are separable in this equation, and an elementary integration subject to the condition that $\theta = 1$ when $\xi = 0$ gives θ as a function of ξ:

$$\sigma\theta = \frac{(1+3\sigma)(1-\frac{2}{3}\sigma\xi^2)^{1/2} - (1+\sigma)}{3(1+\sigma) - (1+3\sigma)(1-\frac{2}{3}\sigma\xi^2)^{1/2}}\,. \tag{3.3}$$

To interpret this solution, we notice that when $n = 0$, $\theta = P/P_c$. Equation (3.3) is the dimensionless form of the expression for pressure in units of the central pressure given by Schwarzschild's interior solution (Tolman 1939, p. 247) for a fluid sphere of constant density. The value ξ_1 of ξ for which $\theta = 0$ is given by

$$\xi_1{}^2 = \frac{6(1+2\sigma)}{(1+3\sigma)^2}\,. \tag{3.4}$$

Except for $n = 0$ it does not seem possible to obtain solutions of equations (2.25) and (2.26) in closed form for non-zero σ. Numerical integration must be used to determine θ and v as functions of ξ. Since the pressure and density must both be positive for a configuration in equilibrium, we consider only the range of values of ξ such that θ is positive. Numerical integration starts at $\xi = 0$ where $\theta = 1$, $v = 0$, and proceeds by finite small steps along the positive ξ-axis until the first zero of ξ is reached at ξ_1. It may be seen from equation (2.26) that v is a monotonically increasing function of ξ. Therefore $v(\xi_1)$ is positive. For a given n, ξ_1 and $v(\xi_1)$ can be regarded as functions of σ. The form of these functions is to be determined by numerical integration of equations (2.26) and (2.27).

b) *The Mass-Radius Relation*

The requirement that $u = 1$ at the boundary of the sphere leads to a mass-radius relation in terms of the relativity parameter σ. To show this we rewrite equation (2.23) with A^2 evaluated according to equation (2.24) and use equation (2.15), getting

$$v(\xi) = \frac{M}{4\pi\rho_c}\frac{\xi}{r}A^2u(r) = \frac{GM}{c^2\sigma(n+1)}\frac{\xi u(r)}{r}.$$

When $r = R$, $\xi = \xi_1$ and $u(R) = 1$, so

$$\frac{GM}{c^2R} = \frac{\sigma(n+1)v(\xi_1)}{\xi_1}. \tag{3.5}$$

This equation gives the ratio of the gravitational radius $2GM/c^2$ to the coordinate radius R when σ is prescribed, since the right-hand side is a function of σ for given n.

The value R of the coordinate r at the boundary of the sphere does not represent the radius of the configuration as measured by an outside observer. In general the value of R depends on the coordinate system used. A quantity $\bar{r}$ which is invariant with respect to coordinate transformations preserving the spherical symmetry, and which therefore represents the true distance from the center of a point with radial coordinate r, can be defined by means of the transformation

$$d\bar{r}^2 = g_{rr}dr^2 .$$

Since $g_{rr} = e^{\lambda}$, $\bar{r}$ is given by

$$\bar{r} = \int_0^r e^{\lambda(r)/2}dr . \tag{3.6}$$

Equation (2.9) may be expressed in terms of dimensionless variables by using equations (2.23) and (2.24). The result is

$$e^{-\lambda} = 1 - 2\sigma(n+1)v(\xi)/\xi . \tag{3.7}$$

Writing $\bar{\xi} = A\bar{r}$ and using the solution (3.7) for e^{λ}, we obtain from equation (3.6)

$$\bar{\xi} = \int_0^{\xi}[1 - 2\sigma(n+1)v(\xi)/\xi]^{-1/2}d\xi . \tag{3.8}$$

The value $\bar{\xi}_1 = A\bar{R}$ corresponding to the physical radius $\bar{R}$ of the sphere is obtained by integrating from 0 to ξ_1 in equation (3.8).

The internal structure of the configuration can be found in terms of the true distance $\bar{r}$ from the center using (3.8). Expression of the structure in terms of $\bar{r}$ rather than r takes into account the non-Euclidean spatial geometry within the gravitating body. The volume of a sphere of radius $\bar{r}$ is not $(4/3)\pi\bar{r}^3$ but rather $(4/3)\pi r^3$, where $\bar{r}$ and r are related by equation (3.6). The deviation from Euclidean geometry depends, through equation (3.8), on the distribution of gravitating matter within the sphere. Since $e^{\lambda} \geq 1$ (Landau and Lifshitz 1962, p. 238), $r < \bar{r}$, and the volume of the sphere is smaller than what it would be if Euclidean geometry were applicable.

The mass and invariant radius are related by equation (3.5) rewritten in terms of $\bar{R}$ and $\bar{\xi}_1$:

$$\frac{GM}{c^2\bar{R}}=\frac{\sigma(n+1)\,v(\xi_1)}{\bar{\xi}_1}. \tag{3.9}$$

This is the form of the mass-radius relation we shall use later in deriving the internal structure of a polytropic sphere with given mass and radius.

c) Structural Parameters

Suppose n, σ, and the central density ρ_c are given. Then the mass, radius, and internal structure of the polytrope may be determined as follows. We calculate the characteristic inverse length A from equation (2.24), writing $K\rho_c^{1/n} = \sigma c^2$:

$$A=\left[\frac{4\pi G\rho_c}{(n+1)\,\sigma c^2}\right]^{1/2}. \tag{3.10}$$

By numerical integration of the system (2.25)–(2.26) we find θ and v as functions of ξ, and determine ξ_1 and $v(\xi_1)$. From ξ_1 and A we find the coordinate radius of the sphere by using the equation $R = \xi_1/A$. By evaluation of the integral in equation (3.8) using $v(\xi)$ already found by numerical integration, we determine $\bar{\xi}_1$ and hence the invariant radius $\bar{R} = \bar{\xi}_1/A$. The mass of the configuration may be found from either equation (3.5) or equation (3.9). The distributions of density and pressure throughout the sphere are determined from the equations $\rho = \rho_c\theta^n$, $P = \sigma\rho_c c^2\theta^{n+1}$, and $\bar{r}/\bar{R} = \bar{\xi}/\bar{\xi}_1$. Since the mass $M(r)$ within a sphere of coordinate radius r is proportional to $v(\xi)$, the mass distribution is given by

$$M(r)/M = v(\xi)/v(\xi_1). \tag{3.11}$$

The components of the metric tensor are found from equations (2.19) and (3.7). Equation (2.19) becomes, when equation (3.5) is used to substitute for GM/c^2R,

$$e^{\nu} = (1+\sigma\theta)^{-2(n+1)}[1-2\sigma(n+1)v(\xi_1)/\xi_1]. \tag{3.12}$$

A quantity of interest in pointing out the difference in structure between the general relativistic polytropic models and those constructed using Newtonian gravitation is the ratio of central density to average density. We define the average density as

$$\bar{\rho}=\frac{M}{(4/3)\,\pi R^3}=\frac{3MA^3}{4\pi\xi_1^3}. \tag{3.13}$$

Equation (2.23) with $\xi = \xi_1$ and $u(R) = 1$ can be used to express the quantity $MA^3/4\pi$ in terms of the central density ρ_c and the boundary value $v(\xi_1)$ of the mass function. We obtain for the density concentration

$$\frac{\rho_c}{\bar{\rho}}=\frac{\xi_1^3}{3\,v(\xi_1)}. \tag{3.14}$$

The concentration of mass toward the center is a function only of n and σ, and does not depend on the central density ρ_c.

d) Physical Interpretation of the Pressure-Density Relation

The equations of equilibrium (2.25) and (2.26) were derived under the assumption that the pressure P is related to the energy density $\epsilon = \rho c^2$ by a simple power law $P = K\rho^{1+(1/n)}$. The total pressure consists of gas pressure plus radiation pressure. The relativistic energy density is the sum of the rest energy $\rho_0 c^2$ of the gas particles plus the density of internal energy. The internal energy of the system can be thought of as divided between the radiation energy and the microscopic kinetic energy of the gas particles. For the special case of an adiabatic process the pressure-density equation (2.11) implies a unique relation between the gas density ρ_g and the total mass density ρ, or alternatively between ρ_g and θ. In this section we derive this relation and investigate some of its consequences. The assumption of an adiabatic process is consistent with the absence of heat flow terms in the energy-momentum tensor (2.1).

For an adiabatic process the first law of thermodynamics takes the form

$$d\epsilon + (P+\epsilon)\frac{dV}{V} = 0\,, \tag{3.15}$$

where $d\epsilon$ is the change in the energy density ϵ due to a change dV in the specific volume V. Equation (3.15) may be regarded in the present context as a natural relativistic generalization of the first law of thermodynamics. Since $dV/V = -d\rho_g/\rho_g$, equation (3.15) leads to a differential equation for ρ_g in terms of P and ϵ:

$$\frac{d\rho_g}{\rho_g} = \frac{d\epsilon}{P+\epsilon}. \tag{3.16}$$

With $\epsilon = \rho_c c^2 \theta^n$ and $P = \sigma \rho_c c^2 \theta^{n+1}$, equation (3.16) becomes

$$\frac{d\rho_g}{\rho_g} = \frac{n\,d\theta}{\theta(1+\sigma\theta)}. \tag{3.17}$$

Integrating, we find

$$\rho_g = \rho_{gc}\left[\frac{(1+\sigma)\theta}{1+\sigma\theta}\right]^n, \tag{3.18}$$

where ρ_{gc} is the value of the gas density ρ_g at the center of the configuration.

The integration constant ρ_{gc} may be evaluated as follows. Near the boundary of the configuration, where $\theta \ll 1$, equation (3.18) becomes approximately

$$\rho_g \simeq \rho_{gc}(1+\sigma)^n \theta^n\,. \tag{3.19}$$

Near the boundary the internal energy density is small compared to the rest-mass energy density, so that $\rho \simeq \rho_g$ there. Comparing equation (3.19) with the equation $\rho = \rho_c \theta^n$, we obtain

$$\rho_{gc} = \frac{\rho_c}{(1+\sigma)^n}, \tag{3.20}$$

and equation (3.18) becomes

$$\rho_g = \rho_c\left(\frac{\theta}{1+\sigma\theta}\right)^n. \tag{3.21}$$

Thus the gas density is equal to the total mass-density divided by $(1+\sigma\theta)^n$; the denominator is in general greater than unity. The gas density and total density are equal in the non-relativistic limit $\sigma \ll 1$.

For a general thermodynamical process the phase velocity of sound v_s in a relativistic fluid is given by (Taub 1948)

$$\frac{v_s^2}{c^2} = \frac{\rho_g}{\mu}\frac{d\mu}{d\rho_g}, \tag{3.22}$$

where

$$\mu = \frac{P+\epsilon}{\rho_g c^2} = \frac{P+\rho c^2}{\rho_g c^2}. \tag{3.23}$$

With $\epsilon = \rho_c c^2 \theta^n$, $P = \sigma \rho_c c^2 \theta^{n+1}$, and ρ_g given by equation (3.21), we find

$$\mu = (1 + \sigma\theta)^{n+1} . \tag{3.24}$$

Hence, according to equation (3.22),

$$\frac{v_s^2}{c^2} = \frac{n+1}{n} \sigma\theta . \tag{3.25}$$

This is the same as the expression for the velocity of sound obtained using the formula (Landau and Lifshitz 1959; Curtis 1949)

$$v_s^2 = (dP/d\rho)_{\text{ad}} , \tag{3.26}$$

where the subscript "ad" means that the derivative is to be taken for an adiabatic process. Equations (3.22) and (3.26) yield the same result because the adiabatic relation (3.16) was used to derive equation (3.21) for ρ_g. The phase velocity of sound at the center of the sphere is

$$v_{sc} = \pm c\left(\frac{n+1}{n} \sigma\right)^{1/2} . \tag{3.27}$$

For a given n there is a maximum value of σ for which the velocity of sound at the center is equal to the speed of light in free space. However, since equation (3.27) gives the phase velocity, which is not necessarily the group velocity, this limitation on σ should perhaps not be taken literally. Further studies of the propagation of energy by sound waves in relativistic media with strong density gradients are indicated.

e) Gravitational Energy

In accordance with the equivalence of matter and energy in the theory of relativity, the total energy E of a body, including the internal energy and gravitational potential energy, is Mc^2, where M is the field-producing mass of the body as determined by the motion of an external test particle. For a fluid sphere, using the coordinate system defined by equation (2.2),

$$E = M c^2 = 4\pi \int_0^R \rho c^2 r^2 d r \tag{3.28}$$

The proper energy E_0 is defined as the integral of the energy density over elements of proper volume

$$[g^{(3)}]^{1/2} dr\, d\vartheta\, d\varphi = e^{\lambda/2} r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi . \tag{3.29}$$

Here $g^{(3)}$ is the determinant formed from the components of the three-dimensional metric tensor. For a spherical distribution of matter the proper energy is (Buchdahl 1959)

$$E_0 = 4\pi \int_0^R \rho c^2 e^{\lambda/2} r^2 d r . \tag{3.30}$$

The gravitational potential energy Ω is then given by[1]

$$E = E_0 + \Omega . \tag{3.31}$$

Since $e^\lambda \geq 1$, in general $E_0 \geq E$ and $\Omega \leq 0$. The metric component e^λ is given in terms of the mass $M(r) = u(r)M$ interior to r by equation (2.9). In the non-relativistic limit

[1] The notation for gravitational potential energy used here has been chosen to coincide with that of Chandrasekhar (1939, p. 64); in Buchdahl's (1959) paper the symbol Ω corresponds to our $-\Omega$, i.e., Buchdahl's Ω represents the "gravitational binding energy."

this expression may be expanded as a power series in $GM(r)/c^2r$. Equation (3.30) then becomes

$$E_0 \simeq 4\pi \int_0^R \rho(r)\, c^2 \left[1 + \frac{GM(r)}{c^2 r}\right] r^2 dr = E + \int_0^R \frac{GM(r)\, dM(r)}{r} \tag{3.32}$$

to a first approximation. The second term on the right of (3.32) represents the work that would have to be done on the system to disperse the matter to infinity against gravitational forces, and is therefore the magnitude of the gravitational potential energy. This concept may be carried over to the full relativistic case with the understanding that the absolute value of the potential energy Ω defined by equation (3.31) represents the work necessary to disperse the entire mass of the system, including the internal energy in mass units, to infinity against gravitation. In general relativity gravitation is due to the mass equivalent of the internal energy of the system as well as to its rest mass.

For a polytropic sphere the proper energy (3.30) becomes, using equations (2.12), (2.22), and (3.7),

$$E_0 = \frac{4\pi \rho_c c^2}{A^3} \int_0^{\xi_1} \frac{\theta^n \xi^2 d\xi}{[1 - 2\sigma(n+1) v(\xi)/\xi]^{1/2}}. \tag{3.33}$$

Using equation (2.23) evaluated at the boundary, we see that the coefficient of the integral in (3.33) is equal to $Mc^2/v(\xi_1)$. Therefore the gravitational potential energy is, in units of Mc^2,

$$\frac{\Omega}{Mc^2} = 1 - \frac{1}{v(\xi_1)} \int_0^{\xi_1} \frac{\theta^n \xi^2 d\xi}{[1 - 2\sigma(n+1) v(\xi)/\xi]^{1/2}}. \tag{3.34}$$

In the Newtonian limit $\sigma \ll 1$ the integral in equation (3.34) is, according to (2.26), nearly equal to $v(\xi_1)$, and the magnitude of the gravitational potential energy is small compared to the rest energy. By expanding the integral in equation (3.34) as a power series in σ, it can be shown that for $\sigma \ll 1$ the potential energy is given by the nonrelativistic expression (Chandrasekhar 1939, p. 101)

$$\Omega \simeq \frac{3}{n-5} \frac{GM^2}{R}. \qquad (\sigma \ll 1) \tag{3.35}$$

The proper energy and proper mass of the gas in a spherically symmetric system may be defined by

$$E_{0g} = M_{0g} c^2 = 4\pi \int_0^R \rho_g c^2 e^{\lambda/2} r^2 dr. \tag{3.36}$$

The quantity E_{0g} represents the sum of the rest masses of the elementary particles in the system expressed in energy units. The quantity M_{0g} is approximately equal to the number of nucleons in the configuration multiplied by the nucleon rest mass. For a polytropic sphere the proper energy of the gas (3.36) in units of the total energy $E = Mc^2$ is, using equations (3.21) and (3.7),

$$\frac{E_{0g}}{E} = \frac{1}{v(\xi_1)} \int_0^{\xi_1} \frac{\theta^n \xi^2 d\xi}{[1 + \sigma\theta]^n [1 - 2\sigma(n+1) v/\xi]^{1/2}}. \tag{3.37}$$

The binding energy of the system may be defined as

$$\text{B.E.} = E_{0g} - E \tag{3.38}$$

(Iben 1963). If one imagines an "initial" state where the particles are widely dispersed and the system has zero internal energy, and if the number of nucleons is conserved, then the quantity B.E. represents the difference in energy between the "initial" state and the "final" state in which the particles and the internal energy are bound by gravitational forces.

It is not apparent by inspection of (3.28) and (3.36) whether the binding energy is positive or negative. The gas density ρ_g is smaller than the total density ρ, but the factor $e^{\lambda/2}$ in equation (3.36) is in general greater than unity. Some information about the binding energy can be gained by examining equation (3.37) in the Newtonian approximation. By expanding the right-hand side of this equation as a power series in σ we find

$$\text{B.E.} \simeq \frac{3-n}{5-n}\frac{GM^2}{R} \simeq \frac{n-3}{3}\,\Omega\,. \qquad (\sigma \ll 1) \quad (3.39)$$

In the Newtonian approximation for $n < 3$ the binding energy is expected to be positive, whereas for $3 < n < 5$ it is expected to be negative.

The gravitational potential energy Ω defined by equation (3.31) represents the work done *by the system* in expanding to a state of infinite diffusion. Since Ω is negative, the work that would have to be done *on the system* to diffuse its mass, associated with internal energy as well as matter, to infinity is positive. From these considerations it appears that a negative binding energy is not a sufficient condition for instability of the system against an expansion to infinity. Negative binding energies in general relativity were first noted by Zeldovich (1962). Equation (3.39) shows that negative binding energies can also occur in Newtonian gravitation, provided the internal energy is included in the mass density in accordance with special relativity.

IV. NUMERICAL RESULTS

a) Integration of the Equations of Equilibrium

Equations (2.25)–(2.26) were integrated numerically for five values of n using the Runge-Kutta method on an IBM 7090 computer. The solutions were started at the initial values $\xi = 0$, $\theta = 1$, and $v = 0$ and proceeded forward using a finite step-size $\Delta\xi$. The value ξ_1 at which $\theta = 0$ was determined by integrating along the positive ξ-axis until a negative value of θ was found. Then integration was resumed with a smaller step size starting from the last value of ξ yielding a positive θ. This process was repeated several times until a value (recorded as ξ_1) of ξ was reached such that $0 \leq \theta \leq 10^{-8}$. The relativistic Lane-Emden functions θ and v found in this way are well behaved and can be tabulated readily for a range of values of σ. Simultaneously with the determination of θ and v, the integrals (3.8) and (3.34) for the invariant radius and gravitational potential energy were computed using the trapezoidal rule. Accuracy was determined in several typical cases by rerunning using a smaller $\Delta\xi$ and noting the change in the values of ξ_1, $\bar{\xi}_1$, and Ω/Mc^2. In most cases a step-size $\Delta\xi = 1/512$ was found small enough to give results accurate to four significant figures.

The computed ξ_1, $\bar{\xi}_1$, $v(\xi_1)$, $\rho_c/\bar{\rho}$, and $-\Omega/Mc^2$ are listed as functions of n and σ in Table 1. The values for the non-relativistic case $\sigma = 0$ have taken been from Chandrasekhar (1939, p. 96), and are presented for comparison with the relativistic results.[2] For each n the maximum σ for which numerical integration was performed was taken as $\sigma_{\max} = n/(n+1)$. For $\sigma = \sigma_{\max}$ the speed of sound evaluated at the center of the sphere becomes just equal to the speed of light c in free space (cf. eq. [3.27]).

As σ increases, and hence as the pressure becomes increasingly important in the barometric equation (2.6), the behavior of the results listed in Table 1 differs in several respects from what would be predicted by Newtonian gravitational theory with the same equation of state. The most striking feature is perhaps the more marked concentration of matter toward the center in the relativistic cases. This is shown by the variation of

[2] From equations (3.35) and (3.5) we see that $\Omega = 0$ when $\sigma = 0$. In the Newtonian limit σ is non-zero but is very small compared to unity, so that Ω is not exactly zero but is small compared to Mc^2.

TABLE 1

PARAMETERS OF THE RELATIVISTIC LANE-EMDEN FUNCTIONS

σ	ξ_1	$\bar{\xi}_1$	$v(\xi_1)$	$\rho_c/\bar{\rho}$	$-\Omega/Mc^2$
			$n=1\ 0$		
0 0	3 1416	3 1416	3 1416	3 2899	0 0000
1	2 599	2 853	1 751	3 342	1240
2	2 277	2 657	1 143	3 443	2108
3	2 064	2 517	0 8192	3 579	2744
4	1 913	2 412	0 6249	3 736	3228
0 5	1 801	2 330	0 4981	3 909	0 3605
			$n=1\ 5$		
0 0	3 6538	3 6538	2.7141	5 9907	0 0000
1	3 088	3 381	1 482	6 310	1307
.2	2 699	3 219	0 9604	6 827	2207
.3	2 493	3 120	0 6883	7 504	2857
.4	2 361	3 061	0 5270	8 326	3344
.5	2 275	3 029	0 4226	9 286	3719
0 6	2 219	3 018	0 3505	10 394	0 4014
			$n=2\ 0$		
0 0	4 3529	4 3529	2 4110	11 403	0 0000
.1	3 699	4 150	1 299	12 99	1355
.2	3 398	4 093	0 8403	15 57	2274
.3	3 271	4 128	0 6055	19 27	2926
.4	3 248	4 228	0 4680	24 41	3405
5	3 296	4 381	0 3800	31 42	3764
6	3 399	4 577	0 3201	40 90	4036
0 66667	3 493	4 730	0 2901	48 97	0 4181
			$n=2\ 5$		
0 0	5 3553	5 3553	2 1872	$2\ 3406\times10^{1}$	0 0000
.1	4 782	5 372	1 169	$3\ 118\ \times10^{1}$	1388
.2	4 721	5 657	0 7606	$4\ 610\ \times10^{1}$	2312
.3	4 986	6 183	0 5556	$7\ 435\ \times10^{1}$	2950
4	5 545	6 976	0 4386	$1\ 296\ \times10^{2}$	3396
5	6 434	8 095	0 3664	$2\ 423\ \times10^{2}$	3702
6	7 728	9 632	0 3202	$4\ 804\ \times10^{2}$	3900
7	9 523	11 69	0 2904	$9\ 912\ \times10^{2}$	4008
0 71429	9 985	12 03	0 2872	$1\ 101\ \times10^{3}$	0 4018
			$n=3\ 0$		
0.0	6 8968	6 8968	2 0182	$5\ 418\ \times10^{1}$	0 0000
1	6 826	7 606	1 078	$9\ 835\ \times10^{1}$	1408
2	7 951	9 261	0 7130	$2\ 350\ \times10^{2}$	2318
3	10 83	12 65	0 5386	$7\ 868\ \times10^{2}$	2900
4	17 82	20 26	0 4516	$4\ 177\ \times10^{3}$	3222
5	37 21	40 55	0 4214	$4\ 075\ \times10^{4}$	3273
6	91 08	95 83	0 4493	$5\ 606\ \times10^{5}$	3000
7	162 6	169 0	0 5266	$2\ 722\ \times10^{6}$	2582
0 75	180 5	187 6	0 5654	$3\ 465\ \times10^{6}$	0 2433

the ratio $\rho_c/\bar{\rho}$ of density at the center to average density, particularly for $n = 2.5$ and $n = 3.0$. The behavior of the dimensionless invariant radius $\bar{\xi}_1$ is not always analogous to the dimensionless coordinate radius ξ_1. For instance, both ξ_1 and $\bar{\xi}_1$ are monotonically decreasing functions of σ for $n = 1.0$ and $n = 1.5$. For $n = 2.0$ both ξ_1 and $\bar{\xi}_1$ first decrease and then increase as σ increases, but the minimum of $\bar{\xi}_1$ does not occur at the same σ as the minimum of ξ_1. For $n = 2.5$ and $n = 3.0$ the coordinate radius first decreases and then increases with rising σ, but the invariant radius increases monotonically.

The last column of Table 1 gives the negative gravitational potential energy in units of the total energy Mc^2. The potential energy is almost independent of n for a given σ, except for $n = 3.0$, where it begins to fall off for values of σ greater than about 0.5. The effect of non-linearities in the field equations is felt more strongly for $n = 3.0$ than for any of the other values of n considered.

b) Internal Structure

Internal structures of some typical polytropic spheres are shown in Figures 1–5. The independent variable chosen is the relative invariant radius $\bar{r}/\bar{R} = \bar{\xi}/\bar{\xi}_1$, so that the

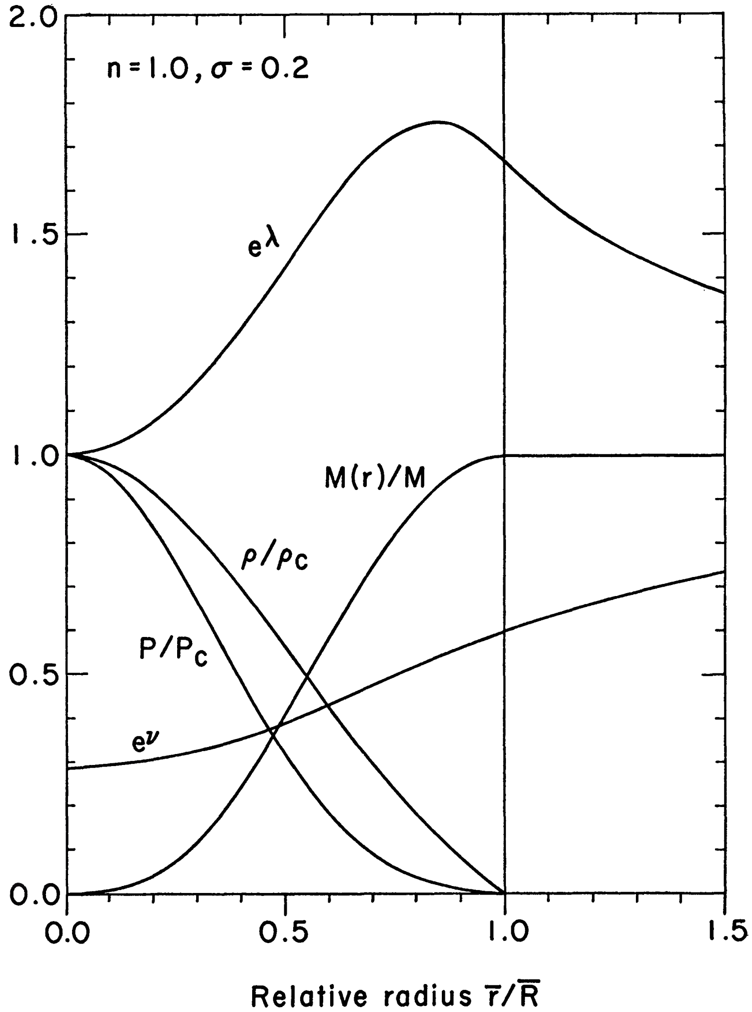

FIG 1.—Internal structure of a typical general-relativistic polytrope with $n = 1.0$. The distributions of density, pressure, mass, and metric tensor components are given in terms of the relative invariant radius $\bar{r}/\bar{R}$.

geometrical effects associated with distortion of the underlying space by the presence of gravitating matter have been taken into account in preparing the graphs. The metric components have also been plotted against $\bar{r}/\bar{R}$, using the solutions (3.7) and (3.12) inside the sphere and the Schwarzschild exterior solution (2.7) outside. The component $g_{rr} = e^{\lambda}$ is always greater than or equal to unity and has a maximum inside the sphere for some finite value of $\bar{r}/\bar{R}$. The component $g_{00} = -e^{\nu}$ is always less than unity and has a minimum at $\bar{r}/\bar{R} = 0$.

For relativistic values of σ the structural features indicate a greater concentration of matter toward the center than in the non-relativistic case $\sigma = 0$, as was already predicted by the behavior of the central condensation $\rho_c/\bar{\rho}$. The normalized density ρ/ρ_c and pressure P/P_c fall off more rapidly as functions of radius in the relativistic cases, and the mass function $M(r)/M$ rises sooner. This behavior becomes more pronounced as the polytropic index increases, and for $n = 3.0$ it is extreme even for relatively small values of σ. For $n = 2.5$ and $n = 3.0$ the density near the boundary is down three or four orders of magnitude from the density at the center. It is an advantage of the present method of calculation that the boundary can be determined with good accuracy from the func-

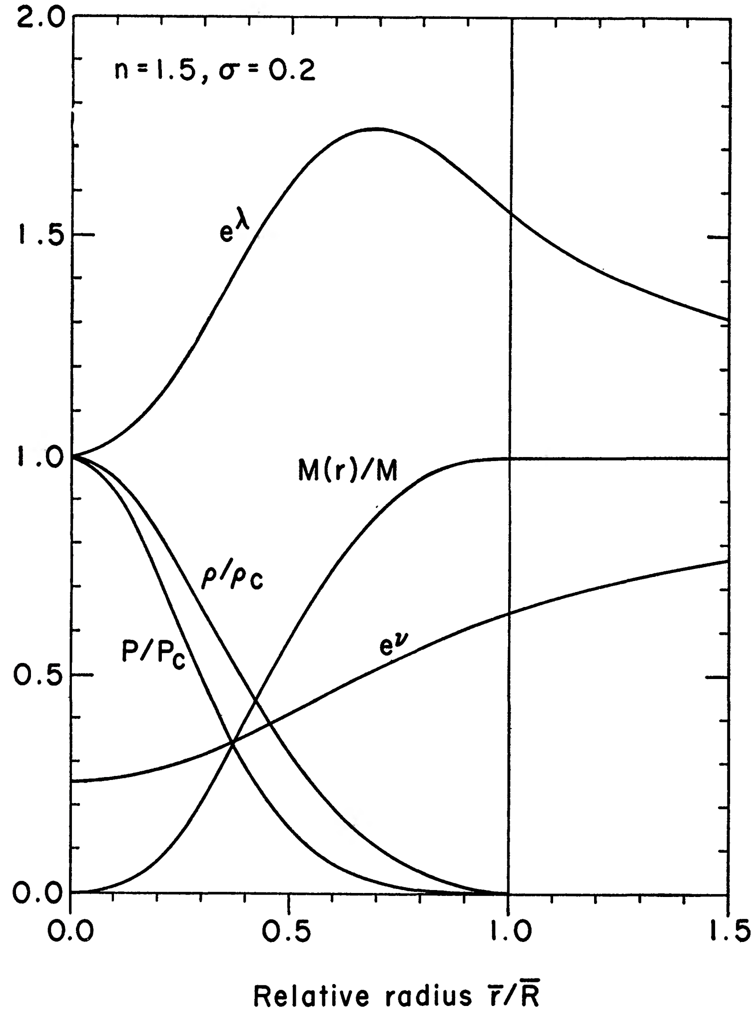

FIG. 2.—Internal structure of a typical general-relativistic polytrope with $n = 1.5$. The density, pressure, mass, and metric components are plotted as functions of the relative invariant radius.

tion θ being computed as the solution to a differential equation rather than from the density or pressure, which are high powers of θ and fall off rapidly with increasing r.

The higher values of σ yield solutions with striking deviation from Euclidean geometry, and the metric components e^{λ} and e^{ν} differ strongly from unity. Table 2 lists the maximum values of e^{λ} and minimum values of e^{ν} as functions of n and σ. For the maximum of e^{λ} the corresponding relative radius $\bar{r}/\bar{R}$ is given; the minimum of e^{ν} always occurs at $\bar{r}/\bar{R} = 0$.

c) Determination of σ for Given Mass and Radius

Previously we considered the relativity parameter σ and the central density ρ_c as given quantities. This contrasts with the usual situation in the non-relativistic theory of polytropes, where one determines the internal structure of a sphere of given mass, radius, and polytropic index (Chandrasekhar 1939, p. 99). If in the relativistic case M, $\bar{R}$, and n are considered as given quantities, then the determination of σ becomes a characteristic value problem.

The most convenient way of finding σ for a given mass and radius is by a graphical

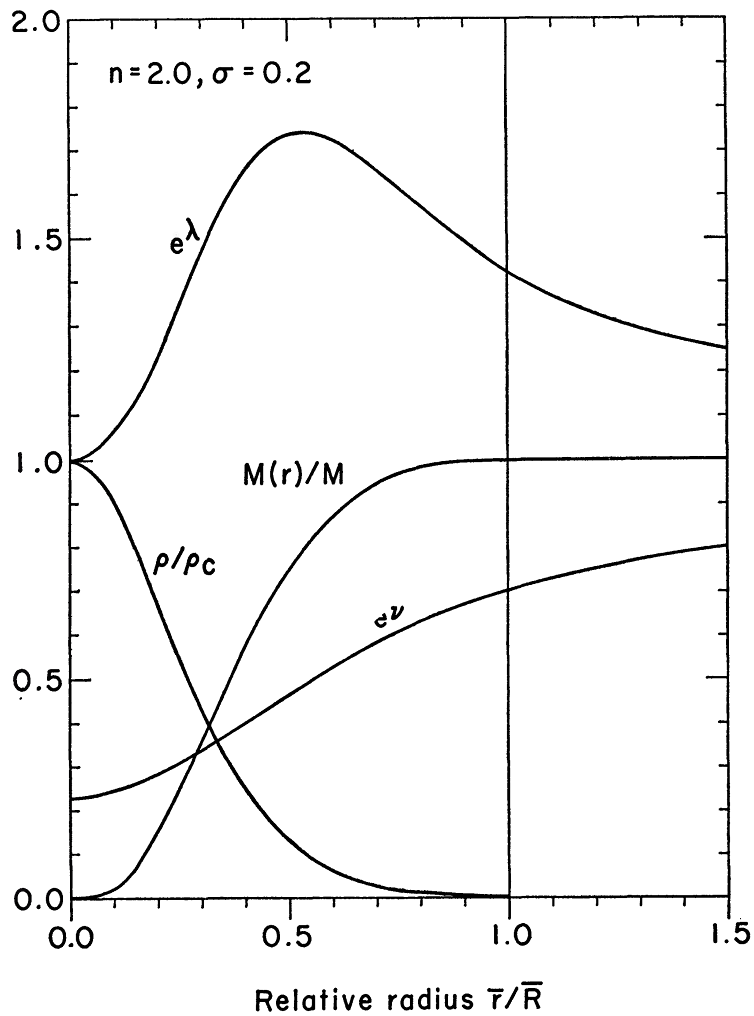

FIG. 3 —Internal structure of a general-relativistic polytrope with $n = 2.0$, $\sigma = 0.2$. The density, mass distribution, and metric components are plotted as functions of the relative invariant radius.

TABLE 2

EXTREMUM VALUES OF THE METRIC TENSOR COMPONENTS

σ	Maximum Value of e^{λ}	Corresponding $\bar{r}/\bar{R}$	Minimum Value of e^{ν} (at $\bar{r}/\bar{R}=0$)
		$n=1$ 0	
0 0	1 000	....	1 0000
1	1 404	0 856	0 4989
2	1 754	832	0 2887
3	2 050	824	0 1834
4	2 296	822	0 1243
0 5	2 500	0 800	0 08826
		$n=1$ 5	
0 0	1 000	....	1 0000
1	1 405	0 727	0 4695
2	1 746	694	0 2589
3	2 023	673	0 1578
4	2 245	625	0 1029
5	2 424	623	0 07053
0 6	2 565	0 578	0 05017
		$n=2$ 0	
0 0	1 000	....	1 0000
1	1 407	0 578	0 4455
2	1 739	551	0 2355
3	2 006	492	0 1382
4	2 214	458	0 08688
5	2 375	413	0 05743
6	2 500	385	0 03940
0 66667	2 567	0 358	0 03116
		$n=2$ 5	
0 0	1 000	...	1 0000
1	1 408	0 435	0 4253
2	1 738	372	0 2161
3.	1 994	317	0 1221
4	2 191	267	0 07385
5	2 340	216	0 04686
6	2 452	174	0 03077
7.	2 537	138	0 02073
0 71429	2 548	0 135	0 01962
		$n=3$ 0	
0 0	1 000	...	1 0000
1	1 410	0 298	0 4075
2	1 735	220	0 1992
3	1 985	149	0 1080
4	2 173	0880	0 06227
5	2 314	0431	0 03725
6	2 418	0177	0 02273
7.	2 495	00964	0 01407
0 75	2 524	0 00824	0 01115

method. First we rewrite equation (3.9) in terms of numerical values for the solar mass $M_\odot$ and radius $R_\odot$, and take logarithms of both sides of the resulting equation.

$$\log_{10}(M/M_\odot) - \log_{10}(\bar{R}/R_\odot) = 5.674 + \log_{10}[\sigma(n+1)v(\xi_1)/\xi_1] . \quad (4.1)$$

Similarly we rewrite equation (3.10) for $A = \xi_1/\bar{R}$ in terms of ρ_c, obtaining the following formula for the central density.

$$\log_{10}(\rho_c/\rho^*) = \log_{10}[\sigma(n+1)\xi_1^2] - 2\log_{10}(\bar{R}/R_\odot) . \quad (4.2)$$

Here ρ^* is a reference density given by

$$\rho^* = c^2/4\pi G R_\odot^2 = 2.215 \times 10^5 \text{ gm/cm}^3 . \quad (4.3)$$

The quantities $\log_{10}[\sigma(n+1)v(\xi_1)/\xi_1]$ and $\log_{10}[\sigma(n+1)\xi_1^2]$ appearing in equations (4.1) and (4.2) are plotted as functions of σ in Figures 6 and 7 for the various values of n considered. For $\sigma \ll 0.1$ the boundary values ξ_1 and $v(\xi_1)$ go over to their non-

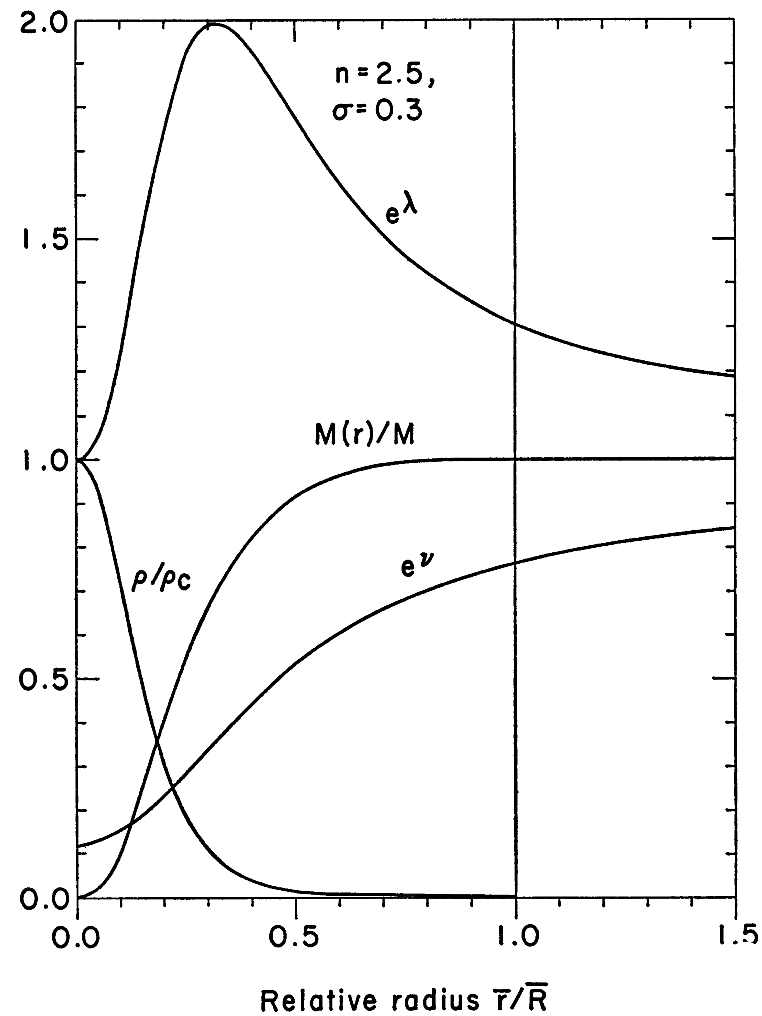

FIG. 4.—Internal structure of a general-relativistic polytrope for $n = 2.5$, $\sigma = 0.3$. The density, mass, and components of the metric tensor are plotted as functions of the relative invariant radius $\bar{r}/\bar{R}$.

relativistic equivalents ξ_1 and $-\xi_1^2\theta'(\xi_1)$ obtained from the solution of the Lane-Emden equation (2.27), so that $\log_{10}[\sigma(n+1)v(\xi_1)/\bar{\xi}_1]$ and $\log_{10}[\sigma(n+1)\bar{\xi}_1^2]$ become constants + $\log_{10}\sigma$. For a pair of values of M and $\bar{R}$ and given polytropic index n, equations (4.1)–(4.2) permit the relativity parameter σ and central density ρ_c to be determined from Figures 6 and 7. Then the internal structure of the model is fixed and can be found by integrating equations (2.25) and (2.26) with the appropriate value of σ. The constant K in the pressure-density relation (2.11) is determined since $\sigma = K\rho_c^{1/n}/c^2$ and ρ_c are known.

Figure 6 points out an important characteristic of the relativistic solutions. For each pair of values of the physical radius $\bar{R}$ and mass M a single value of $\log_{10}[\sigma(n+1)v(\xi_1)/\bar{\xi}_1]$ is obtained according to equation (4.1). But for $n = 2.0$ and $n = 2.5$ there correspond, for some pairs of values of M and $\bar{R}$, *two* values of σ, and for $n = 3.0$ there may be as many as *three* values. For example, in the case $n = 3.0$ for $\sigma \ll 1$ we have from the non-relativistic solution $v(\xi_1) \simeq 2.018$ and $\bar{\xi}_1 \simeq \xi_1 \simeq 6.897$. Thus for small σ we obtain $\log_{10}[\sigma(n+1)v(\xi_1)/\bar{\xi}_1] \simeq \log_{10}\sigma + 0.068$. Now suppose M and $\bar{R}$ are such that $\log_{10}$

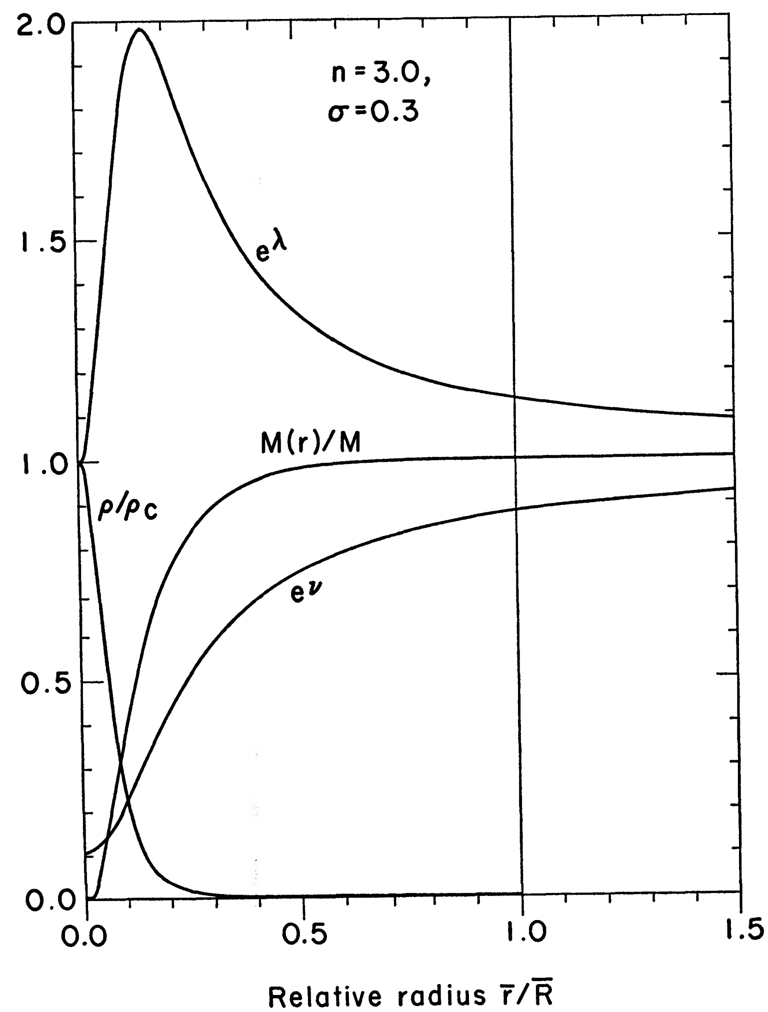

FIG. 5.—Internal structure of a typical general-relativistic polytrope for $n = 3.0$. Even for the relatively modest $\sigma = 0.3$ shown here, the density at $\bar{r} = \frac{1}{2}\bar{R}$ is down several orders of magnitude from its central value, and there is extreme deviation of the metric tensor from the flat space-time of special relativity.

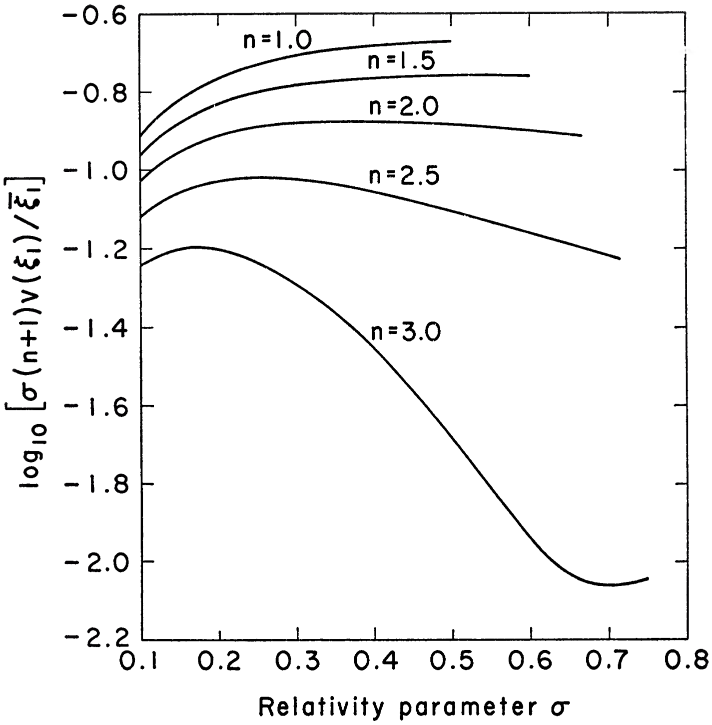

FIG 6 —The logarithmic ratio of gravitational radius to geometrical radius as a function of the relativity parameter σ for various values of the polytropic index n.

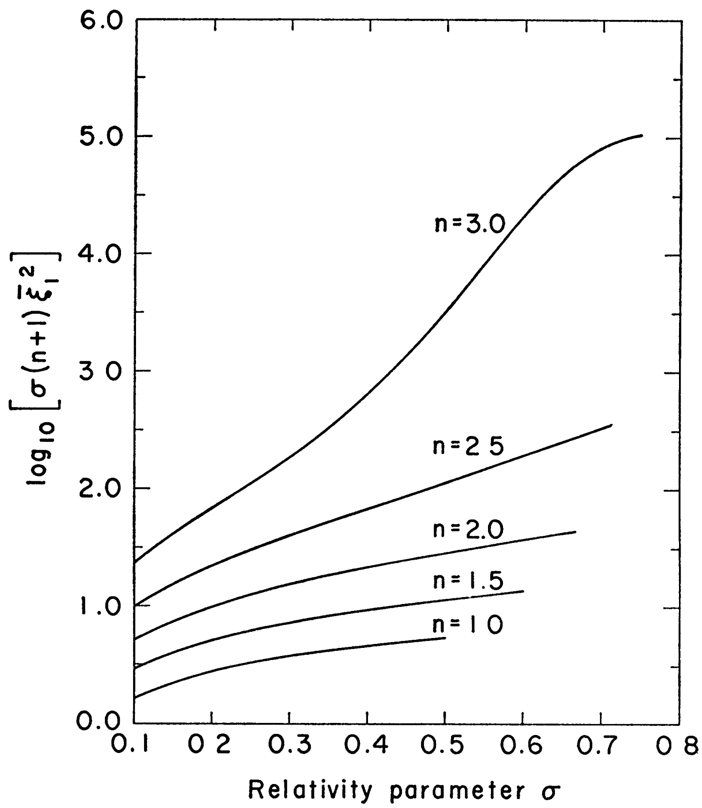

FIG 7 —The logarithmic dimensionless density function from equation (4 2) plotted against σ for various values of n.

$(M/M_\odot) - \log_{10} (\bar{R}/R_\odot) = 5.674 - 2.050$; equation (4.1) will then give a value $\sigma \simeq 7.62 \times 10^{-3}$ small compared to unity. But from Figure 6 we obtain two other values of σ corresponding to $\log_{10} [\sigma(n+1)v(\xi_1)/\bar{\xi}_1] = -2.050$, namely, $\sigma \simeq 0.665$ and $\sigma \simeq 0.740$. Since each value of σ corresponds to a distinct internal structure, it is possible in the relativistic case to have as many as three spherical polytropic configurations each having the same mass and radius but widely different internal distributions of density, pressure, and gravitational potential (components of the metric tensor).

In principle it should be possible to distinguish between three model stars each having the same mass and radius by considerations which go beyond the scope of the simple polytropic models discussed here. We have called the model configurations "polytropic fluid spheres" instead of "stars" because no assumptions have been made concerning energy generation by nuclear reactions. If energy generation as a function of density were properly taken into account, it would be possible to distinguish observationally between the alternate structures predicted by a general relativistic theory of the stellar interior by means of the luminosity, or total energy output per unit time. The luminosity would depend on the internal density-energy generation distribution and would be different for each of the alternate models of given mass and radius.

d) Limiting Mass-Radius Relations

Contrary to the situation obtaining in the non-relativistic case, for a given polytropic index n there exist pairs of values of mass and radius for which *no static solutions exist.* For example, if $n = 1.0$, the maximum value of $\log_{10} [\sigma(n+1)v(\xi_1)/\bar{\xi}_1]$ is about -0.670 (cf. Fig. 6), achieved at the value of σ for which the speed of sound at the center is just equal to the speed of light. This means $\sigma(n+1)v(\xi_1)/\bar{\xi}_1 \leq 0.214$, which, according to equations (3.5) and (3.9), is equivalent to

$$\frac{GM}{c^2} \leq 0.214\bar{R} = 0.276R. \qquad (n = 1.0) \quad (4.4)$$

Thus the gravitational radius $2GM/c^2$ is at most about 43 per cent of the invariant radius $\bar{R}$ for a polytrope of index 1.0. The limiting ratio 0.214 for $GM/c^2\bar{R}$ is smaller than the corresponding ratio 0.340 predicted by the Schwarzschild interior solution for a fluid sphere of uniform density (Wyman 1946; Rosen and Newman 1946). As we have seen, the uniform sphere can be regarded as a polytrope of zero index. As a second example, consider the polytrope with index $n = 3.0$. Here the maximum value of $\log_{10} [\sigma(n+1)v(\xi_1)/\bar{\xi}_1]$ is about -1.200, and equations (3.5) and (3.9) give

$$\frac{GM}{c^2} \leq 0.0631\bar{R} \simeq 0.0729R. \qquad (n = 3.0) \quad (4.5)$$

This is somewhat smaller than the limiting values obtained for either $n = 0.0$ (uniform density) or $n = 1.0$. It may be noted that for $n = 3.0$ the presence of an absolute maximum at $\sigma \simeq 0.175$ depends on the upper limit $\sigma_{max} = 0.75$ imposed by the requirement that the speed of sound should be less than or equal to the speed of light. In the absence of this condition the mass-radius function $\sigma(n+1)v(\xi_1)/\bar{\xi}_1$ would continue to rise for $\sigma \geq 0.7$ and would eventually surpass the relative maximum at $\sigma \simeq 0.175$.

The limiting forms (4.4) and (4.5) of the mass-radius relation were obtained by requiring the phase velocity of sound in the medium to be always less than the velocity of light in empty space. It is conceivable that a study of the dispersion of sound waves in the inhomogeneous medium could lead to a revision of these limits, based on the requirement that the group velocity or velocity of energy transport must always be less than the speed of light.

e) Stability of Models with Large σ

In this section we show by means of an energy principle that some of the general relativistic polytropic models with sufficiently high σ are probably unstable, a feature not shared by the corresponding Newtonian models. If instabilities occur, they may provide a way of explaining the large energies (10^{58}–10^{60} ergs) associated with strong radio sources by means of the sudden collapse of massive stars (10^6 to $10^8\ M_\odot$), as postulated by Hoyle and Fowler (1963*a*, *b*), from an unstable state to a state of lower energy. The likelihood of instability with concomitant energy release follows directly from the polytropic equation of state using the equilibrium equations of general relativity, and does not occur with the same equation of state using Newtonian gravitation. The possibility accordingly arises that strong radio sources receive their energy from the gravitational collapse of large starlike objects near the centers of galaxies, and that this collapse is specifically a general-relativistic effect.

The present discussion of stability is based on the properties of static solutions to the gravitational field equations, and must rely on energy considerations to provide criteria for stability or instability of the various equilibrium configurations. A rigorous treatment of the stability of general-relativistic polytropes utilizing a linearized analysis of time-dependent perturbations on the equilibrium states has been given by Chandrasekhar (1964*b*). Stability is established for those states in which small deviations from the static solutions oscillate with real frequencies. Caution is necessary when making statements concerning stability based on static solutions alone. Instabilities which arise from thermal properties (e.g., the ratio of specific heats) of the configurations may not be predicted by energy considerations. Thermal properties are, however, important in determining the dynamical stability of stars (Ledoux and Walraven 1958). The energy considerations on which the following discussion is based are to be regarded as giving a strong likelihood of instability for certain values of the relativity parameter σ. They also provide estimates of the energy released if transitions based on general relativistic instabilities do in fact occur in nature.

To examine the stability of a set of poytropes having the same values of K and n in the equation of state (2.11), we make a plot of the active gravitational mass M versus the rest mass of the constituent particles

$$M_{0g} = M(E_{0g}/E) \tag{4.6}$$

for a range of values of the relativity parameter σ. To do this we must give an explicit expression for M in terms of σ, n, and K. Writing $R = \xi_1/A$ in the mass-radius relation (3.5), evaluating A in terms of σ and ρ_c by equation (3.10), and remembering that $\rho_c = (\sigma c^2/K)^n$, we obtain for the total mass

$$M = (4\pi)^{-1/2}(n+1)^{3/2}G^{-3/2}K^{n/2}(\sigma c^2)^{(3-n)/2}v(\xi_1)\ . \tag{4.7}$$

For constant K and n the mass M is proportional to $\sigma^{(3-n)/2}\, v(\xi_1)$, and the proper mass of gas M_{0g} is proportional to $\sigma^{(3-n)/2}v(\xi_1)(E_{0g}/E)$. Both masses are functions only of σ if the same equation of state is to hold for all models in the sequence.

A plot of $v(\xi_1)$ versus $v(\xi_1)(E_{0g}/E)$ for $n = 3.0$ is given in Figure 8. Since $\sigma^{(3-n)/2}$ is independent of σ for $n = 3.0$, the abscissa is proportional to M_{0g} and the ordinate is proportional to M. As σ increases the curve doubles back on itself and two values of the total mass M (and hence the total energy E) are seen to correspond to some values of the rest mass M_{0g}. This is a general-relativistic feature; in the Newtonian limit the curve in Figure 8 degenerates to a single point with abscissa and ordinate both equal to 2.018.

We consider transitions between points on the upper branch of the curve in Figure 8 to points on the lower branch with the same values of rest mass, i.e., with the same abscissa. Although the dynamics of these transitions can be determined only by solving the time-dependent equations of general relativity, they are possible in principle because the system would go from an equilibrium state of higher energy (higher total mass) to one of lower energy. Since the total energy, which is proportional to $v(\xi_1)$, first decreases and then increases when going from $\sigma = 0.7$ to $\sigma = 0.31$ (see Fig. 9), there is no energy barrier separating the two states. Thus the equilibrium of the upper state (with the higher value of σ) is energetically unstable but that of the lower state is stable.

As an example we consider a transition between the state corresponding to $\sigma = 0.7$, for which $v(\xi_1) = 0.527$, and the lower energy state of equal rest mass, for which $\sigma \simeq 0.31$ and $v(\xi_1) \simeq 0.523$. The total mass after the transition is $0.523/0.527 = 0.992$ of the original mass. Suppose the original mass is $10^6\ M_\odot$, corresponding to a massive starlike body near the center of a galaxy as postulated by Hoyle and Fowler (1963*a*). Suppose further that a polytropic equation of state with $n = 3.0$ is satisfied, and the radius is such that $\sigma = 0.7$. The equilibrium of this object would be unstable, and a relatively sudden transition to a lower mass $9.992 \times 10^5\ M_\odot$ could take place. The energy equivalent of $8 \times 10^3\ M_\odot$, or 1.4×10^{58} ergs, would be released in the process. This energy is near the low end of the range (10^{58}–10^{60} ergs) of energies observed for strong radio sources. If the original mass were $10^8\ M_\odot$ instead of $10^6\ M_\odot$, the energy released would be 1.4×10^{60} ergs.

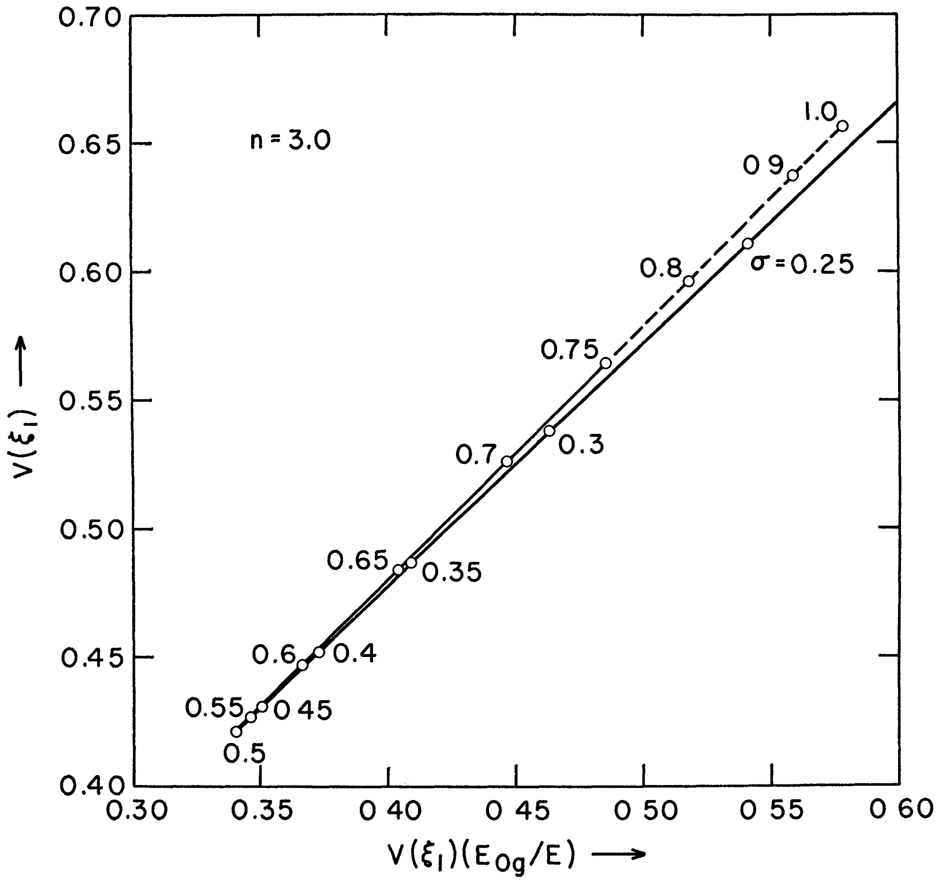

FIG. 8 —The terminal value $v(\xi_1)$ plotted versus $v(\xi_1)(E_{0g}/E)$ for $n = 3.0$. For $n = 3.0$, $v(\xi_1)$ is proportional to the active gravitational mass and $v(\xi_1)(E_{0g}/E)$ is proportional to the rest mass of the constituent particles (cf. eq. [4 6] and [4.7]). The numbers near the curve denote the value of the relativity parameter σ at the indicated points. The dotted portion of the curve denotes an extension beyond the limit $\sigma_{max} = n/(n + 1) = 0.75$ imposed by requiring the phase velocity of sound at the center to be less than the speed of light.

The order of magnitude of the energy released from a general-relativistic transition from a point on the upper branch in Figure 8 to a point on the lower branch is relatively independent of the initial value of σ. By contrast the initial and final radii and the internal structures of the two configurations depend rather strongly on σ for a given initial mass. However, to get some idea of the dimensions of the massive objects being considered, we give the radii before and after the transition from $\sigma = 0.7$ to $\sigma = 0.31$ for an initial mass of $10^6\, M_\odot$, the first of the two examples considered. For $\sigma = 0.7$, we have $\log_{10} [\sigma(n + 1)v(\xi_1)/\xi_1] = -2.06$ from Figure 6, and equation (4.1) gives $\log_{10} (\bar{R}/R_\odot) = 2.39$ or $\bar{R} = 245\, R_\odot$. For $\sigma = 0.31$, Figure 6 gives $\log_{10} [\sigma(n + 1)v(\xi_1)/\xi_1] = -1.31$ and we have $\log_{10} (\bar{R}/R_\odot) = 1.64$, or $\bar{R} = 43.6\, R_\odot$. Therefore the transition is from a larger body with a pronounced concentration of mass toward the center to a smaller body with less concentration. If the initial mass were $10^8\, M_\odot$ the respective radii would be a factor 10^2 greater.

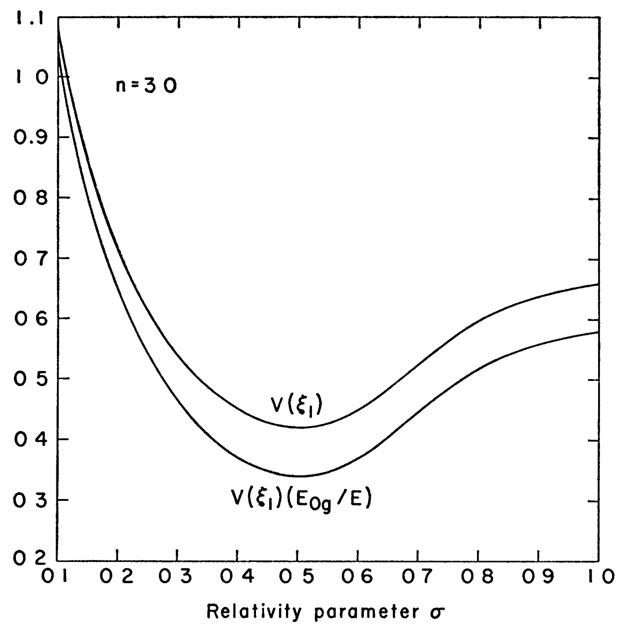

FIG. 9.—The dimensionless total mass $v(\xi_1)$ and rest mass $v(\xi_1)(E_{0g}/E)$ plotted as functions of the relativity parameter σ for $n = 3.0$.

The instabilities predicted for $n = 3$, $\sigma > 0.5$, arise as a consequence of the distortion of the spatial metric in a strong gravitational field. The energy is released by gravitational collapse from a state of unstable equilibrium to one of stable equilibrium having the same rest mass. The work that would have to be done against gravitation in order to disperse the configuration to infinity is positive before and after the transition. The magnitude of the potential energy is greater after the system has changed to the more stable configuration.

Arguments for instability of very massive stars in which radiation pressure dominates must be formulated along lines similar to the treatment given here for the polytrope $n = 3$. For each energetically unstable equilibrium state there must be a stable state having the same proper energy. Energy is released when the star undergoes gravitational collapse to a state of lower total energy. In the Newtonian gravitational theory the assumption that the ratio of radiation pressure to ideal gas pressure is constant throughout the star (the standard model) leads to a polytropic relation (with index $n = 3$) between the total pressure and the gas density. In the relativistic theory of gravitation the total energy density contains the density of radiation and kinetic energy in addition to the gas density. As the radiation pressure becomes increasingly important compared to the gas pressure, the equation of state approaches the form $P = \frac{1}{3}\rho c^2$ appropriate to radiation only. This limiting equation of state is of polytropic form with infinite index n. Thus if radiation density is included in the standard model, the polytropic index is effectively raised from $n = 3$ and the general-relativistic effects become more pronounced. The conclusions regarding the existence of unstable energy states and the order of magnitude of the energy released by transitions between states remain essentially the same as those presented here for the polytrope $n = 3$. Numerical modifications would be introduced by the specific form of the relation between the rest mass M_{0g} and the total mass M.

V. SUMMARY AND CONCLUSIONS

In the preceding discussion the general-relativistic equations of equilibrium for a compressible fluid sphere have been derived under the assumption that the energy density is related to the pressure by a power law. It was shown how the internal structure of such a fluid sphere could be determined for given mass, radius, and polytropic index in terms of the relativity parameter σ, equal to the ratio of pressure to energy density at the center of the sphere. We draw the following conclusions from the analysis and the results obtained by numerical integration:

1. The general-relativistic models have a higher concentration of mass toward the center than the corresponding Newtonian models. This is shown by the behavior of the ratio of central density to average density as a function of σ for given polytropic index n, and by the rapid falling-off of the relativistic Lane-Emden function $\theta(\xi)$ for the higher values of σ.

2. For $1.0 < n < 2.5$ the negative gravitational potential energy (in units of the total energy) of a polytropic sphere is a monotonically increasing function of σ. It is almost independent of n for $\sigma < n/(n + 1)$. For $n = 3.0$ the negative potential energy first rises and then falls as σ increases, with a maximum at $\sigma \simeq 0.5$. For the range of n considered, the magnitude of the potential energy is at most about 42 per cent of the total energy. This contrasts with the Newtonian case where the gravitational potential energy is negligible compared to the total energy Mc^2.

3. Deviation from Euclidean geometry is large even for rather modest values of the relativity parameter σ. This is shown by differences between the coordinate radius and invariant radius (which are proportional to ξ_1 and $\bar{\xi}_1$ in Table 1) and by deviations of the metric components from unity (shown in Table 2).

4. With a given index n the internal structure of a polytropic fluid sphere with given mass and radius can be determined subject to the following restrictions. If we require the phase velocity of sound at the center of the sphere to be less than or equal to the velocity of light, there exists for each n a maximum ratio of mass to invariant radius beyond which no static solutions exist. Moreover for certain masses and radii multiple solutions are possible, with two or (for $n = 3.0$) as many as three internal structures corresponding to the same mass and radius.

5. For $n = 3.0$ and $\sigma > 0.5$ the solutions are energetically unstable, with the possibility of sudden transitions from a state of higher total energy to a lower state. There is no energy barrier between these two states. General-relativistic instabilities may be a mechanism by which the high energies (up to about 10^{60} ergs) observed for some radio sources are produced. The energy can be derived from gravitational collapse of the very massive stars (up to about $10^8 M_\odot$) proposed by Hoyle and Fowler (1963*a*). If these massive starlike objects can be regarded to first approximation as polytropes of index 3.0, the collapse appears as a general-relativistic effect and would not occur for the same equation of state in Newtonian gravitational theory.

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