

THE SOLAR-FLARE PHENOMENON AND THE THEORY OF RECONNECTION AND ANNIHILATION OF MAGNETIC FIELDS

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ABSTRACT

The paper presents a study of the observations and theory which are relevant to the presently popular belief that the solar-flare phenomenon is a direct consequence of the annihilation of magnetic fields on the sun. There is very little in the observations to support such views. A systematic study of the known theoretical mechanisms for the diffusion, reconnection, and annihilation of magnetic fields is presented. The mechanisms discussed are Joule dissipation, ambipolar diffusion, and various forms of Sweet's mechanism. It is shown that Sweet's mechanism is much more effective in a highly compressible medium if the merging magnetic fields are *exactly* antiparallel. None of the known mechanisms are sufficiently rapid to account for the solar flare from the annihilation of magnetic fields. Errors and omissions in earlier discussions of the problem are cited. Since no possible energy source other than magnetic fields has yet been developed, the question of field annihilation remains both open and pressing.

The possibilities of runaway electrons and hydrodynamic instability, as means for hastening magnetic-field annihilation, are considered. It is shown that there is no reason to expect runaway electrons and no effective instability unless the fields are *exactly* antiparallel.

The observational and theoretical difficulties with the hypothesis of magnetic-field annihilation suggest that other alternatives for the flare must be explored.

I. INTRODUCTION

The solar-flare phenomenon is made up of a loose association of diverse dynamical processes occurring in the solar atmosphere. At the time of a flare, there is a visible brightening in line emission, seen both against the disk and in profile at the limb, which constitutes the *visible flare*. There are also enhanced radio-noise emission, enhanced X-ray emission, enhanced coronal heating, explosive outbursts of matter, prominence activity, acceleration of electrons and ions to subrelativistic and relativistic energies, etc. Observation does not yet make clear the interrelation of these various processes. They occur in widely varying proportions from one flare to the next. To complicate the matter further, the intensity of individual flares forms a continuous spectrum extending all the way down to what are only bright plages. Thus the theoretical problem of understanding the solar-flare phenomenon presents many facets. One of the basic questions is whether the flare is in fact the occurrence of some unique dynamical process which suddenly liberates the enormous quantities of energy consumed in the total flare, or whether it is merely an intensification of something like coronal heating which is going on in some degree all the time. Only after a provisional answer has been given to this question can one proceed to look for mechanisms to explain the detailed operation of the flare.

At the present time the popular view seems to be that a flare occurs as the result of some unique dynamical process which suddenly makes available hitherto latent energies. The view is based, quite reasonably, on the rather explosive properties of many of the larger flares. If one adopts this point of view, then the question of the energy supply for the flare seems to take on definite shape, viz., the energy for the explosion must either be *in situ* in some latent form or be supplied by a violent eruption from the vast storehouse of thermal energy in the solar interior. The latter appears dubious because of

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the failure of most flares to be visible in white light and because of the failure to observe violent disturbance in the photosphere beneath the visible flare. The assumption that the energy is initially *in situ* leads to the conclusion that it is magnetic energy, for the reason that thermal energies form the only apparent alternative and are completely inadequate. On the basis of these ideas, the central problem in the theory of the solar flare is to understand the mechanism by which magnetic energy, contained presumably in the very strong fields associated with the sunspots near the flare, can be converted quickly into thermal energy or its equivalent. A great deal of thought has gone into this problem in the past decade, and many interesting ideas have been turned up (see discussion in Cowling 1953 of some of the early ideas). Indeed, there appears to be some popular belief that the necessary annihilation of magnetic fields has been accounted for by quantitative theory.

It is the purpose of the present paper to show that, in fact, there are several difficulties, both theoretical and observational, with the idea that the solar flare is simply a consequence of the sudden annihilation of magnetic fields in an active sunspot region. The observational objections are rather straightforward and well known to many observers. The theoretical objections are based on the fact that when the known mechanisms for the conversion of magnetic energy are worked out quantitatively, they are found to be inadequate by several powers of 10 to supply the rapid energy consumption of the flare (Parker 1957*c*). The bulk of the paper is devoted to an extended quantitative exposition of the presently known mechanisms for annihilation of magnetic fields. Numerical examples are given in the last section to illustrate the inadequacy of the mechanisms. Some alternative possibilities are suggested and discussed very briefly.

Let us review some of the gross observational features of the solar-flare phenomenon. We pay particular attention to those observations which give an indication of (i) the total energy expenditure, (ii) the magnetic environment, and (iii) the magnetic changes occurring at the time of the visible flare. The observations which seem to contradict the idea that the energy for the flare comes from the annihilation of magnetic fields will be presented in italics.

Against the disk of the sun the visible flare appears as a brightening in the chromospheric line emission, showing usually a contorted region in among the individual spots of opposite polarity in an active group. The brightening always occurs in a plage region, and, in fact, it is not easy to define the difference between a bright plage and some of the fainter portions of the flare (see discussion and photographs in Kiepenheuer 1953 and in Athay and Moreton 1961). There is also the interesting phenomenon of sympathetic brightening of plages at some distance from the flare. The emission from the visible flare rises to a maximum usually within 1–15 minutes, thereafter declining, to give a total life of the order of half an hour. The rates of extension of brightening are commonly of the order of 100–400 km/sec. The visible areas of flares range from as little as 10^{17} cm² to something of the order of 0.5×10^{20} cm² for the largest. The emission region is evidently a luminous cloud as high as it is broad, with a characteristic vertical scale of 2×10^9 cm (Warwick 1955). When seen in profile on the solar limb, the flare often appears as the sudden brightening of active loop prominences over the associated sunspot group.

It is not easy to estimate the total radiative output from a solar flare. The total radiative output can be estimated in principle by integrating over the entire electromagnetic spectrum, but in practice this is very difficult because the flare is usually extremely faint compared with the solar disk. Many of the emission lines are strongly enhanced, so that they stand out clearly against the photosphere, but the continuum is much more difficult to observe, and it cannot be stated at the present time what is the sign or magnitude of its contribution.¹ The extra emission in the lines indicates a radiative output of the order of 10^{32} ergs during the life of a very large flare (Parker 1957*c*) and 10^{30} – 10^{31} ergs during the more common class 2 flare (Billings and Roberts 1953).

¹ The author is indebted to Professor R. G. Athay for this important point.

Analysis of the emission lines and the continuum suggests (Jeffries 1957; Jeffries *et al.* 1959; Jeffries and Orrall 1961*a, b*) that the mean density in the visible flare is perhaps a few times 10^{11} electrons/cm³, with a temperature somewhere between one and two times 10^4 °K. It follows that the gas is highly ionized, so that there are very few neutral atoms present. The emission lines from the visible flare show many peculiar characteristics, such as very broad wings. The broad wings have led some investigators (see, for instance, Tandberg-Hanssen and Zirin 1959) to suggest gas temperatures much higher than the $1\text{--}2 \times 10^4$ °K. It is pointed by Jeffries and Orrall (1961*a, b*), however, that the broad wings, if they represent Doppler width, are more probably the result of high-speed irregular gas motions.

It has been observed that hydrodynamic disturbances sometimes emanate from the site of the visible flare with velocities of the order of 1000–2000 km/sec across the disk of the sun (Athay and Moreton 1961) during the explosive onset of the flare. The disturbances are presumably responsible for the temporary fading of prominences at distances up to 4×10^5 km from the flare.

At the time of the visible flare there are accelerated somewhere in the vicinity of the active region copious quantities of relativistic electrons, whose synchrotron radiation gives rise to the type IV and V radio bursts observed in association with the visible flare. There are accelerated also large quantities of positive ions, some portion of which escape into interplanetary space, where they are observed at the orbit of earth. The peak flux of energetic protons above about 30 Mev may be 10^4 protons/cm² sec (see, for instance, Meyer *et al.* 1956, and the summary of proton events in Malitson and Webber 1962). The energy spectrum of the protons is sufficiently steep that most of the particle energy is near the lowest observed energy. A proton of 30 Mev has a velocity of about 0.7×10^{10} cm/sec, so that a particle flux of 10^4 protons/cm² implies a particle density of the order of 3×10^{-6} /cm³ and a total energy density of the order of 1.6×10^{-10} ergs/cm³, which is more than 10^2 times greater than the normal galactic cosmic-ray energy density in interplanetary space. Assuming that the particles at the time of peak intensity at earth occupied 1 cubic a.u., it follows that the energy of the solar protons above 30 Mev is of the order of 10^{30} ergs.

It is observed that a blast wave often arrives at earth 1 or 2 days after a flare on the visible hemisphere of the sun. The inference is that the blast wave originated at the sun at the time of the flare, presumably from the explosive heating of the local solar atmosphere to several times 10^6 °K. The velocity of the blast wave in interplanetary space is inferred to be 1 or 2 times 10^3 km/sec, based on the arrival time at earth 1 or 2 days after the flare. The blast wave is responsible for the geomagnetic storm and for the Forbush decrease in the galactic cosmic-ray intensity. Judging from the frequency with which the geomagnetic field is buffeted by such a blast wave, compared with the number of large flares on the visible hemisphere of the sun, the blast wave must be rather broad, occupying at least 1 steradian as seen from the sun. The initial phase of the geomagnetic storm, during which time the blast is pressing in on the geomagnetic field (see, for instance, Parker 1962), usually lasts several hours, suggesting that the blast-wave thickness is at least 0.1 a.u. Theoretical considerations (Parker 1961) of interplanetary blast waves suggest, too, that the thickness is probably not less than 0.1 a.u. Altogether, then, the blast wave occupies a volume not less than 0.1 a.u.³ as it reaches the orbit of earth. The observed compression of the geomagnetic field suggests, in association with an assumed velocity of 1500 km/sec, that the density of the wave is of the order of 30 atoms/cm³. Blackwell and Ingham (1961) suggest 300 atoms/cm³ on the basis of their zodiacal light observations, but the observation seems a little dubious for several reasons. Using 30 protons/cm³, which is at best only an order-of-magnitude estimate, yields a kinetic energy density of 0.6×10^{-6} ergs/cm³, a total energy of 2×10^{32} ergs, and a total mass of 2×10^{15} gm. Note that the total energy is at least as large as any of the estimates of the energy in the other features of the flare, so that we may conjecture

that one of the principal manifestations of a large solar flare is an interplanetary blast wave. It is not clear whether the interplanetary blast wave has any connection with the sudden expulsion or surge of gas from the flare during the explosive phase (Athay and Moreton 1961) or whether it arises solely from the expansion over a larger period of time of the enhanced corona associated with the active region and the flare. We are inclined to the view that, whether there is any connection with the surge or not, a major portion of the energy and momentum of the interplanetary blast wave is imparted by expansion of the enhanced corona.

This completes the summary of the observed and inferred properties of the solar flare. We take 10^{32} ergs as a very rough order-of-magnitude estimate of the energy expenditure in a large (class 3) flare event. This amount of energy corresponds to a mean flux 2×10^9 ergs/cm² sec through a typical large visible flare area (if, in fact, the visible flare area has any relevance at all) of 5×10^{19} cm² for a typical characteristic flare life of 10^3 sec. This energy flux may be compared with the flux of radiant energy 6×10^{10} ergs/cm² sec from the photosphere and with the estimated chromospheric and coronal heating requirement of 10^8 ergs/cm² sec. In terms of magnetic energy, 10^{32} ergs is equivalent to the complete annihilation of a 500-gauss field occupying a cube 2×10^4 km on a side, or 100 gauss over a cube 6×10^4 km on a side, or the reduction of a 500-gauss field to 400 gauss over a cube 3×10^4 km on a side, etc. These numbers serve to illustrate the magnitude of the magnetic changes that might be expected in a flare sight if the flare energy is magnetic in origin. Magnetic observations of active regions have been carried on for some time (Hale *et al.*, 1919; Babcock and Babcock 1955; Babcock 1959), but only recently have the sensitivity and resolution been sufficient to see the variations in field across the site of a flare. And, of course, there is need for considerable further improvement before some of the existing questions can be resolved. Briefly, Severny (1958) reports that flares seem to occur near regions where the longitudinal component of the magnetic field shows a reversal in sign and that the magnetic field is significantly changed during the course of the flare. Evans (1959) finds much the same thing except that in his observations *the field returned quickly to its initial value during the declining phase of the flare*. Evans estimated the magnetic change to be of the order of 4×10^{31} ergs during a class 1+ flare, before the sudden return to the initial value. Howard and Babcock (1960) also find *no net change in the magnetic field over the course of the flare*. Hanson and Gordon (1960) have shown that *succeeding flares in an active region may occur with identical visible profiles and shapes*, suggesting that if the flare shape is determined by the local magnetic-field configuration, then the field is not changed much from one flare to the next. Then there is the remarkable observation that the time dependence and magnitude of the X-ray emission (Fortini 1963) and of the radio emission (de Feiter and Fokker 1961) are the same for a sequence of flares in the same site, whereas they are strikingly different between flares in different sites. The observations all suggest that in some way succeeding flares follow some blueprint that is characteristic of the site in which they occur. The blueprint does not seem to be destroyed by the individual flares.

Altogether, then, it would seem that there is very little in the direct observations of solar magnetic fields to support the notion that the flare is generated by the catastrophic annihilation of magnetic fields. We turn now to the question of whether there is any known theoretical mechanism that might permit the annihilation of magnetic fields on the scale observed on the sun to produce 10^{32} ergs in the short period of 10^2 – 10^3 sec seemingly required by the visible flare. Since the problem of the annihilation and reconnection of magnetic lines of force is a subject of some interest in itself and has been conspicuously neglected in spite of its importance to a number of natural phenomena, it will be developed at some length in the sections which follow.

A brief listing of the known processes for the diffusion, reconnection, and annihilation of magnetic lines of force will serve as a guide through the theoretical sections that follow. Application of the theory to specific phenomena, such as the solar flare, is given in Section IV.

The basic means for reconnection of magnetic lines of force arises from the electrical resistivity of the gas in which the field is imbedded. If the resistivity is written $1/\sigma$ (esu), then the magnetic field in a static medium satisfies the diffusion equation

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{c^2}{4\pi\sigma} \nabla^2 \mathbf{B}. \quad (1)$$

The magnetic lines of force interdiffuse across a distance L with a characteristic time $L^2\sigma/c^2$ and characteristic velocity $c^2/\sigma L$. A number of elementary examples of the resistive reconnection of lines of force may be found in Parker and Krook (1956). It turns out that the reconnection of lines by resistive diffusion is much too slow to be of great interest in most solar and astrophysical phenomena; this remark is equivalent to the usual statement that the magnetic Reynolds number, $L\sigma/c^2$ (Elsasser 1954), is very large under most astrophysical circumstances.

There are other effects which can hasten the diffusion beyond the slow rate $c^2/L\sigma$ given above. For instance, there is the phenomenon of ambipolar diffusion, which may increase the diffusion rate by many factors of 10 in a tenuous, partially ionized gas (Piddington 1954; Cowling 1956, 1957). There is Sweet's mechanism by which the gas is squeezed out from between two antiparallel fields, allowing them to approach more closely, decreasing the scale L , and increasing the diffusion rate (Sweet 1958*a, b*). There is the possibility of diffusive transport of magnetic fields by turbulent motions in the fluid. There is the possibility of runaway electrons, leading to plasma instabilities that may greatly increase the effective resistivity.

II. AMBIPOLAR DIFFUSION

a) General Discussion

Ambipolar diffusion is by now a familiar concept, but unfortunately it has been misused in some of its applications, so we feel that an elementary presentation from a somewhat different point of view will help to clarify the problems that turn up in later discussion. Consider a gas of N_a neutral atoms/cm³ with a relatively slight degree of ionization, Δ , so that the ion and electron densities are $N_i = N_e = \Delta N_a$ particles per cm³ and $N_i \ll N_a$. Denote by σ the electrical conductivity of the medium. The electrical conductivity resides in the ionized component, of course, so regard the ionized component as one fluid with velocity $\mathbf{u}$, and the neutral component as a second fluid with velocity $\mathbf{v}$. The physical essence of ambipolar diffusion is simply that the magnetic field moves with the electrically conducting ionized component, whose motion relative to the dominant neutral component is impeded by the friction due to collisions of the ions with the neutral atoms. The magnetic-field stresses drive the ionized component toward field configurations of lower energy as rapidly as the friction will permit.

To treat the problem quantitatively, we employ the hydromagnetic equation for the magnetic field $\mathbf{B}$ carried by the conducting ionized component,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \cdot \mathbf{B}) + \frac{c^2}{4\pi\sigma} \nabla^2 \mathbf{B}. \quad (2)$$

The magnetic field is not directly affected by the non-conducting neutral component.² If the two fluids are in relative motion, there is friction between them as a consequence of collisions of the ionized component with the neutral atoms. Denote the electron and ion collision frequencies with the neutral atoms by ν_e and ν_i , respectively. Then the frictional force on the ionized component is approximately

$$\mathbf{F} = -N_i(M\nu_i + m\nu_e)(\mathbf{u} - \mathbf{v}),$$

² The electrical conductivity is reduced only slightly by the additional collisions of the electrons with the neutral component.

assuming that all relative momentum is lost at each collision. Here M represents the ion mass and m the electron mass. Since the collision cross-section of the neutrals is approximately the same for incident electrons and for incident protons, the second term in the parentheses may be neglected as small $O(m^{1/2}/M^{1/2})$ compared with the first. In addition to the frictional forces, there are the usual Maxwell stresses $(\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi$ exerted on the ionized component, so that, altogether, the motion of the ionized component is

$$N_i M_i \frac{d\mathbf{u}}{dt} \cong -\nabla p_i + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} - N_i M \nu_i (\mathbf{u} - \mathbf{v}), \quad (3)$$

where p_i is the hydrostatic pressure of the ionized component.

Assuming that the gases are of sufficiently large scale that the friction maintains $\mathbf{u}$ of the same order as $\mathbf{v}$ and remembering that $N_i \ll N_a$, it follows that the inertia of the ionized gas may be neglected in relation (3), so that relation (3) may be solved algebraically for $\mathbf{u}$ and substituted in equation (1), yielding

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{c^2}{4\pi\sigma} \nabla^2 \mathbf{B} + \nabla \times \left[\frac{1}{N_i M \nu_i} \left\{ -\nabla p \times \mathbf{B} + \frac{1}{4\pi} [(\nabla \times \mathbf{B}) \times \mathbf{B}] \times \mathbf{B} \right\} \right]. \quad (4)$$

This expression is essentially that given by Cowling (1957), indicating that the present derivation is equivalent to the somewhat different approach given by him. The terms in braces lead to dissipation of the magnetic-field energy by the simple mechanism that $\mathbf{B}$ is allowed to spring slowly back through the impeding neutral gas toward elastic equilibrium of the Maxwell stresses. The springing-back is propelled by the force $(\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi$ and is hindered only by the friction with the dense neutral component. The magnetic energy decreases while the field relaxes toward a configuration of lower strain. But note that, except for the ordinary resistivity of the gas, $1/\sigma$, equation (2) shows that the lines of force remain permanently connected in the ionized component, with motion $\mathbf{u}$. Thus ambipolar diffusion does not by itself produce reconnection of magnetic lines of force, nor does it produce any change in the total flux. Lines of force cannot be annihilated, as they can by the resistivity, but can only be shortened and their spacing readjusted as the field slowly relaxes its kinks, compressions, and extensions. On the other hand, it will be shown in later examples how the rapid relaxation of the field permitted by ambipolar diffusion may put the field into such a configuration that the resistivity can rapidly reconnect the lines of force.

Ambipolar diffusion is effective in partially ionized gases ($T \lesssim 10^4$ °K) of sufficiently low density that the ions and electrons may slip quickly through the neutral component. The characteristic time for relaxation of the field is of the order of the second term in the brackets on the right-hand side of relation (4) divided by the left-hand side. Then, if L is the scale of the field, so that ∇ is of the order of $1/L$, we have

$$\tau = \frac{L^2 N_i M \nu_i}{B^2} = \frac{L^2 w_i}{\lambda_i C^2} = \frac{L^2 N_i A_i w_i}{C^2}, \quad (5)$$

where λ_i is the mean-free path for the collision of an ion with the neutral atoms, A_i is the collision cross-section, w_i is the ion thermal velocity ($\lambda_i = 1/N_a A_i$, $\nu_i = w_i/\lambda_i$) and C is the Alfvén velocity $B/(4\pi N_a M_i)^{1/2}$ (which form tacitly assumes that the ion and neutral atom masses are equal). This diffusion time is to be compared with $L^2\sigma/c^2$ for resistive diffusion, where

$$\frac{\sigma}{c^2} = \frac{N_e e^2}{m c^2 (\nu_e + \nu)} \cong 10^{-14} T^{3/2} \left(1 + \frac{\nu_e}{\nu} \right)^{-1} \text{ sec/cm}^2, \quad (6)$$

where ν_e is again the collision frequency of an electron with the neutral atoms and ν is the collision frequency of an electron with the ions (Cowling 1953; Spitzer 1956). The

Coulomb collision cross-section is usually so large at low temperatures that $\nu_e \lesssim \nu$ even when the ionization is a few per cent or less. Note that $L^2\sigma/c^2$ is essentially independent of the gas density, whereas the ambipolar diffusion time varies as the product $N_i N_a$ when $N_i \ll N_a$. Thus ambipolar diffusion may be made arbitrarily rapid compared with resistive diffusion by going to sufficiently small densities.

b) Diffusion in One Dimension

To illustrate the field distributions which develop from ambipolar diffusion, consider the simple case $\mathbf{B} = \mathbf{e}_z B(x, t)$, where $\mathbf{e}_z$ represents a unit vector in the z -direction. Assuming no motion of the neutral gas ($\mathbf{v} = 0$) and neglecting the pressure of the ionized component ($p_i \ll B^2/8\pi$), equation (4) reduces to

$$\frac{\partial B}{\partial t} = \frac{c^2}{4\pi\sigma} \frac{\partial^2 B}{\partial x^2} + \frac{1}{4\pi\kappa} \frac{\partial}{\partial x} \left(B^2 \frac{\partial B}{\partial x} \right), \quad (7)$$

where $\kappa = N_i N_a M A_i w_i$. Assuming that σ and κ are independent of time and position, the stationary solution is

$$b \left(b^2 + \frac{3c^2\kappa}{\sigma B_0^2} \right) = \xi, \quad (8)$$

where $b \equiv B/B_0$ and $\xi = x/L$, so that B_0 represents the field density at a characteristic distance L ; B_0^3/L is a constant of integration. The second integration constant has been made to vanish by suitable choice of the origin $x = 0$. To demonstrate how in a sufficiently tenuous gas the ambipolar diffusion enhances the resistive or Joule dissipation, note that, for low gas densities, $q^2 \equiv \sigma B_0^2/c^2\kappa \gg 1$. The field varies linearly, $b \cong \xi q^2/3$, and very steeply out to the vicinity of $b = 3^{1/2}/q$, where $\xi_1 = 3^{3/2}/q^3 \ll 1$. Beyond this region the field varies as $b \sim \xi^{1/3} [1 - 1/q^2 \xi^{2/3} + \dots]$ so that it increases only rather slowly. The Joule dissipation $\mathfrak{E} = j^2/\sigma$ is thus

$$\begin{aligned} \mathfrak{E}(\xi) &= \frac{c^2 B_0^2}{16\pi^2 \sigma L^2} \left(\frac{\partial b}{\partial \xi} \right)^2 \\ &= \frac{c^2 B_0^2}{144\pi^2 \sigma L^2} \frac{1}{(b^2 + 1/q^2)^2}, \end{aligned}$$

which is sharply peaked in the vicinity of the origin. The total Joule dissipation, J_1 , in the linear interval $(0, \xi_1)$ is

$$\begin{aligned} J_1 &= L \int_0^{\xi_1} d\xi \mathfrak{E}(\xi) \\ &\cong \frac{c^2 B_0^2}{144\pi^2 \sigma L} \frac{3}{q^2} \int_0^{3^{1/2}/q} \frac{db}{(b^2 + 1/q^2)^2} \\ &= \frac{c^2 B_0^2}{16\pi^2 \sigma L} \frac{q}{6} \left(\frac{\pi}{3} + \frac{3^{1/2}}{4} \right), \end{aligned}$$

which is a measure of the total Joule dissipation. This is to be compared with the total Joule dissipation in the interval $(0, L)$ in the absence of ambipolar diffusion. The stationary solution of equation (7) is then $b = \xi$, assuming the same field strength B_0 at $x = L$, and the dissipation is readily shown to be $c^2 B_0^2/16\pi^2 \sigma L$. The presence of ambipolar diffusion, then, increases the Joule dissipation by a factor of the order of $q = (\sigma B_0^2/c^2\kappa)^{1/2} \gg 1$. The cause of the enhanced Joule dissipation is rather simple. The drift of the magnetic lines of force, representing ambipolar diffusion, proceeds rapidly except

near where the field goes to zero. There the driving force $(\nabla \times \mathbf{B}) \times \mathbf{B}$ goes to zero and the lines of force pile up, giving a very steep gradient $\partial B/\partial x$ near the origin. The Joule dissipation goes as the square of the gradient, so that the total dissipation is enhanced and, of course, concentrated near the zero of $B(\xi)$ instead of spread uniformly as in the presence of Joule dissipation alone. It is by concentrating $\partial B/\partial x$ that ambipolar diffusion enhances the Joule dissipation and the reconnection of the magnetic lines of force.

It is interesting to note that the field $B(\xi)$ outside the narrow region $\xi < \xi_1$ of intense Joule dissipation is essentially unaffected by the precise value of the electrical conductivity. This may be seen directly from the asymptotic form of $b(\xi)$ for $\xi \gg \xi_1$, in which q does not appear in the leading term. The importance of this point is that it is not necessary to include the first term on the right-hand side of equation (7) when considering ambipolar diffusion unless we are concerned with the field in the immediate vicinity of a zero of $B(\xi)$. The Joule term does not affect the general drift of the field in regions far removed from the neighborhood of vanishing B . We will take advantage of this fact in some of the calculations presented later on.

Let us return briefly to a point made earlier, that ambipolar diffusion does not annihilate or reconnect magnetic lines of force. Within the context of the present one-dimensional example, note from equation (7) that the total flux between any two points $x = a$ and $x = b$ varies as

$$\frac{\partial}{\partial t} \int_a^b dx B(x) = \frac{1}{4\pi\kappa} B^2 \frac{\partial B}{\partial x} \Big|_a^b,$$

so far as the effects of ambipolar diffusion alone are concerned. The quantity $(4\pi\kappa)^{-1} B^2 \partial B/\partial x$ on the right-hand side is just B times the drift velocity $\mathbf{u}$ computed from equation (3). If a and b are chosen so as to be zeros of $B(x)$, then the rate of transport of magnetic lines of force, proportional to $B^2 \partial B/\partial x$, vanishes at a and b , provided that $\partial B/\partial x$ is finite at a and b . But the presence of Joule dissipation, no matter how small, is sufficient to guarantee that $\partial B/\partial x$ is everywhere finite. Hence $B^2 \partial B/\partial x$ will always vanish at the zeros of $B(x)$, and the total magnetic flux between the zeros is conserved,

$$\frac{\partial}{\partial t} \int_a^b dx B(x) = 0.$$

In sharp contrast with this conservation of magnetic lines of force by ambipolar diffusion is the well-known annihilation of magnetic lines of force by the resistivity, Joule dissipation. The net transport of flux then is

$$\frac{c^2}{4\pi\sigma} \frac{\partial B}{\partial x} \Big|_a^b,$$

which clearly does not vanish when a and b are zeros of $B(x)$.

Additional solutions of equation (7) are given in Appendix I. The propagation of a wave of the form $B(x - vt)$ is discussed to demonstrate the diffusion velocity of fields by ambipolar diffusion. The asymptotic decay of regions of magnetic field with alternate signs is considered as an example of the decay of magnetic filaments in the solar atmosphere.

c) General Energy Equation

Consider the development of the general energy relation for ambipolar diffusion. Take the scalar product of $\mathbf{B}$ with relation (4). Suppose that the neutral component is motionless ($\mathbf{v} = 0$), and neglect Joule dissipation and the pressure of the ionized component. Then it can be shown, upon applying well-known vector identities, that

$$\frac{\partial}{\partial t} \left(\frac{B^2}{8\pi} \right) = -\frac{1}{4\pi} \nabla \cdot (B^2 \mathbf{u}) - \kappa u^2. \quad (9)$$

Since $\mathbf{u}$ is perpendicular to $\mathbf{B}$, the condition that $\mathbf{E} = -\mathbf{u} \times \mathbf{B}/c$ yields $\mathbf{u} = c\mathbf{E} \times \mathbf{B}/B^2$, so the first term on the right-hand side is the divergence of the Poynting vector, representing the convection of magnetic energy by the drift velocity $\mathbf{u}$. The second term on the right-hand side is the energy loss due to friction with the neutral medium. If we integrate over the entire volume of the field, so that B vanishes everywhere on the boundary, then the convection term $\nabla \cdot (B^2\mathbf{u})$ drops out and the total energy loss is just the total frictional dissipation $\kappa \int dV u^2$, which energy comes from the relaxing magnetic field.

III. SWEET'S MECHANISM

a) General Remarks

Sweet's mechanism (Sweet 1958*a, b*) for enhanced diffusion of magnetic fields is most simply described by considering two regions of conducting fluid containing approximately uniform magnetic fields. Let one of the regions be oriented so that its magnetic field is antiparallel to the field of the other. Then press the two regions together firmly along

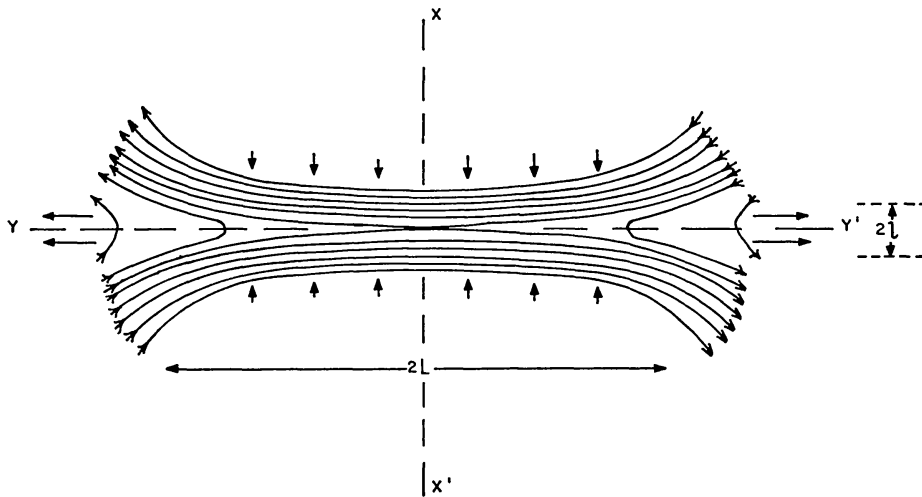


FIG. 1.—Schematic representation of the magnetic lines of force of two antiparallel regions of magnetic field pressed firmly together, with the fluid caught between free to escape out the ends Y and Y' . The short arrows show the direction of the fluid motion as the fields diffuse together. The characteristic scales l and L are indicated. The short arrows show the direction of the fluid motion as the fields diffuse together.

their sides. Unless we take special measures to prevent it, the fluid will immediately begin to squeeze out from between the two regions of field. The squeezing of the fluid from between the two fields allows the fields to approach more and more closely to each other. The field gradient in the region between the two fields increases without limit as the fluid continues to squeeze from between the fields. The Joule dissipation increases without limit, and the fields may be rapidly merged.

A simple order-of-magnitude discussion will serve to illustrate the physical principles involved (Parker 1957*b*) before going on to more quantitative calculations. Figure 1 is a schematic representation of the lines of force of two oppositely directed magnetic fields pressed firmly together. There is a neutral point at the center and a plane of symmetry between the two regions of field, marked YY' , on which the tangential component of the magnetic field is zero. The short, straight arrows indicate the motion of the fluid, carried slowly with velocity v by the fields in toward YY' over the width of the region and squirting out rapidly with velocity V at each end from between the fields. Let $2L$ represent the width of the regions of field, and $2l$ the width of the thin transition layer between the fields from which the fluid is squeezing out.

Application of the four principles—conservation of mass, momentum balance, conservation of energy, and diffusion of magnetic fields—serves to determine the merging

velocity v . For incompressible flow, mass is carried into the transition region by the merging fields at a rate of the order of $2LN_0Mv$, where N_0 is the number of atoms/cm³ and M is the mass of each atom. Mass flows out from between the fields with a velocity V over a width of the order of $2l$, so that the efflux is $2lN_0MV$. Equating the influx and efflux for steady conditions leads to $vL = Vl$ for conservation of mass.

The conditions of momentum balance and conservation of energy combine to give the obvious result that in the absence of viscous dissipation the fluid will be ejected from between the two fields with a velocity V of the order of the Alfvén velocity $C_0 = B_0/(4\pi N_0 M)^{1/2}$, where B_0 is the strength of the field outside the thin transition region.

Finally, the velocity with which the fields are diffusing into each other is $v = c^2/l\sigma$, as discussed earlier.

It is a simple matter to solve the three equations $Lv = lV$, $V = C_0$, $v = c^2/l\sigma$, obtaining

$$v = c \left(\frac{C_0}{L\sigma} \right)^{1/2}, \quad l = \left(\frac{c^2 L}{\sigma C_0} \right)^{1/2}. \quad (10)$$

This result was obtained earlier (Parker 1957*b*). An alternative form for these relations is in terms of the magnetic Reynolds number R_0 , defined as $R_0 = L\sigma C_0/c^2$ for the characteristic scale L and velocity C_0 . Then

$$v = C_0 R_0^{-1/2} = \frac{c^2}{L\sigma} R_0^{+1/2}.$$

With Sweet's mechanism the merging velocity is smaller than the Alfvén velocity by $R_0^{-1/2}$ and larger than the ordinary diffusion velocity $c^2/L\sigma$ by $R_0^{+1/2}$.

To treat the case of compressible flow in a similar manner, note that if the ambient gas density of N_0 atoms/cm³ is compressed to N_m /cm³ as it is carried with velocity v into the region between the fields, then conservation of matter becomes $N_0Lv = N_m lV$. The density N_m is such that $N_m kT \cong B_0^2/8\pi$. The velocity V with which the fluid is ejected is the Alfvén velocity computed for the density N_m , and hence is essentially the thermal velocity or the speed of sound in the transition region. Denoting this by a , we obtain

$$v = c \left(\frac{N_m}{N_0} \right)^{1/2} \left(\frac{a}{L\sigma} \right)^{1/2}, \quad l = \left(\frac{N_0}{N_m} \right)^{1/2} \left(\frac{c^2 L}{\sigma a} \right)^{1/2} \quad (11)$$

in place of equation (10). This order-of-magnitude result has been given previously by Parker (1957) and Hoyle and Wickramasinghe (1961). The velocity is larger, for the same ejection velocity, by the square root of the compression factor $(N_m/N_0)^{1/2}$. The thickness of the transition region is correspondingly smaller. The enhanced diffusion resulting from compressibility has not been presented in previous discussions. The physical reason for the enhanced merging velocity is the increased mass ejection rate $N_m lV$ from between the fields with the higher density N_m .

To include ambipolar diffusion, for which the diffusion coefficient is characterized by $B_0^2/4\pi\kappa$ (see the second term on the right-hand side of eq. [7]), it is only necessary to substitute for the Joule diffusion coefficient c^2/σ in equations (10) and (11). The friction coefficient κ should be computed for the density N_m at the center of the transition region, so that if Δ is the fraction of atoms that are ionized, we have $N_i N_a \cong \Delta N_a^2 \cong \Delta N_m^2$. Then, since $\kappa = N_i N_a M_i A_i w_i$ and $B_0^2/(4\pi N_m M)^{1/2} \cong a$, we obtain the expressions

$$v \cong a \frac{1}{(\Delta L A_i N_0)^{1/2}}, \quad l = \left(\frac{N_0}{N_m} \right)^{1/2} \left(\frac{L}{\Delta N_m A_i} \right)^{1/2} \quad (12)$$

upon assuming that $w_i \cong a$.

It has been assumed so far that the merging magnetic fields are *exactly* antiparallel.

This seems somewhat improbable in nature, so it is instructive to consider the possibility that there may be a general component of magnetic field in the z -direction, i.e., in the direction perpendicular to the plane occupied by the antiparallel fields discussed so far. If this field is assumed to have a magnitude $B_1 \ll B_0$, then the two main fields will make an angle $\pi - 2\theta$, where $\theta = B_1/B_0$, instead of being exactly antiparallel with an angle π between their directions. The z -component diffuses rather more slowly than the antiparallel fields $\pm B_0$ because it does not reverse sign across the transition region. To a first approximation, neglect the diffusion of the z -component B_1 . It follows that B_1 is compressed along with the fluid, so that $B_z \cong B_1(N/N_0)$. The total pressure at the center of the transition region is the gas pressure $N_m kT$ plus the magnetic pressure $(B_1^2/8\pi)(N_m/N_0)^2$. Since in many applications $N_m/N_0 \gg 1$, it follows that the magnetic pressure of the z -component may be large, even if $B_1 \ll B_0$, seriously limiting the compression N_m/N_0 of the gas. A glance at equations (11) indicates that the merging velocity v may then be significantly reduced by the presence of B_1 .

With this brief exposition of the various forms of Sweet's mechanism to be treated quantitatively in the sections which follow, now consider the basic hydrodynamic and hydromagnetic equations on which a quantitative treatment may be founded. The physical principles involved are, of course, just those stated above in the qualitative discussion. Conservation of mass under quasi-stationary conditions is represented by

$$\nabla \cdot (N\mathbf{v}) = 0. \quad (13)$$

Momentum balance is described by

$$NM(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \left(p + \frac{B^2}{8\pi} \right) + (\mathbf{B} \cdot \nabla) \frac{\mathbf{B}}{4\pi}.$$

Noting that under most circumstances the merging velocity of the antiparallel fields is so slow that its inertia $(\mathbf{v} \cdot \nabla)\mathbf{v}$ may be neglected, and the lines of force in the vicinity of the transition region are so straight and uniform that the tension term $(\mathbf{B} \cdot \nabla)\mathbf{B}/4\pi$ has negligible effect, the momentum equation reduces to

$$\frac{\partial}{\partial x} \left(p + \frac{B^2}{8\pi} \right) = 0$$

along XX' (see Fig. 1) across the transition region. If the ambient gas pressure outside the transition region is p_0 , then this expression can be integrated to give

$$p(x) + \frac{B^2(x)}{8\pi} = p_0 + \frac{B_0^2}{8\pi} \quad (14)$$

along XX' . At the center of the transition region ($x = 0$) we have $B = 0$, so that the gas pressure is $p(0) = p_0 + B_0^2/8\pi$ and is higher than the ambient gas pressure by $B_0^2/8\pi$. It is this pressure differential along the transition region YY' (see Fig. 1) between the neutral point and the surrounding space that causes the fluid to flow rapidly out from between the fields. The inertial term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ in the momentum equation is not negligible for the flow in the y -direction out from between the fields. It will be necessary to use energy considerations to obtain an integral for the motion in the y -direction in the sections that follow. Note from Figure 1 that the few lines of force that cross YY' are in such a configuration that their principal effect is for their tension term $(\mathbf{B} \cdot \nabla)\mathbf{B}$ to assist the expulsion of fluid from between the fields. This is a weak effect, however, and will be omitted from the calculations.

Finally, consider the stationary hydromagnetic diffusion equation,

$$0 = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nu \nabla \times (\nabla \times \mathbf{B}),$$

where $\nu \equiv c^2/4\pi\sigma$. Integrating this equation yields

$$0 = \mathbf{v} \times \mathbf{B} - \nu \nabla \times \mathbf{B} - \nabla \psi ,$$

where ψ is a general scalar function of position. In the two-dimensional case considered here $\mathbf{v}$ and $\mathbf{B}$ are confined to the xy -plane and are functions only of x and y (near XX' they are functions only of x). Hence both $\mathbf{v} \times \mathbf{B}$ and $\nabla \times \mathbf{B}$ are in the z -direction. Hence ψ must be independent of x and y . Put $-\nabla \psi = \mathbf{e}_z v_0 B_0$. Then in the vicinity of XX' , where $\mathbf{v} \cong \mathbf{e}_x v(x)$ and $\mathbf{B} \cong \mathbf{e}_y B(x)$, the equation becomes

$$\nu \frac{dB}{dx} = vB - v_0 B_0 . \quad (15)$$

Since dB/dx becomes small at large x , where $B \rightarrow B_0$, it follows that v_0 represents the fluid velocity there ($v_0 < 0$).

Equations (13), (14), and (15), together with an energy principle to describe the fluid motion in the y -direction, form the basis for the quantitative examples which follow.

b) Incompressible Flow, Joule Dissipation

The simplest example of Sweet's mechanism is an incompressible fluid with constant electrical resistivity enmeshed in two exactly antiparallel fields. For incompressible flow, equation (13) yields $\partial v_x / \partial x = -\partial v_y / \partial y$, where y is the co-ordinate measured along YY' from the neutral point in Figure 1. To estimate $\partial v_y / \partial y$ in the vicinity of XX' ($y = 0$), note from Bernoulli's law (conservation of energy) that the pressure differential $p(x) - p_0$, between the hydrostatic pressure $p(x)$ on XX' and the pressure p_0 in the surrounding space, leads to an expulsion velocity V along the lines of force through x , given by

$$\frac{1}{2} \rho V^2(x) = p(x) - p_0 . \quad (16)$$

It will be assumed that

$$\frac{\partial v_y}{\partial y} \cong \alpha \frac{V(x)}{L} = \frac{\alpha}{L} \left[\frac{2(p - p_0)}{\rho} \right]^{1/2}$$

in the vicinity of XX' , where α is a constant of the order of unity. Denote by $v(x)$ the merging velocity v_x of the fluid along XX' . It follows from equation (13) that

$$\frac{dv}{dx} = -\frac{\alpha C_0}{L} \left[1 - \frac{B^2(x)}{B_0^2} \right]^{1/2} \quad (17)$$

after using equation (14) to express $p(x)$ in terms of $B(x)$.

The magnetic field is determined by the hydromagnetic diffusion equation (15). Equations (15) and (17) together are sufficient to determine $v(x)$ and $B(x)$. The boundary conditions are that $v(0) = B(0) = 0$ and that v reaches v_0 when B becomes equal to B_0 .

Equation (17) is an approximate relation because of the manner in which $\partial v_y / \partial y$ was estimated, from energy considerations and the characteristic scale L . The energy arguments leading up to equation (17) were essentially those for a system without dissipation, i.e., no turbulence or viscosity. Had there been viscous dissipation, for instance, the expulsion would have been more nearly proportional to $p - p_0$ instead of $(p - p_0)^{1/2}$. To show that the form of the pressure dependence has little effect on the merging velocity v , we rewrite equation (16) as

$$\frac{dv}{dx} = -\frac{\alpha^2 C_0}{L} \left[1 - \frac{B^2(x)}{B_0^2} \right]^\gamma , \quad (18)$$

where γ is a constant whose effective value is expected to lie somewhere between 0.5 and 1.0. The values $\gamma = 0.5$ and $\alpha \cong 1$ are expected for the system in the absence of dissipation in the fluid motions.

For the mathematical discussion it will be convenient to express equations (15) and (18) in terms of the dimensionless velocity $u = v/v_0$, field $b = B/B_0$, and distance $\xi = -v_0 x/\nu$ ($v_0 < 0$), the last defined being in effect the magnetic Reynolds number for the merging velocity v_0 out to the distance x . We obtain

$$\frac{db}{d\xi} = 1 - ub, \quad (19)$$

$$\frac{du}{d\xi} = S^2(1 - b^2)^\gamma, \quad (20)$$

where $S^2 \equiv \alpha^2 C_0 \nu / L v_0^2 > 0$. Note that for all values of γ there exists the equilibrium solution $u(\xi) = b(\xi) = 1$.

Divide equation (20) by equation (19), obtaining the differential equation

$$\frac{du}{db} = S^2 \frac{(1 - b^2)^\gamma}{1 - ub} \quad (21)$$

for $u(b)$. The problem is to determine the numerical value of S such that the solution $u(b)$ of equation (21) through the origin passes through $u(1) = 1$. The merging velocity v_0 then follows from S . The dependence of u and b on position follows from either equation (19) or equation (20) by quadrature, once $u(b)$ is known. It will be sufficient for the present purposes to consider the three cases $\gamma = 0, 0.5$, and 1.0 . It is readily shown for $\gamma = 0$ that

$$b(u) = \frac{1}{S^2} \int_0^u dw \exp\left(\frac{w^2 - u^2}{2S^2}\right), \quad u(\xi) = S^2 \xi, \quad S = 0.76.$$

For $\gamma = 1$ it is readily shown that

$$u(b) = b, \quad b = \tanh \xi, \quad S = 1.0.$$

For $\gamma = 0.5$ it is necessary to use approximate methods. Noting that near the origin u and b are both small compared with 1, equation (21) reduces to $du/db = S^2$ as a first approximation. Hence $u \cong S^2 b$. To obtain a better approximation, it is a simple matter to eliminate u from the denominator of the right-hand side of equation (21), using $u = S^2 b$, obtaining

$$\frac{du}{db} = S^2 \frac{(1 - b^2)^\gamma}{1 - S^2 b^2}. \quad (22)$$

This approximation, with the present boundary conditions, is sufficient to estimate S^2 . Integrating equation (22) for $\gamma = 0.5$ gives

$$u(b) = \arcsin b - (1 - S^2)^{1/2} \arctan \left[\frac{(1 - S^2)^{1/2} b}{(1 - b^2)^{1/2}} \right]. \quad (23)$$

Putting $u(1) = 1$ gives $S = 0.93$. On the other hand, equation (22) is a poor representation of $u(b)$ in the immediate vicinity of $b = 1$, which is important for the computation of $u(\xi)$ and $b(\xi)$. To compute $u(b)$ near $b = 1$, let $u = 1 - \epsilon$, $b = 1 - \delta$. Then it follows from equation (20) that

$$\epsilon(\delta) = \frac{S2^{(\gamma+1)/2}}{(\gamma+1)^{1/2}} \delta^{(\gamma+1)/2} \{ 1 + O[\delta^{(1-\gamma)/2}] \} \quad (24)$$

for $0 < \gamma < 1$. It follows from equation (19) that

$$\xi = \xi_1 - \int_b^1 \frac{db}{1 - ub} \quad (25)$$

$$\begin{aligned} &\cong \xi_1 - \int_0^\delta \frac{d\delta}{\epsilon(\delta)} \\ &= \xi_1 - \frac{(\gamma + 1)^{1/2} (1 - b)^{(1-\gamma)/2}}{2^{(\gamma-1)/2} (1 - \gamma) S}, \end{aligned} \quad (26)$$

where ξ_1 is the position at which b becomes equal to 1.0. It is obvious that the distance $\xi_1 - \xi$ beyond ξ at which b reaches 1.0 is finite for all $0 \leq \gamma < 1$. Hence for $0 \leq \gamma < 1$ the magnetic field B and the velocity v reach B_0 and v_0 at some finite distance from the center of the transition. Beyond ξ_1 the solution is just the uniform drift $v = v_0$, $B = B_0$ all the way to the outer boundary of the region of field.

Note that, for $\gamma = 0$, ξ_1 has the value 1.7; for $\gamma = 0.5$, ξ_1 is something of the order of 3. For $\gamma = 1$, ξ_1 is infinite, but the value of $db/d\xi$ at $\xi = 0$ is 1.0. This fixes the scale l of the transition region as $0(\nu/v_0)$ for all γ between 0 and 1.0, as assumed in the earlier order-of-magnitude treatment.

It is possible to use equation (24) in combination with equation (23) to obtain a better approximation for S than that given by equation (23) alone. Requiring that equations (23) and (24) join at, say, $\xi = 0.9$ leads to $S = 0.87$. The value 0.93 was obtained from equation (23) alone, indicating only a 7 per cent correction. It is important to note, for the approximate solutions that follow in the next section, that the approximation represented by equation (22) gives a fairly good approximation for S for all values of γ from 0 to 1.0. For $\gamma = 0$, equation (22) gives $Sb = \tanh(u/S)$, so that $S = 0.833$. This is to be compared with the exact value $S = 0.76$. For $\gamma = 1$, equation (22) is exact, giving $u = b$ and $S = 1.0$.

Altogether, then, we have the results $S = 0.76$, $S = 0.87$, and $S = 1.0$ for $\gamma = 0$, 0.5, and 1.0, respectively. The merging velocity v_0 is computed from the definition of S as

$$v_0 = -\frac{a}{S} \left(\frac{C_0 \nu}{L} \right)^{1/2}. \quad (27)$$

This expression differs from the order-of-magnitude value (10) by a factor of the order of only 2 or 3. Note that S varies only about 30 per cent between $\gamma = 0$ and $\gamma = 1$. It follows that v_0 varies only by about 30 per cent and is relatively insensitive to γ . The physical reason for the insensitivity is that most of the fluid squeezes out from between the fields near $x = 0$, where b^2 is rather less than 1. The factor $(1 - b^2)^\gamma$ is then approximately 1.0 regardless of the value of γ .

The function $u(b)$ is plotted in Figure 2 for $\gamma = 0$, 0.5, and 1.0. The curve labeled $\gamma = 0.5$ is computed from equation (23). The unlabeled curve is the composite of equations (23) and (24) for $\gamma = 0.5$ joined at $\xi = 0.9$. The functions $u(\xi)$ and $b(\xi)$ are given in Figure 3.

As we have already noted, the present calculations are approximate because they are based in part on equation (20). We have shown that there is no critical dependence of the merging velocity v_0 on γ . Nonetheless, it would be instructive if the exact stationary equations could be integrated on a machine. Not only would such a computation check the detailed accuracy of the values of v_0 obtained in the present discussion, but presumably they would show quantitatively the flow pattern of the fluid squeezing out from between the fields, which the present formulation is forced to avoid. Because of the interest in the exact problem, the dynamical equations are noted in Appendix II.

c) Incompressible Flow, Ambipolar Diffusion

When ambipolar diffusion is present, we must add to equation (15) the second term on the right-hand side of equation (7), so that the equation equivalent to (19) becomes

$$\frac{db}{d\xi} = \frac{1 - ub}{1 + q^2 b^2}, \quad (28)$$

where $q^2 = B_0^2/(4\pi\kappa\nu)$, representing the ratio of ambipolar diffusion to Joule dissipation. Assume q to be constant, then in place of equation (21) we have

$$\frac{du}{db} = S^2 \frac{(1 - b^2)^\gamma (1 + q^2 b^2)}{1 - ub}. \quad (29)$$

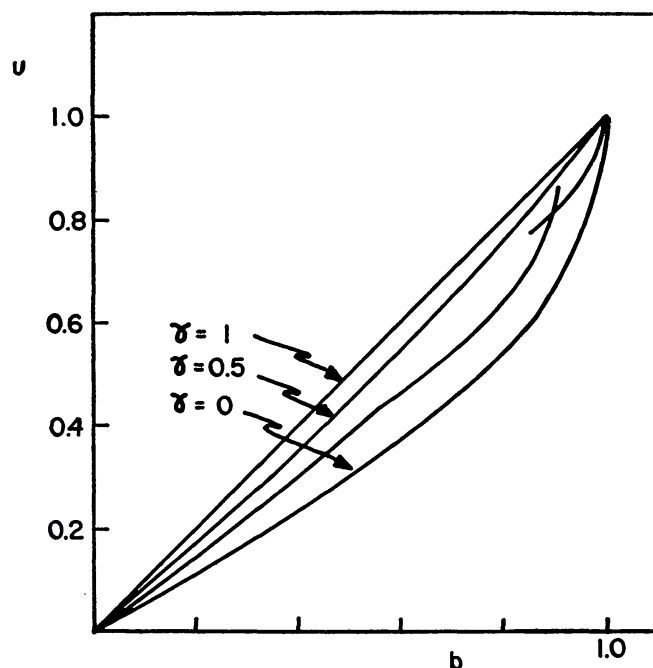


FIG. 2.—Plot of $u(b)$ in incompressible flow with Joule dissipation for $\gamma = 0, 0.5$, and 1.0 in eq. (20). The unlabeled curve, in two pieces, is the approximate form obtained by joining eqs. (23) and (25) at $\xi = 0.9$.

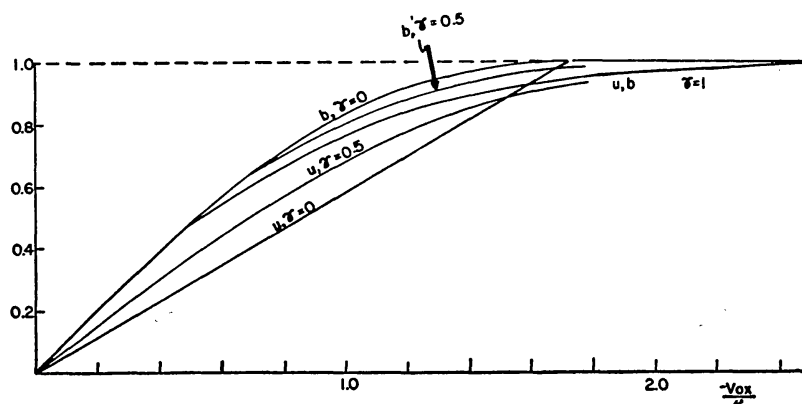


FIG. 3.—Plot of $u(\xi)$ and $b(\xi)$ in incompressible flow with Joule dissipation for $\gamma = 0, 0.5$, and 1.0 in eq. (20).

If ambipolar diffusion is important, then $q \gg 1$, and 1 may be neglected compared with $q^2 b^2$ except for $b \lesssim 1/q$. In this region, $b \lesssim 1/q$, we have

$$\frac{du}{db} = S^2(1 + q^2 b^2) \left[1 + O\left(\frac{1}{q^2}\right) \right],$$

$$u \cong S^2 b \left(1 + \frac{1}{3} q^2 b^2 \right). \quad (30)$$

The first term in the parentheses represents the narrow linear portion of the curve in the vicinity of the origin. It was shown in Section II that we can ignore it, in the limit

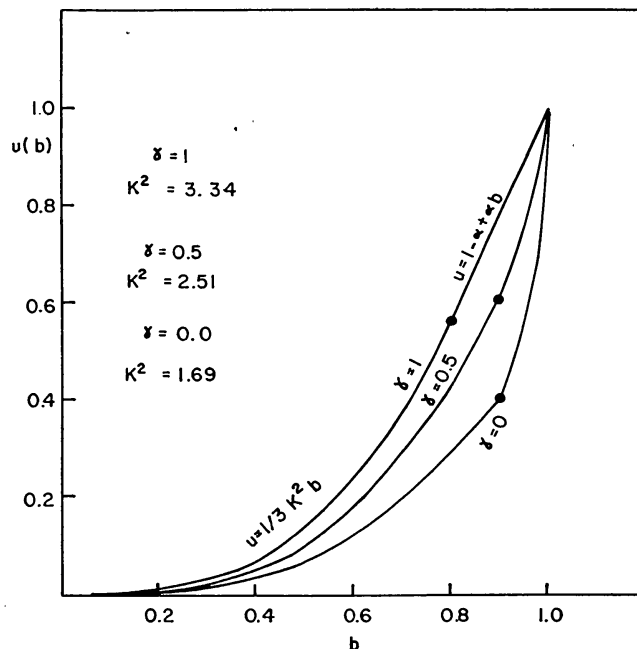


FIG. 4.—Plot of $u(b)$ in incompressible flow with ambipolar diffusion for $\gamma = 0, 0.5$, and 1.0 in eq. (20). The corresponding values of K are indicated.

of large q , without significantly affecting the computation of the merging velocity or the fields elsewhere in $b \gg 1/q$. Thus we confine our attention to solution of

$$\frac{du}{db} = K^2 \frac{b^2(1 - b^2)^\gamma}{1 - ub}, \quad (31)$$

where $K^2 \equiv q^2 S^2$.

To obtain an estimate of K^2 , note that, for $b^2 \ll 1$, we have $u \sim \frac{1}{3} K^2 b^3$. For $1 - b^2 \ll 1$ and $\gamma \neq 1$, we have equation (24) with S replaced by K ; for $\gamma = 1$, we have $Z\epsilon = [(1 + 8K^2)^{1/2} - 1]\delta$. Joining the solutions for $b^2 \ll 1$ and $1 - b^2 \ll 1$ at $b = 0.9$ for $\gamma = 0, 0.5$, and at $b = 0.8$ for $\gamma = 1$ yields $K = 1.30, 1.58$, and 1.83 , respectively. The resulting functions $u(b)$ are plotted in Figure 4. An alternative computation, expanding $u(b)$ as

$$u(b) = \frac{1}{3} K^2 b^3 (1 + a_1 b^2 + a_2 b^4 + a_3 b^6 + \dots)$$

and requiring that $u(1) = 1$ gives $K = 1.41$ and $K = 1.77$ for $\gamma = 0$ and 0.5 , respectively, suggesting that these simple estimates of K may be good to 10 or 20 per cent. More accuracy is hardly justified in view of the approximate form of equation (20). Note again the relative insensitivity of K^2 to γ .

The merging velocity follows from the definition of S as

$$v_0 = -\frac{a}{K} \left(\frac{C_0 \nu}{L} \right)^{1/2} q, \quad (32)$$

in agreement with equation (12) except for a factor of the order of unity. Comparing this with equation (24) and remembering that, with Joule dissipation alone, S is of the order of unity, whereas, with ambipolar diffusion, K is of the order of unity, we note that v_0 is increased by the presence of ambipolar diffusion by the factor q , i.e., by the square root of the ratio of ambipolar diffusion to Joule dissipation.

When we come to compute $u(\xi)$ and $b(\xi)$, we again obtain the asymptotic form (24), except that S is replaced by K and there is a factor q^2 in the numerator. Thus the point ξ_1 at which $u(\xi)$ and $b(\xi)$ reach 1.0 is again finite for $\gamma < 1$, but ξ_1 is larger by the factor q^2 . Beyond ξ_1 the solution is again $u(\xi) = b(\xi) = 1$.

Now v_0 is increased by the factor q , and ξ_1 by the factor q^2 , above the magnitudes with Joule dissipation alone. It follows that x_1 (the scale of the transition region in cm) is larger by the factor q . On the other hand, since $b \cong \xi$ in the narrow region ($b \lesssim 1/q$) of Joule dissipation, it follows that the width $\xi \cong 1/q$ of the Joule dissipation is $x \cong \nu/v_0 q$, and is narrower by the factor $1/q^2$. The dissipation is proportional to the square of dB/dx , which is proportional to $v_0 db/d\xi$, and is more intense by the factor q^2 . Hence the total power dissipated by the electrical resistivity is of the same order of magnitude as in the absence of ambipolar diffusion. The total rate at which field energy is annihilated is proportional to v_0 , which is increased by the factor q . Most of this energy goes into the friction between the ionized and neutral components in the region $x < x_1$. The flux of magnetic energy into the region of Joule dissipation is of the order of $1/q$ of the total.

d) Compressible Flow

Thus far the discussion has dealt with incompressible flows, which are applicable to problems in nature where the ambient pressure p_0 is large compared with the magnetic pressure $B_0^2/8\pi$. There are many instances, however, when, instead of p_0 being large compared with $B_0^2/8\pi$, it is small. The result is that the gas carried into the transition region by the merging fields may be enormously compressed, by a factor of 10^4 , or more, in some cases as the gas pressure, given by equation (14), increases to its maximum value $p_0 + B_0^2/8\pi$ at the neutral point. It was shown in equation (11) that the compression enhances the merging velocity of the fields by the square root of the compression factor, so the effect is of considerable importance. It will be sufficient for the present to limit the quantitative discussion to the simple case of isothermal compression. We might imagine that radiative equilibrium maintains the temperature in the vicinity of some uniform value, T_0 , while the gas is being carried into the transition region. It will be convenient to replace the pressure $p(x)$ with the density $N(x)$ ($p = NkT_0$). It will also be convenient to use the density $N_m = B_0^2/8\pi kT_0$, which is the density at the neutral point if $p_0 = 0$, employing N_m as a unit of particle density, defining the dimensionless number density $n(x) = N(x)/N_m$. Then equation (14) may be rewritten

$$n - n_0 = 1 - b^2, \quad (33)$$

where $n_0 = N_0/N_m$ and N_0 is the ambient density before compression in the field. In place of equation (20), let a represent the characteristic ejection velocity, so that the gas is ejected with a velocity

$$v_y = a \frac{N - N_0}{N}.$$

The mass flow then is $Nv_y = a(N - N_0)$. Presumably, a is of the order of the sound velocity $(3kT_0/M)^{1/2}$, which is of the order of the Alfvén velocity $B_0/(4\pi N_m M)^{1/2}$. It follows that

$$\frac{\partial}{\partial y} Nv_y \cong \frac{a^2 a}{L} (N - N_0)$$

in the vicinity of XX' ($y = 0$ in Fig. 1). From equation (13) it follows that

$$\frac{\partial}{\partial x} (Nv_x) = -\frac{a^2 a}{L} (N - N_0)$$

or

$$\frac{d(nu)}{d\xi} = R^2 (n - n_0), \quad (34)$$

where $R^2 = a^2 a \nu / L v_0^2$. Finally, note that the generalized diffusion equation (28) should be modified to take account of the fact that q^2 can be expected to depend upon n , instead of being a constant. In particular, the ambipolar diffusion rate is proportional to $1/N_i N_a$. Since it is usually assumed, when ambipolar diffusion is important, that the medium is only slightly ionized, put $N_a \cong N$ and $N_i = N\Delta$, where $\Delta (\ll 1)$ represents the fraction of the atoms that are ionized. The ionization may increase or decrease with increasing N . Thus put $q^2 = Q^2/n^\beta$, so that $\beta = 0$ represents N_i varying inversely with N_a ; $\beta = 1$ represents N_i constant; and $\beta = 2$ represents Δ constant.³ In place of equation (28) we have

$$\frac{db}{d\xi} = \frac{1 - ub}{1 + Q^2 b^2 / n^\beta}. \quad (35)$$

Note that $b = u = 1$, $n = n_0$ is a solution of equations (34) and (35), as in the previous cases.

Divide equation (34) by equation (35) and use equation (33) to eliminate n . The result is

$$\frac{d}{db} [(n_0 + 1 - b^2)u] = \frac{R^2 (1 - b^2) [1 + Q^2 b^2 (n_0 + 1 - b^2)^{-\beta}]}{1 - ub}. \quad (36)$$

We must now integrate equation (36) and determine the value of R such that the solution through $u = b = 0$ passes through $u = b = 1$. We are particularly interested in the limit of large compression, i.e., $n_0 \ll 1$. It will be shown presently that, in the limit as n_0 becomes small, R becomes small like a positive power of n_0 . It follows that $u \ll b$ except as $b \rightarrow 1$. Thus, except as $b \rightarrow 1$, we have $ub \ll 1$ and

$$\frac{d}{db} [(n_0 + 1 - b^2)u] \cong R^2 (1 - b^2) [1 + Q^2 b^2 (n_0 + 1 - b^2)^{-\beta}]. \quad (37)$$

But by the time b has approached sufficiently close to 1.0 that ub is non-negligible, we find that $1 - b^2$ is essentially zero, $(1 - b^2) \ll (1 - ub)$, and the right-hand side of equation (36) or (37) may be neglected entirely. Thus, altogether, it follows that equation (37) will give $u(b)$ correctly over the entire range $0 \leq b \leq 1$ in the limit of small n_0 . This will be demonstrated quantitatively following integration of equation (37).

The physical basis for this approximation is the fact that the magnetic field is essentially unaffected by the convection, ub , of lines of force in the transition layer where the fluid is being squeezed out of the region along the lines of force. In the region where fluid is being squeezed out ($nu \neq \text{constant}$) the field is in a state of static diffusion, described by equation (7). On the other hand, convection of the lines of force by the fluid

³ Q^2 is then the value of q^2 computed for $N_a = N_m$ near the center of the transition region.

motion $v(x)$ is important only outside the transition layer, where $n \ll 1$, i.e., where the loss of fluid is insignificant compared with the loss in the vicinity of $n = 1$. There we have $nu \cong n_0$, and, by the time n is even a very small fraction of 1.0, u is so small that convection ub can be neglected.

Now relation (37) can be reduced immediately to quadrature,

$$u = \frac{R^2 [b(1 - \frac{1}{3}b^2) + Q^2 I(b, \beta)]}{n_0 + 1 - b^2}, \quad (38)$$

where

$$I(b, \beta) \equiv \int_0^b \frac{ds(1 - s^2)s^2}{(n_0 + 1 - s^2)^\beta}. \quad (39)$$

The integral $I(b, \beta)$ is easily evaluated for integral values of β ,

$$I(b, 0) = \frac{1}{3}b^3 - \frac{1}{5}b^5,$$

$$I(b, 1) = n_0 b + \frac{1}{3}b^3 - \frac{1}{2}n_0(1 + n_0)^{1/2} \ln \frac{(1 + n_0)^{1/2} + b}{(1 + n_0)^{1/2} - b},$$

$$I(b, 2) = \frac{2b^3 - b(2 + 3n_0)}{2(n_0 + 1 - b^2)} + \frac{2 + 3n_0}{4(1 + n_0)^{1/2}} \ln \frac{(1 + n_0)^{1/2} + b}{(1 + n_0)^{1/2} - b},$$

$$I(b, 3) = -\frac{n_0 b}{4(n_0 + 1 - b^2)^2} + \frac{(4 + 5n_0)b}{8(1 + n_0)(1 + n_0 - b^2)} \\ - \frac{4 + 3n_0}{16(1 + n_0)^{3/2}} \ln \frac{(1 + n_0)^{1/2} + b}{(1 + n_0)^{1/2} - b}.$$

It then follows that

$$I(1, 0) = \frac{2}{15}, \quad I(1, 1) = \frac{1}{3} - \frac{1}{2}n_0 \ln \frac{4}{n_0} + O(n_0),$$

$$I(1, 2) = \frac{1}{2} \ln \frac{4}{n_0} - \frac{3}{2} + O(n_0 \ln n_0),$$

$$I(1, 3) = \frac{1}{4n_0} - \frac{1}{4} \ln \frac{4}{n_0} + O(n_0 \ln n_0).$$

To determine R , put $u = b = 1$ in equation (38), obtaining

$$R^2 = \frac{n_0}{\frac{2}{3} + Q^2 I(1, \beta)}. \quad (40)$$

In the absence of ambipolar diffusion ($Q = 0$) this reduces to $R^2 = 3n_0/2$. In the presence of strong ambipolar diffusion ($Q \gg 1$) the result is $R^2 Q^2 = n_0/I(1, \beta)$, and for $\beta = 0, 1, 2$ the result is $R^2 = 15n_0/2Q^2, 3n_0/Q^2, 2n_0/Q^2 \ln 4/n_0$, respectively. Since the merging velocity v_0 is proportional to $1/R$, it follows that, with or without ambipolar diffusion, the merging velocity is enhanced by a factor $1/n_0^{1/2}$ in the presence of high compressibility ($n_0 \ll 1$). As noted previously, the physical basis for the enhanced merging velocity is the high rate at which matter in compressed form is squeezed out from between the two fields. The density N_m , where the fluid is squeezing out, is so large compared with the density N_0 being carried in by $v(x)$ that only a very thin (scale l) transition region is required in which to squeeze out the material. A thin transition region leads to more rapid diffusion.

Now consider once again the approximation on which the calculation was carried out from equation (36). For $1 - b^2$ rather greater than n_0 , equation (36) reduces to

$$\begin{aligned}\frac{du}{db} &= O[R^2(1 + Q^2b^2)] \\ &= O\left\{\frac{3}{2}n_0 \frac{(1 + Q^2b^2)}{[1 + \frac{3}{2}Q^2I(1, \beta)]}\right\} \\ &\lesssim O(n_0)\end{aligned}$$

in the limit of small n_0 . Integrating this equation leads to $ub \leq 0(n_0b^2)$, demonstrating that the factor $1 - ub$ may be neglected even when b^2 becomes of the order of 1.0. This justifies the first part of the approximation, in which the factor $1 - ub$ was omitted from equation (36), provided only that $1 - b^2 > n_0$.

The second part of the approximation involves the demonstration that the right-hand side of the differential equation is negligible when $1 - b^2 < n_0$. Toward this end we put $u = 1 - \epsilon$ and $b = 1 - \delta$ again and expand about the point $u = b = 1$. Equation (36) becomes

$$n_0 \frac{d\epsilon}{d\delta} - 2 \cong \frac{2R^2\delta}{\epsilon + \delta} \left[1 + \frac{Q^2}{(n_0 + 2\delta)^\beta} \right].$$

Let it be assumed for the moment that the right-hand side of the equation, which we denote by $\mathfrak{R}$, is in fact small, as we have supposed. Then, neglecting $\mathfrak{R}$, we obtain $\epsilon = 2\delta/n_0$ upon integration of the left-hand side. It follows that

$$\begin{aligned}\mathfrak{R} &\cong R^2n_0 \left[1 + \frac{Q^2}{(n_0 + 2\delta)^\beta} \right] \\ &< R^2n_0 \left(1 + \frac{Q^2}{n_0^\beta} \right) \\ &< \frac{3n_0(1 + Q^2/n_0^\beta)}{2[1 + \frac{3}{2}Q^2I(1, \beta)]}.\end{aligned}$$

If $Q = 0$, then $\mathfrak{R} = 0(n_0^2)$ and is small, as required. If $Q \neq 0$, then in the limit of small n_0 we have

$$\begin{aligned}\mathfrak{R} &< \frac{3n_0^{2-\beta}Q^2}{2[1 + \frac{3}{2}Q^2I(1, \beta)]} \\ &< \frac{n_0^{2-\beta}}{I(1, \beta)}.\end{aligned}$$

It is readily seen from the values given for $I(1, \beta)$ that $\mathfrak{R}$ is small, provided that $\beta \leq 2$. For $\beta > 2$, we have $\mathfrak{R}$ of the order of unity. Thus, for $\beta \leq 2$, the right-hand side of equation (36) is small compared with unity in the neighborhood $\delta = 0(n_0)$ of $b = 1$. The right-hand side may then be neglected, compared with the left-hand side, which is of the order of unity, and the error introduced in equation (37) by neglecting the factor $1 - ub$ drops out. The solution is $\epsilon = 2\delta/n_0$, computed from equation (36), and it is readily shown that the same relation is obtained from equation (38), which is the solution of equation (37). Thus, altogether, we have demonstrated that in the limit of small n_0 , equation (37) is a valid approximation of equation (36) over the entire interval $0 \cong b \leq 1$, provided only that $\beta \leq 2$. It is interesting to note that the asymptotic form of the solution $u(b)$ as $n_0 \rightarrow 0$ is $u = 0$ for $0 \leq b < 1$, and then $b = 1$ for $0 \leq u \leq 1$.

The dependence of b on ξ is readily determined by quadrature from equation (35), knowing $u(b)$. From the asymptotic relation $\epsilon = 2\delta/n_0$, equation (35) yields

$$\xi \sim \frac{1 + Q^2/n_0^6}{1 + 2/n_0} \ln \delta + O(1), \quad (41)$$

so that b approaches 1.0 only in the limit as $\xi \rightarrow \infty$. Apart from this asymptotic limit, we are interested in $b(\xi)$ for $b < 1$ so as to demonstrate the form of the transition region. For the simple case that $Q = 0$, we obtain, upon using equations (38) and (40),

$$\begin{aligned} \xi &= \int_0^b \frac{db}{1 - ub} \\ &= \frac{1}{1 + n_0/2} \left[\left(1 - \frac{1}{2}n_0\right)(1 + n_0)^{1/2} \left(\frac{2}{n_0}\right)^{1/2} \operatorname{arctanh} \left(\frac{n_0}{2}\right)^{1/2} \frac{b}{(1 + n_0)^{1/2}} \right. \\ &\quad \left. + n_0 \operatorname{arctanh} b \right]. \end{aligned}$$

Since $n_0 \ll 1$, this expression may be simplified to

$$\xi \cong b + n_0 \operatorname{arctanh} b + O(n_0). \quad (42)$$

The functions $u(\xi)$ and $b(\xi)$ are plotted in Figure 5 for $n_0 = 0, 0.01$, and 0.1 . The asymptotic form as $b \rightarrow 1$ is

$$b(\xi) \sim 1 - 2 \exp \left[\frac{-2(\xi - 1)}{n_0} \right], \quad (43)$$

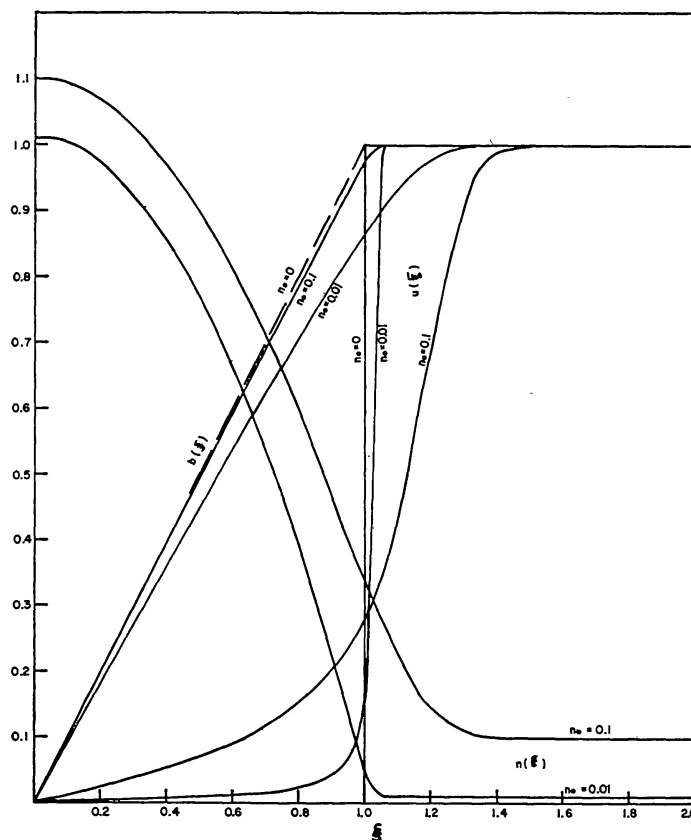


FIG. 5.—Plot of $u(\xi)$ and $b(\xi)$ in isothermal compressible flow for various compression factors $n_0 = N_0/N_m$.

which is just relation (41). The thickness of the transition region is readily obtained from relation (42) by ignoring the second term on the right-hand side and extrapolating the result to $b = 1$, obtaining $\xi = 1$. If this value of ξ is put equal to $-lv_0/\nu$, then it is readily shown from equation (40) and the definition of R that

$$\frac{l}{L} = \left(\frac{3\nu n_0}{2a^2La} \right)^{1/2}. \quad (44)$$

For the case that $Q \gg 1$, neglect the factor $1 - ub$ in equation (35), obtaining

$$b(\xi) = \left(\frac{3\xi}{Q^2} \right)^{1/3} \quad (45)$$

for $\beta = 0$;

$$\xi = Q^2 \left[\frac{1}{2} (1 + n_0)^{1/2} \ln \frac{(1 + n_0)^{1/2} + b}{(1 + n_0)^{1/2} - b} - b \right] \quad (46)$$

for $\beta = 1$; and

$$\xi = Q^2 \left[\frac{b}{2(n_0 + 1 - b^2)} - \frac{1}{4(1 + n_0)^{1/2}} \ln \frac{(1 + n_0)^{1/2} + b}{(1 + n_0)^{1/2} - b} \right] \quad (47)$$

for $\beta = 2$. These relations are approximately valid only in $0 < b < 1 - n_0$, because of the neglect of $1 - ub$. In the region $1 - Kn_0 < b \leq 1$, the asymptotic form (41) or (43) is applicable.

It is readily shown that when $b \ll 1$, both equations (46) and (47) reduce to equation (45). Thus the characteristic scale l of the transition region may be estimated from equation (45). Putting $b(\xi) = 1$ leads to $\xi = \frac{1}{3}Q^2$, which we take to represent the thickness of the transition region. Then with $\xi = -lv_0/\nu$, we have $l = \nu Q^2/3v_0$. For $\beta = 1$, this is

$$\frac{l}{L} = \left(\frac{\nu Q^2 n_0}{3a^2La} \right)^{1/2} = \left(\frac{B_0^2}{6\pi a^2 M A_i w_i N_m^2 L a \Delta} \right)^{1/2}, \quad (48)$$

where Δ is the fraction of atoms that are ionized at $\xi = \infty$, where $N = N_0$; $\beta = 1$ assumes that recombination limits N_i to ΔN_0 for all ξ . For $\beta = 2$,

$$\begin{aligned} \frac{l}{L} &= \left[\frac{2\nu Q^2 n_0}{9a^2La \ln(4/n_0)} \right]^{1/2} \\ &= \left[\frac{B_0^2 n_0}{9\pi a^2 M A_i w_i N_m^2 \ln(4/n_0) L a \Delta} \right]^{1/2}. \end{aligned} \quad (49)$$

In this case, $\beta = 2$, it is assumed that N_i increases in proportion to N , $N_i = \Delta N$, with the result that l/L is smaller by $n_0^{1/2}$ than when $\beta = 1$. These estimates of l will be of importance to a later discussion of the possibility of catastrophic field annihilation before a steady state is reached.

To visualize the physical picture presented by the foregoing calculations on the enhanced Sweet's mechanism with high compressibility ($N_0 kT \ll B_0^2/8\pi$), think of the magnetic field as a river of magnetic flux moving slowly in under the pressure $B_0^2/8\pi$ toward the transition region and carrying with it the tenuous gas with which it is permeated. The gas piles up at $\xi = 0$, where it meets a river of flux and gas flowing in from the opposite direction. The magnetic fields are annihilated at $\xi = 0$ because they have opposite sign. The material is not annihilated, so it accumulates to large density $N_0 + N_m$ ($N_m kT = B_0^2/8\pi$), and is squirted out along $\xi = 0$, so that an efflux is achieved which balances the continual inward flow $N_0 v_0$ of new material in the two rivers of magnetic flux. It is important to note that the adverse pressure gradient of the gas ac-

cumulating in the vicinity of $\xi = 0$, illustrated in Figure 5, causes the gas to slip backward through the rivers of flux (because of Joule dissipation and/or ambipolar diffusion), as they approach the transition region. The gas motion begins to decrease abruptly before any significant decrease in the river of flux appears; it is readily seen from Figure 5 that with $n_0 = 0.01$, for instance, u has fallen to 0.17, where b has decreased only from 1.00 to 0.98.

e) Transient and Instability Effects

There are a number of conceivable transient and instability effects associated with Sweet's mechanism which might enhance the dissipation under some circumstances. Dungey (1961) has suggested that under some conditions runaway electrons may develop in the transition region, leading to two-stream plasma instabilities which may enhance the effective resistivity of the gas. A number of people (see, for instance, Dungey 1961; Severny 1958, 1959, 1960) have suggested the possibility of a pinch instability resulting from the high current densities in the transition layer. To these we may add the hydrodynamic instability of the sheet of gas squirting out along the transition region, which may mix up the opposite fields and enhance their dissipation. Then there is the question of approach to the stationary conditions discussed so far. If the ambient gas density N_0 is sufficiently tenuous, there simply may not be enough gas to compress into the density N_m across the transition region of the computed stationary scale l . The result would be a much thinner transition region, filled to the density N_m , and much more rapid diffusion and dissipation.

1. *Runaway electrons.*—Let us begin with the possibility of runaway electrons. Run away electrons are the result of the decreasing effect of Coulomb collisions with increasing electron velocity. Hence the gas must be significantly ionized in order for runaways to occur, with the result that we probably do not expect to find runaway electrons under most conditions where ambipolar diffusion is very important. The discussion will be limited, therefore, to Sweet's mechanism with Joule dissipation alone. It will be assumed that the necessary and sufficient condition for the onset of runaway electrons is that the electron conduction velocity w must exceed the electron thermal velocity w' by a factor μ of the order of unity. The electron thermal velocity is $w' = (3kT/m)^{1/2}$. The electron conduction velocity is related to the current density j , which in turn is related to the gradient of the magnetic field by $4\pi j = c\nabla \times B$. Putting $|\nabla \times B_0| = B_0/l$, it follows that

$$w = \frac{cB_0}{4\pi l N_e e},$$

where N_e is the electron density. The scale l is given, in order of magnitude, by equation (11). Putting $N_e = N_m$ and $a^2 = 3kT/M$, we obtain

$$\frac{w^2}{w_1^2} = \frac{1}{6\pi} \frac{m\sigma}{e^2 L N_0} \left(\frac{3kT}{M} \right)^{1/2}.$$

Since $\sigma \cong 1 \times 10^7 T^{3/2}$ esu for ionized hydrogen, the numerical expression is

$$\frac{w}{w'} \cong \frac{6T}{(LN_0)^{1/2}}. \quad (50)$$

Runaway electrons may occur when this ratio is as large as μ , of the order of unity. It would appear that runaway electrons may be accomplished when T is high and LN_0 sufficiently low. Applications of this formula will be discussed in a later section. The possibility of runaway electrons is of particular interest in the problem of charged-particle acceleration, which occurs widely in active astrophysical phenomena, such as solar flares and the terrestrial aurora, but for which a convincing theory is still lacking.

2. *Hydrodynamic instability*.—Now consider the possibility of hydrodynamic instability in the thin sheet of material squeezing out from between the two merging fields. Keeping with the notation used so far, the fluid is flowing in the y -direction in a thin sheet of thickness $2l$ centered about $x = 0$. The magnetic field on each side of the fluid sheet is also in the y -direction. One may imagine two distinct unstable transverse wave motions that could arise, one with its wave vector in the y -direction and the other with its wave vector in the z -direction. The wave with its vector in the y -direction leads to waves in the fluid sheet and in the lines of force. The perturbations are essentially Alfvén waves propagating along the fluid sheet and the neighboring field. We have been unable to demonstrate, on the basis of simple models, how such waves will significantly enhance

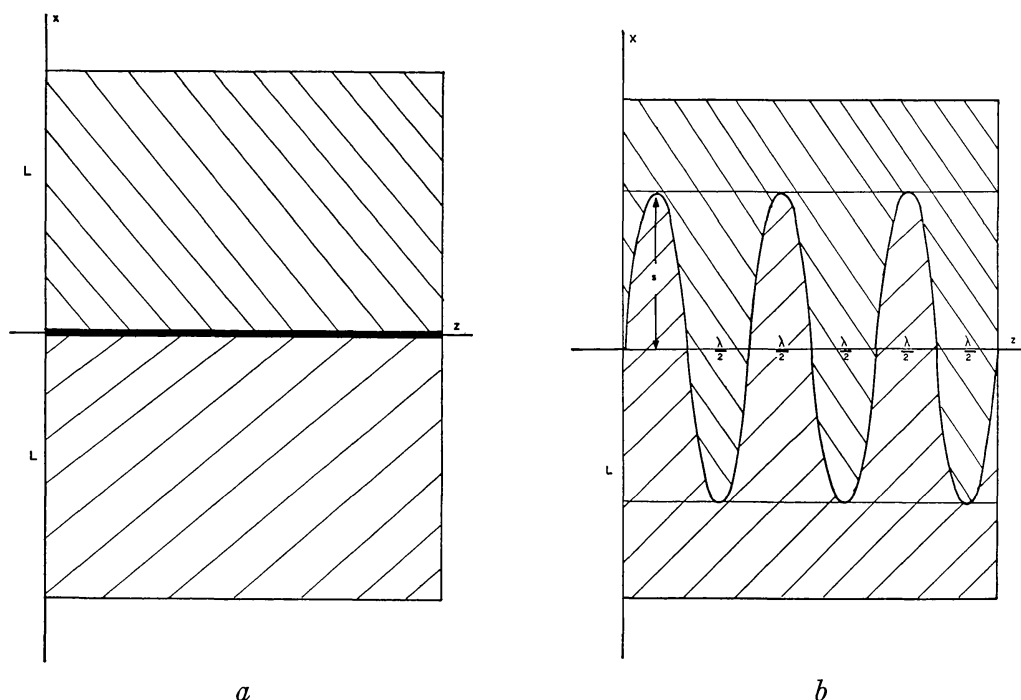


FIG. 6.—Schematic representation of the section $y = 0$ through the merging antiparallel regions of field. The magnetic lines of force are perpendicular to the plane of the page and in (a) have one sign above the z -axis and the opposite sign below, as indicated by the two different crosshatchings. The first diagram (a) represents the unperturbed situation, with the heavy line along the axis indicating the transition region between the two fields. The second diagram (b) represents the exchange perturbation of wavelength λ and amplitude s , carrying lines of force deep into regions of opposite sign.

the dissipation. Their only obvious effect is to consume energy that might otherwise have gone into the flow of fluid out of the transition region, thereby inhibiting Sweet's mechanism. The wave with its vector in the z -direction, on the other hand, exchanges magnetic flux between opposite sides of the transition while doing little or no work on the field. The situation is shown in simplified form in Figure 6. The heavy line across the middle of Figure 6, *a*, represents an end view, looking in the y -direction, of the unperturbed transition region. The two crosshatchings represent magnetic fields of opposite sign, the field being perpendicular to the plane of the page. The perturbed field is illustrated in Figure 6, *b*, where the transition region is carried into a sinusoidal wave of large amplitudes. If the field is assumed to be of great length in the y -direction (perpendicular to the page), then negligible work need be done on the field. If the wave length λ of the perturbation is large compared with the thickness l of the transition layer, as illustrated in Figure 6, then the diffusion is enhanced because the transition layer is stretched to greater length, by the factor $(1 + s^2/\lambda^2)^{1/2}$, assuming that l is relatively unaffected. If

the wavelength should prove to be small compared with l , then note that, for large amplitude ($s > \lambda$), the shortest scale is λ , and the fields will dissipate to a distance s on each side of $x = 0$ in a time of the order of $\lambda^2 \sigma / c^2$. Without the perturbation, the dissipation to a distance s would have taken a period $s l \sigma / c^2$. Thus the dissipation and the merging velocity are enhanced by a factor of the order of $1 + s l / \lambda^2$.

It should be noted that the existence of such an exchange perturbation of large amplitude requires that the fields on opposite sides of the unperturbed transition region be almost exactly antiparallel. This question is taken up in Section *f*.

3. *Onset of stationary conditions*.—Consider, finally, the time required for the onset of the stationary conditions to which the discussion of Sweet's mechanism has so far been limited. In particular, we are interested in the length of time required to accumulate the quantity $l N_m$ of the compressed gas that keeps the fields apart in the transition zone. The total amount of material in the field initially is $l N_0$. If $l N_m \ll l N_0$, then only a small portion of the field would be annihilated before the necessary accumulation of gas takes place and quasi-stationary conditions prevail. On the other hand, if $l N_m \gtrsim l N_0$, there is not enough gas in the field to supply the stationary requirement. The available gas would compress to the density N_m , forming a transition layer much thinner than the l estimated in equation (32), and the dissipation would proceed correspondingly faster. The importance of this initial transient dissipation depends on the ratio $l / l N_0$. If the ratio is of the order of unity, the effect would be large. The ratio l / L is given by equations (11) and (44) for Joule dissipation and by equations (12), (48), and (49) for ambipolar diffusion. For Joule dissipation it follows that

$$\frac{l}{l N_0} \cong \frac{4 \times 10^{11}}{a} \frac{B_0}{(l N_0 T^3)^{1/2}} \quad (51)$$

for ionized hydrogen. For ambipolar diffusion, use equations (48) and (49), putting $w_i = a = 3kT/M$. Then, with $A_i = 10^{-16} \text{ cm}^2$, equation (48) for $\beta = 1$ gives

$$\frac{l}{l N_0} = \frac{1.1 \times 10^{15}}{a} \frac{B_0}{N_0 (L T \Delta)^{1/2}}, \quad (52)$$

and equation (49) for $\beta = 2$ gives

$$\frac{l}{l N_0} = \frac{5.4 \times 10^7}{a [N_0 L \Delta \ln(4/n_0)]^{1/2}}, \quad (53)$$

respectively. It is obvious that if L and N_0 are sufficiently small, the transient dissipation before a steady state is reached may be important. With Joule dissipation, small temperatures enhance the effect, but with ambipolar diffusion for $\beta = 2$, temperature appears only in the logarithm and has very little effect when $n_0 \ll 1$.

We would estimate that the length of time required to achieve the steady state is characterized by $(L/v_0)(l/l N_0) = l/n_0 v_0$.

4. *Pinch instability*.—The possibility of a pinch instability is based on the assumption that a current may be driven along the neutral line between the two regions of oppositely directed field to such an intensity that the stresses of the self-field of the current overpower everything else and cause a violent contraction of the fields and plasma in the vicinity of the current. Pinches are produced in laboratory plasmas with a high voltage, low impedance, external electromotive force across the plasma. It is not obvious that the astrophysical situation is at all analogous, because there is generally no *external* electromotive force. There are only induced electromotive forces available to drive currents. In the absence of an external electromotive force, the hydromagnetic equations are applicable, and the question is simply whether any violent magnetic instabilities are likely. This question has been discussed, though perhaps inadequately, in the foregoing para-

graphs. It is to be hoped that the proponents of the pinch instability can work out a quantitative model of the pinch, if there is any truth in it, as such an instability would go far in helping to explain the solar flare.

f) Effect of Non-parallel Field Directions

Thus far the discussion has been limited to the two-dimensional problem of the annihilation of antiparallel magnetic fields. Obviously, two dimensions overlook the "end effects" brought about by the finite scale of the field in the z , or third, direction. Such effects are probably of no qualitative importance. But there is one characteristic of the two-dimensional problem that is of considerable importance, and that is the fact that

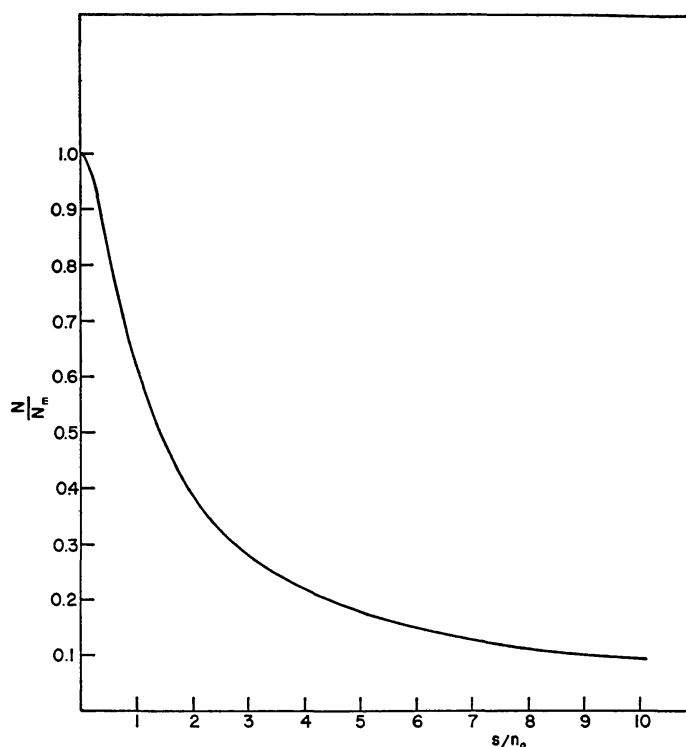


FIG. 7.—Reduction of the compression ratio N_m/N_0 by the field B_1 in the z -direction

the two magnetic fields being annihilated are *exactly* antiparallel when limited to two dimensions. In the three-dimensional world, to which the results of the calculations will be applied, the two fields will in general not be exactly antiparallel. This is of little or no consequence to the incompressible flow case, considered in Sections *b* and *c*, but it may be of profound importance in compressible flow, as will now be demonstrated.

Adopting the physical approach outlined in Section *a*, suppose that in the simple case of isothermal flow, discussed in Section *d*, there is a weak field B_1 in the z -direction being carried into the transition region along with the usual field $\pm B_0$ in the y -direction. This is equivalent to the statement that the merging fields make an angle $\pi - 2B_1/B_0$, instead of being exactly antiparallel with an angle π . The field in the z -direction has the same sign on each side of the $x = 0$ plane, so there is no annihilation of B_1 . The z -field flowing steadily in, at a rate $v_0 B_1$, can escape only by being convected out along the transition region with the escaping fluid. Thus B_1 is brought in with the fluid and escapes with the fluid, suggesting that B_z is proportional to N , at least in order of magnitude, throughout the entire region. B_z with its initial value B_1 , is compressed as it approaches

the transition region, just as is N , and unless $B_1 < n_0 B_0$, the pressure of B_z in the transition region may completely dominate the gas pressure and greatly reduce the compression factor N_m/N_0 previously computed. It was shown in equation (11) that the merging velocity v_0 is proportional to $(N_m/N_0)^{1/2}$, so that v_0 may be seriously reduced.

To show under what conditions B_1 may decrease v_0 from the value given in equation (11), consider the condition equivalent to equation (33) for pressure equilibrium across the transition region. For isothermal conditions,

$$N(x) kT_0 + \frac{B^2(x) + B_z^2(x)}{8\pi} = N_0 kT_0 + \frac{B_0^2 + B_1^2}{8\pi}. \quad (54)$$

At the center of the transition region, the field in the y -direction, which has been denoted by $B(x)$, vanishes. Putting $B_z(x) \cong B_1[N(x)/N_1]$ and neglecting $N_0 kT$ and $B_1^2/8\pi$ compared with $B_0^2/8\pi$, the relation can be solved for $N(0)$ to give

$$\frac{N(0)}{N_m} = \frac{n_0^2}{2s^2} \left\{ \left[1 + 4 \left(\frac{s}{n_0} \right)^2 \right]^{1/2} - 1 \right\}, \quad (54a)$$

where $N(0)$ is the gas density at $x = 0$, $N_m \equiv B_0^2/8\pi kT$ is the gas density at $x = 0$ if there were no field in the z -direction, and s is the ratio B_1/B_0 ($\ll 1$). The merging velocity given by equation (32) is reduced by the square root of this ratio $N(0)/N_m$, plotted in Figure 7 as a function of s/n_0 . Roughly speaking, $N(0)$ is significantly less than N_m , and the merging velocity is significantly reduced by B_1 , if $s > n_0$. For $s \gg n_0$, we have $N(0)/N_m \sim n_0/s$, so that v_0 is reduced from equation (11) by $(B_0/B_1)^{1/2}$. Since $n_0 = N_0/N_m$ may be as small as 10^{-5} in such places as the solar atmosphere, it is evident that even very small deviations of the two merging fields from antiparallelism may seriously impede their annihilation.

IV. APPLICATIONS

a) Annihilation of Solar Fields

Consider the rate at which magnetic fields may be annihilated in the solar atmosphere by ambipolar diffusion and by Sweet's mechanism.

1. *Photosphere*.—In the solar photosphere, where $T \sim 6000^\circ \text{K}$ and one atom in 10^3 or 10^4 may be ionized, the electrical conductivity is of the order of 10^{12} esu. The diffusivity $\nu = c^2/4\pi\sigma$ is, accordingly, of the order of $10^8 \text{ cm}^2/\text{sec}$. The ambipolar diffusion coefficient $B^2/4\pi\kappa = B^2/4\pi N_i N_a M_i A_i w_i$ may be estimated in specific cases. The picture is complicated by the presence of both positive and negative ions, the positive ions being largely metallic in the cooler layers; but this need not concern us in making the present rough estimates. To obtain an order-of-magnitude estimate of the ambipolar diffusivity, put $w_i = (3kT/M)^{1/2}$. Then remember that the effective value of $B^2/8\pi$ available for driving the ions through the neutral gas cannot exceed the strength of the neutral gas to resist, i.e., $B^2/8\pi < N_a kT$. It follows that the ambipolar diffusion coefficient is bounded by

$$\frac{B^2}{4\pi\kappa} \lesssim \frac{2w_i}{3N_i A_i}. \quad (55)$$

The temperature must be below 10^4°K if the medium is to be largely un-ionized, so that, for hydrogen ions, w_i does not exceed about 10^6 cm/sec and, for metallic ions, $2 \times 10^5 \text{ cm/sec}$. Put $A_i = 10^{-16} \text{ cm}^2$. Then, for metallic ions, the ambipolar diffusion coefficient is not more than $1.3 \times 10^{21}/N_i \text{ (cm}^2/\text{sec)}$. A typical value for N_i in the visible photosphere is $10^{13}/\text{cm}^3$, with N_i generally decreasing upward throughout the entire region (see, for instance, Münch 1953). The result is $1.3 \times 10^9 \text{ cm}^2/\text{sec}$ as an upper limit

on ambipolar diffusion in the photosphere. Comparing this with the 10^8 cm²/sec estimated for the diffusion due to ohmic resistance, it follows that ambipolar diffusion is evidently no more important in the photosphere than Joule dissipation. Elsewhere in the sun we would expect ambipolar diffusion to be even less important because of the higher degree of ionization.

To obtain a rough upper limit for the rate of Sweet's mechanism in the photosphere, put $B = 10^3$ gauss in a typical density of 10^{17} atoms/cm³. For such values the gas pressure is somewhat larger than the magnetic pressure, so that, to a first approximation, the expression (26) may be employed with $S = 0.87$. With $C_0 = 7 \times 10^5$ cm/sec and $\nu \cong 10^8$ cm²/sec, it follows that $|v_0| = 1.0 \alpha 10^7/L^{1/2}$ cm/sec. Over the scale $L \cong 10^8$ cm of a granule or a small feature of a sunspot, the merging velocity of two fields is about 10^3 cm/sec, giving a life $L/|v_0|$ of 10^5 sec. Over other scales the life varies as $L^{3/2}$. It is evident that photospheric fields cannot be annihilated and/or reconnected in less than about 1 day by known mechanisms unless their scale should be considerably less than 10^3 km. Such hypothetical smaller scale fields are quite unobservable at the present time but would be extremely interesting to explore if it should ever become possible. The main fields of sunspot groups could perhaps be destroyed in periods less than their observed life if oppositely directed fields were pressed together in a suitable way. But there is no observational evidence that this occurs in the photosphere.

2. *Chromosphere and corona.*—Of great interest in the solar chromosphere is the possibility of annihilating strong magnetic fields so rapidly as to account for the energy release associated with a solar flare. The energy output from a large solar flare seems to be of the order of 10^{32} ergs, requiring the equivalent of the annihilation of 500 gauss over a cube 2×10^4 km on a side. The really outstanding problem seems to be to understand the mechanism by which a magnetic field might be annihilated and its energy released into the flare phenomenon in the life of 10^2 – 10^3 sec. Consider, then, the optimum circumstances of two exactly antiparallel magnetic fields pressed firmly together in some region of the solar chromosphere or corona.

The magnetic pressure of a 500-gauss field is about 10^4 dynes/cm², which is very large compared with the ambient pressure of the chromospheric gas. The gas is strongly compressed as it is carried into the transition region. The merging velocity v_0 is computed from the value of R given in equation (40). Analysis of flare observations (Jeffries 1957); Jeffries *et al.* 1959; Jeffries and Orrall 1961*a, b*) suggests that the mean density in the visible flare is perhaps a few times 10^{11} electrons per cm³, with a temperature somewhere between one and two times 10^4 °K.⁴ Under these conditions the gas, mainly hydrogen, is fully ionized, and ambipolar diffusion is not expected in the region of the visible flare. It then follows from equation (40) that $R^2 = 3n_0/2$ and

$$v_0 = c \alpha \left(\frac{a}{6\pi n_0 \sigma L} \right)^{1/2}.$$

To obtain an upper limit on v_0 , put $T = 10^4$ °K. Then the speed of sound a is of the order of 16 km/sec, the density N_m in the transition region between the two 500-gauss fields is 0.7×10^{16} /cm³, and $\sigma \cong 10^{13}$ sec⁻¹. With $N_0 = 2 \times 10^{11}$ /cm³, it follows that $1/n_0 = 3.5 \times 10^4$. The merging velocity becomes $v_0 = a 0.6 \times 10^9/L^{1/2}$ cm/sec, so that a characteristic scale of $L = 10^4$ km yields $v_0 \cong 0.2$ km/sec, with $\alpha = 1$. The characteristic life L/v_0 is 5×10^4 sec, or almost 6 hours, which is much too slow an annihilation to be of much use in producing a flare.

The introduction of ambipolar diffusion, which we regard as perhaps dubious in view of the temperatures cited above, adds very little to the merging velocity because of the

⁴ There are other authors who suggest somewhat higher temperatures (see, for instance, Tandberg-Hanssen and Zirin 1959) which the reader may wish to explore with the formulae given here.

high value of N_m . Neglecting Joule dissipation and assuming that the fraction of ionized atoms Δ is uniform across the transition region, put $\beta = 2$ in equation (4). Then

$$Q^2 = \frac{B_0^2}{4\pi\nu\Delta M A_i w_i N_m^2}$$

$$= \frac{2kT}{\nu\Delta M A_i w_i N_m}$$

and, with $w_i = a$,

$$v_0 = a \frac{a}{3^{1/2}} \left[\frac{\ln(4/n_0)}{L\Delta A_i N_0} \right]^{1/2}. \quad (56)$$

The temperature must be less than 10^4 °K if the gas is to be only partially ionized, so put $a = 10^6$ cm/sec. Then $A = 10^{-16}$ cm² and $\Delta = 10^{-2}$ yield $v_0 = 5 \times 10^9/L^{1/2}$ cm/sec, which is only seven times larger than for Joule dissipation alone. A scale of $L = 10^4$ km yields 1.6 km/sec merging velocity and a characteristic life of 6×10^3 sec, or about 2 hours. This is still too slow to agree with the characteristic flare times of 10^2 – 10^3 sec suggested by observation. It is interesting to note that Gold and Hoyle (1960) in their application of ambipolar diffusion to field annihilation in a flare obtained a characteristic annihilation time of 10^2 sec. The rapid annihilation which they compute is the result of their using a density of 10^{12} neutral atoms per cm³ for the density N_m in the transition region. This overlooks the requirement that the gas caught between the fields is compressed until $NkT \sim B^2/8\pi$, with $T \lesssim 10^4$ °K.

The ratio of the electron conduction velocity w to the electron thermal velocity w' is readily computed from the order-of-magnitude relation given numerically in equation (50). A temperature of 10^4 °K with $N_0 = 2 \times 10^{11}$ /cm³ and $L = 10^4$ km yields $w/w' = 4 \times 10^{-6}$. Even coronal temperatures, $T \sim 10^6$ °K, and densities $N_0 \sim 10^8$ /cm³ give $w/w' = 2 \times 10^{-2}$. There does not seem to be any possibility here for the runaway electrons postulated by Dungey (1961).

It is often implied in the literature that the electric field $v_0 B/c$ along the length of the transition region leads to an electrical discharge in the vicinity of the neutral line at the center (see, for instance, Sweet 1958*b*; Dungey 1958; Severny 1960). The argument seems to be based on the large total potential difference $v_0 B_0 L/c$ that the field represents across the entire width of the region. For instance, a merging velocity of $v_0 = 1$ km/sec with a field $B_0 = 5 \times 10^2$ gauss means an electric field of 1.6×10^{-3} statvolts/cm or 0.5 volts/cm. The total potential difference across $L = 10^9$ cm is then 5×10^8 volts, which is an imposing number. But it must be remembered that the electric field $v_0 B_0/c$ is just that electric field which is able to produce the electron drift velocities computed in the previous paragraph. And we saw that these velocities are small compared with the electron thermal velocities. Thus there seems to be no indication of anything approaching an electrical discharge, or runaway electrons, or a pinch instability.

Finally, note that the merging fields should very soon reach the stationary state treated in the quantitative discussion of Section III because the amount of matter accumulated in the transition region under stationary conditions is only a very small fraction of the total, i.e., $LN_m \ll LN_0$. The ratio of these two quantities is given by equation (51) for Joule dissipation and is 1.4×10^{-2} for the chromospheric values $T = 10^4$ °K; $N_0 = 2 \times 10^{11}$ /cm³; $L = 10^9$ cm; and $B_0 = 5 \times 10^2$ gauss. Under coronal conditions of $T = 10^6$ °K and $N_0 = 10^8$ /cm³, the ratio is even smaller, 6×10^{-4} . With ambipolar diffusion, equation (53) gives 1×10^{-2} for 1 per cent ionization, $\Delta = 10^{-2}$. The length of time required to reach a steady state is, accordingly, not more than about $10^{-2} L/v_0$, which is so short as to suggest that the merging can never be far from the stationary state during the whole time that the oppositely directed fields are being brought into position for an impending flare.

To evaluate these calculations of the annihilation of precisely antiparallel magnetic fields in the solar chromosphere and corona, note that the computation indicates that magnetic fields of 500 gauss with a scale of 10^4 km may annihilate and reconnect in periods of perhaps an hour to a day if they are pressed firmly together. If the fields are not exactly antiparallel, then the merging speed will be very much reduced. It is readily shown from equation (54) that if the fields lack being antiparallel by even so little as 10^{-2} radians, then, with the value of n_0 (3×10^{-5}) used in the calculations, the merging velocity is reduced by a factor of the order of 10^2 , and the characteristic reconnection time is at least a week or a month. The conclusion is that the presently known mechanisms for the annihilation of magnetic fields may give reconnection of magnetic lines of force over periods of time short compared with the characteristic relaxation time $L^2\sigma/c^2$, which is 300 years for $L = 10^9$ cm and $\sigma = 10^{13}$ esu. But there is no indication that the known mechanisms can account for the explosive annihilation needed to account for a solar flare. The known mechanisms are sufficiently rapid as to be of interest in the slow re-formation of sunspot fields and of quiescent prominences, with observed lives of weeks.

The presently known annihilation mechanisms would be greatly enhanced if the scale L could be identified with some scale smaller than the over-all 10^4 km characteristic of flares and active regions. If, for instance, the magnetic fields in the site of a flare consisted of many thin ropes of magnetic flux of opposite sense, then L might be very much less than 10^4 km. The characteristic annihilation time varies as $L^{3/2}$, so that, with a reduction to $L \lesssim 10^2$ km, it might be possible to achieve the desired explosive annihilation in 10^2 sec even if the fields were not exactly antiparallel. Both careful observational and theoretical investigation would be required to construct such a model, if, in fact, there is any merit in the suggestion at all. Jeffries *et al.* (1959) find observational evidence for violent small-scale motion such as might be associated with a network of small scale annihilation.

The presently known annihilation mechanisms would be greatly enhanced if some exchange instability across the thin transition region, of width $2l$, were to enhance the diffusion. Biermann (1962) has been interested in the possibility that instabilities of the type found in the steep pressure gradients in laboratory plasma devices might play an important role. We are not in a position to decide this interesting question here, but we can compute the conditions under which such instabilities would have to operate. The width of the transition region is given by equation (44), where $\nu = c^2/4\pi\sigma$. With $a = 1$, $n_0 = 3 \times 10^{-5}$, $a = 16$ km/sec, $\sigma = 10^{13}$ sec $^{-1}$, and $L = 10^4$ km, we obtain a characteristic scale of about 5×10^2 cm. Over this scale of 5 meters the magnetic field changes from, say, 500 gauss to zero, and the plasma density changes by a factor of 3×10^4 , from N_0 to $N_m = 10^{16}$ /cm ^{3} , assuming that the merging fields are antiparallel. If the fields are not precisely antiparallel, then of course these extreme gradients are considerably reduced. It will be extremely interesting to see whether the extrapolation of laboratory diffusion rates to these larger-scale gradients suggests diffusion rates which are large enough to give the desired field annihilation.

3. *Summary.*—To summarize the present state of the theory of the solar-flare phenomenon, we note that the basic problem is to construct an energy source that can supply the 10^{32} ergs that seem to be expended by a large flare. The popular view at the present time is that this energy is released principally in the central portions of the visible flare. On the basis of this view it is supposed that the energy arises from the annihilation of magnetic fields. We seem to find very little support for these ideas either in observation or in theory. The best that can be said for the theory is that there still remains the possibility that the magnetic fields at the site of the visible flare may consist of many very small-scale (10^2 km) flux tubes of mixed sign, and there still remains the possibility of powerful plasma instabilities in the steep gradients across the transition region. Either of these might conceivably enhance the dissipation and annihilation of

magnetic field to the rate required by the observed flare. On the other hand, these possibilities will have to be demonstrated before they can be taken seriously. The present state of the theory does not allow magnetic energy as the source of a flare.

The obvious question at this point concerns the possible alternatives to the idea of explosive magnetic-field annihilation. The one possibility that seems to stand out above all others follows from the fact that the interplanetary blast wave may consume the bulk of the energy put out by the flare. To put the matter briefly, it is not known whether the visible flare expends anywhere near as much energy as goes into the blast wave (see discussion in Introduction). Now if it should ultimately prove that the visible flare does not expend more than, say, 10 per cent of the total flare energy, then the rate at which energy need be supplied (ergs/cm² sec) to the flare would be greatly reduced, to perhaps only 2×10^8 ergs/cm² sec. The blast wave may be supplied over a much larger area and longer time than the visible flare, so that it, too, might need only rather modest energy fluxes. The point here is that 10^8 ergs/cm² sec is of the same order as the flux estimated (Weymann 1960; Osterbrock 1961; Whitaker 1963) for the normal quiet chromosphere and corona. Might not the flare then be no more than some redistribution and/or enhancement of the normal heating of the chromosphere and corona? The magnetic fields of the active region may play some guiding and controlling role, just as they are believed to do in the plage (see, for instance, Osterbrock 1961). The reader must understand that such notions as these are entirely conjectural, but no more so, we feel, than the presently popular view of magnetic annihilation.

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APPENDIX I

FURTHER EXAMPLES OF AMBIPOLAR DIFFUSION

There are a number of simple solutions of equation (7), in addition to the stationary solution considered in the text, that are instructive to write down. As one example consider, for instance, the wave $B(s)$, where $s = x - vt$, superimposed on a uniform background field B_0 . Then, since $\partial/\partial t = -v d/ds$, equation (7) may be integrated to give

$$\frac{dB}{ds} \left[\nu + \frac{(B_0 + B)^2}{4\pi\kappa} \right] = -vB, \quad (\text{A1})$$

where $\nu \equiv c^2/4\pi\sigma$ and the constant of integration has been chosen so that dB/ds vanishes with B . Integrating again gives

$$\left(\nu + \frac{B_0^2}{4\pi\kappa} \right) \ln \frac{B}{B_1} + \frac{1}{8\pi\kappa} (B - B_1)(B + B_1 + 4B_0) = -vs.$$

The constant of integration has been chosen so that $B = B_1$, at $s = 0$. In the limit as $s \rightarrow +\infty$ we have

$$B \sim B_1 \exp \left[\frac{B_1(B_1 + 4B_0)}{2(B_0^2 + 4\pi\nu\kappa)} \right] \exp \left(-\frac{vs}{\nu + B_0^2/4\pi\kappa} \right) \quad (\text{A2})$$

as the asymptotic form for the leading edge of the wave. Note that this represents an exponential precursor. If the background field is not extremely small, then $B_0^2/4\pi\kappa$ may be very large compared with ν , and the characteristic scale of the precursor is $B_0^2/4\pi\kappa v$. But, on the other hand, if B_0 should be identically zero, the precursor is very short, with a scale of only ν/v .

In the limit as $s \rightarrow -\infty$, i.e., for $B \gg B_1$, we have

$$B \sim (-8\pi\kappa v s)^{1/2} - 2B_0 - \frac{4\pi\kappa\nu + B_0^2}{2(-8\pi\kappa v s)^{1/2}} \ln\left(-\frac{8\pi\kappa\nu s}{B_1}\right) + \frac{(B_1 + 2B_0)^2}{2(-8\pi\kappa v s)^{1/2}} + O\left(\frac{1}{s}\right). \quad (\text{A3})$$

If the background field were zero and we neglected Joule dissipation, we would have

$$B^2 = B_1^2 - 8\pi\kappa v s \quad (\text{A4})$$

for all values of s . Note that there is then no precursor for large positive s , the field going to zero like $(s - s_0)^{1/2}$ at the head of the wave $s_0 = B_1^2/8\pi\kappa v$. Far back on the wave (s large and negative) we have $B \sim (-8\pi\kappa v s)^{1/2}[1 + O(1/s)]$, in agreement with the leading term of (A3). This demonstrates again that ν and B_0 do not affect the main portion of the field; they affect B only as $B - B_0 \rightarrow 0$. Finally, note that the velocity of propagation of the wave is related to the magnitude B_2 of the field at some large distance $s = -h$ back from the leading edge by

$$v = \frac{B_2^2}{8\pi\kappa h}.$$

These conditions are to be contrasted with the wave

$$B = B_1 \exp\left(-\frac{v s}{\nu}\right),$$

which results in the absence of ambipolar diffusion when only Joule dissipation is present. In such a wave the velocity of propagation is

$$v = \frac{\nu}{h} \ln \frac{B_2}{B_1}.$$

Not only is v small in this case because of the relatively small value of ν , but it is small for strong fields also because it increases only as $\ln B_2$ instead of as B_2^2 .

As a second example, consider the field with the similarity property that $B(x, t) = X(x)T(t)$. Then, neglecting Joule dissipation, equation (7) may be written

$$\frac{1}{T^3} \frac{dT}{d\tau} = \frac{1}{k^2 X} \frac{d}{dx} \left(X^2 \frac{dX}{dx} \right), \quad (\text{A5})$$

where $\tau \equiv tk^2 B_0^2/4\pi\kappa$. Then separating the variables, so that

$$\frac{1}{T^3} \frac{dT}{d\tau} = -1,$$

we have

$$T = (1 + 2\tau)^{-1/2} \quad (\text{A6})$$

for the time dependence of the relaxation of the field. The spatial dependence is given by

$$\frac{d}{dx} \left(X^2 \frac{dX}{dx} \right) + k^2 X = 0. \quad (\text{A7})$$

To integrate this equation, we take advantage of the well-known fact that any such equation, regardless of the power of X appearing with dX/dx in the parentheses, may be reduced to quadrature by the substitution

$$H(x) = \int_0^x dx X(x).$$

Then integration of the equation gives, in the present case,

$$\left(\frac{dH}{dx}\right)^2 \frac{d^2H}{dx^2} + k^2 H = \frac{1}{2} k^2 C_1, \quad (\text{A8})$$

when C_1 is a constant of the integration. Multiplying through by dH/dx and integrating again gives an expression for dF/dx that can be reduced to the quadrature

$$x = \frac{1}{(2k^2)^{1/4}} \int_0^H \frac{du}{(C_2 + C_1 u - u^2)^{1/4}}, \quad (\text{A9})$$

where C_2 is the second constant of integration. The integral is elliptic and is readily reduced by the substitution

$$\xi = \frac{2(C_2 + C_1 u - u^2)^{1/2}}{(C_1^2 + 4C_2)^{1/2}}$$

to the simple form,

$$x = \frac{(C_1^2 + 4C_2)^{1/4}}{2^{3/4} k^{1/2}} \int_{\xi_1}^{\xi} \frac{d\xi \xi}{[\xi(1 - \xi^2)]^{1/2}},$$

where $\xi_1 = 2C_2^{1/2}/(C_1^2 + 4C_2)^{1/2}$. We now let $\xi = \cos^2 \phi$ and obtain

$$x = \frac{(C_1^2 + 4C_2)^{1/4}}{2^{1/4} k^{1/2}} \{ [F(\phi, 2^{-1/2}) - F(\phi_1, 2^{-1/2})] - 2[E(\phi, 2^{-1/2}) - E(\phi_1, 2^{-1/2})] \}, \quad (\text{A10})$$

where F and E represent incomplete elliptic integrals of the first and second kinds, respectively, and

$$\phi_1 = \arccos \left[\frac{2^{1/2} C_2^{1/4}}{(C_1^2 + 4C_2)^{1/4}} \right], \quad (\text{A11})$$

$$\phi = \arccos \left[\frac{2^{1/2} (C_2 + C_1 F - F^2)^{1/4}}{(C_1^2 + 4C_2)^{1/4}} \right]. \quad (\text{A12})$$

Now suppose that we apply this solution to the asymptotic form of the decay, by ambipolar diffusion, of regions of field of alternate sign. Say that the field vanishes at $x = 2nL$ ($n = 0, 1, 2, 3$) and is positive in $(0, 2L)$, $(4L, 6L)$, etc., and negative in $(-2L, 0)$, $(2L, 4L)$, etc. This is equivalent to the case of oppositely directed fields on each side of $x = 0$ confined by impenetrable boundaries at $x = \pm L$. The boundary conditions are that $X(0) = 0$ and $X'(\pm L) = 0$. In terms of H , this is $H'(0) = H''(\pm L) = 0$. Differentiating (A8) with respect to x and remembering that H was defined so that $H(0) = 0$, it follows that $X(0) = 0$ requires that $C_2 = 0$. Let $B_0 L \Phi$ represent the total flux between $x = 0$ and $x = L$ when $t = 0$. Then it follows that $H(L) = L\Phi$, and if $H''(L) = 0$, (A8) requires that $C_1 = 2L\Phi$. It then follows from (A11) that $\phi_1 = \pi/2$ and $F(\phi_1, 2^{-1/2}) = K(2^{1/2})$, $E(\phi_1, 2^{-1/2}) = E(2^{-1/2})$, in (A10), where K and E represent the complete elliptic integrals.

To obtain an expression for k in terms of L , put $x = L$ in (A10). Then from (A12) it follows that $\phi = 0$, and (A10) becomes

$$kL = 2^{1/2} \Phi [2E(2^{-1/2}) - K(2^{-1/2})]^2 = 1.20 \Phi.$$

Thus, for $0 < x < L$, (A10) becomes

$$\frac{x}{L} = 1 - \frac{2E(\phi, 2^{-1/2}) - F(\phi, 2^{1/2})}{2E(2^{-1/2}) - K(2^{-1/2})}. \quad (\text{A13})$$

The expression (A12) for ϕ may be rewritten as

$$\phi = \arccos \left\{ \left[2 \left(\frac{H}{L\Phi} \right) - \left(\frac{H}{L\Phi} \right)^2 \right]^{1/4} \right\}. \quad (\text{A14})$$

The field X/Φ and the flux $H/L\Phi$ are plotted in Figure 8 as a function of x/L . To obtain an expression for $H(x)$ in the vicinity of $x = 0$, differentiate (A9), obtaining

$$\frac{d(H/L\Phi)}{d(x/L)} = \frac{kL}{\Phi} \left[2 \left(\frac{H}{L\Phi} \right) - \left(\frac{H}{L\Phi} \right)^2 \right]^{1/4}.$$

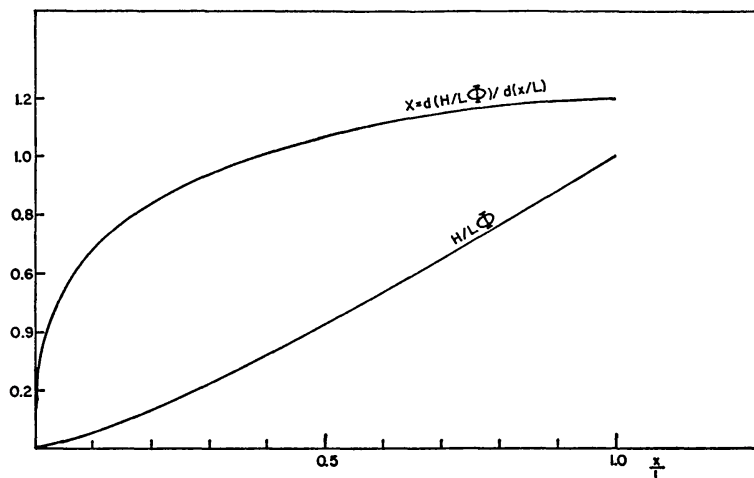


FIG. 8.—Plot of the field density $X(x)$ and the flux $H(x)$ across strips of magnetic field of alternating sign.

Then, if we expand $H/\Phi L$ as $a_1(x/L)^a + a_2(x/L)^\beta + \dots$, we readily find that $a = \frac{4}{3}$ and $\beta = \frac{8}{3}$, so that

$$\frac{H}{\Phi L} = \left(\frac{81}{128} \right)^{1/3} \left(\frac{kL}{\Phi} \right)^{4/3} \left(\frac{x}{L} \right)^{4/3} + O \left[\left(\frac{x}{L} \right)^{8/3} \right] \cong 1.09 \left(\frac{x}{L} \right)^{4/3}$$

and

$$X(x) \cong 1.45\Phi \left(\frac{x}{L} \right)^{1/3}. \quad (\text{A15})$$

This, then, is the asymptotic form, plotted in Figure 8, of a magnetic field of alternating sign decaying by ambipolar diffusion. The example illustrates the configuration that might be found in magnetic models of a solar flare or in adjacent arms of the galaxy.

The inclusion of a small amount of resistivity, in the form of the diffusion coefficient $\nu = c^2/4\pi\sigma$, would have introduced a small linear portion in the field variation across $x = 0$, in place of the $x^{1/3}$ variation of (A15). However, as noted in the text, this very short linear portion does not significantly affect the form of the field elsewhere.

APPENDIX II

DYNAMICAL EQUATIONS FOR INCOMPRESSIBLE FLOW IN TWO DIMENSIONS

Putting $\mathbf{v} = \nabla \times \mathbf{e}_z \Phi$ and $\mathbf{B} = \nabla \times \mathbf{e}_z A$ ($4\pi\rho$)^{1/2} to represent incompressible flow and the magnetic field in the xy -plane, the stationary equations reduce to

$$\frac{\partial(\Phi, \nabla^2 \Phi)}{\partial(x, y)} = \frac{\partial(A, \nabla^2 A)}{\partial(x, y)}, \quad \frac{\partial(\Phi, A)}{\partial(x, y)} + \nu \nabla^2 A = \lambda,$$

after taking the curl to eliminate the gradient terms. The constant λ represents the term $-\nabla\psi$.

To satisfy the first equation, it is sufficient that

$$\nabla^2 \Phi = f(\Phi), \quad \nabla^2 A = h(A).$$

The second may then be written

$$\frac{\partial(\Phi, A)}{\partial(x, y)} = \lambda - \nu h(A).$$

This equation requires, in general, that no functional relation exist between Φ and A . The first difficulty encountered in attempting an analytical solution is the problem of determining the functions f and h which correspond to the problem at hand.

The problem of incompressible flow in two dimensions is very similar to the problem of incompressible flow with axial symmetry, discussed at length in Chandrasekhar (1956 and references therein).

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