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A CLASSIFICATION OF THE INTEGRALS OF MOTION

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ABSTRACT

A distinction is made between isolating, quasi-isolating, and ergodic integrals of motion of a Hamiltonian system In the quasi-isolating case there is eventually a set of isolating integral hypersurfaces. A new proof is given of the theorem that, if a Hamiltonian H is a series in the variables, another Hamiltonian $\bar{H}$ coinciding with the given one up to the terms of any given degree has as many isolating integrals as there are degrees of freedom. Then a distinction between stable, quasi-stable, and unstable integrals is made

I. ISOLATING INTEGRALS

In recent papers (Contopoulos 1960, 1963; van de Hulst 1962; Lynden-Bell 1962*a*, *b;* Barbanis 1962) a number of time-independent integrals of motion of Hamiltonian systems have been used in problems of stellar dynamics. It should be useful to have a systematic classification of such integrals.

Consider a Hamiltonian system,

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i} \qquad (i=1, 2, \ldots, n) , \tag{1}$$

where $H = H(x_i, y_i)$ is a time-independent, analytic, and single-valued function of (x_i, y_i) in a certain region Ω of the phase space. We assume that Ω is composed of a set of hypersurfaces of $2n - 1$ dimensions:

$$H(x_i, y_i) = c , \tag{2}$$

with $c' \leq c \leq c''$; that is, the hypersurfaces form a closed set.

We want to classify the integrals of motion that are time-independent continuous functions of (x_i, y_i) in some part of the phase space.

It is well known that if $P_0(x_{i0}, y_{i0})$ is a point in Ω, there exists one and only one solution,

$$x_i = x_i(x_{i0}, y_{i0}, t) , \qquad y_i = y_i(x_{i0}, y_{i0}, t) , \tag{3}$$

of the system (1) going through P_0 at $t = 0$. Equations (3) are analytic functions of t and (x_{i0}, y_{i0}) as long as the point $P(x_i, y_i)$ is inside Ω. This is the case for all t if Ω is a set of hypersurfaces of the above form.

From equations (3) we get $2n$ time-dependent integrals of motion of the form

$$x_i(x_i, y_i, -t) = x_{i0} , \qquad y_i(x_i, y_i, -t) = y_{i0} . \tag{4}$$

By eliminating t, we get $2n - 1$ time-independent integrals

$$I_k(x_i, y_i) = c_k ; \tag{5}$$

I_k is an analytic function of (x_i, y_i) in some part Ω_k of Ω. If we solve the equation (5) for any variable, say x_1, we get a function,

$$x_1 = \phi_{1k}(x_2. \ldots x_n, y_1 , \ldots , y_n, c_k) , \tag{6}$$

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which is again analytic for some interval of values of the other variables and of the constant c_k. However, ϕ_{1k} may give an infinite number of values x_1 for a given set of values of the other variables and of c_k.

In some cases all these values x_1 cannot be reached by a point moving continuously on the hypersurface (5). Such is the case of the function

$$I_k = \sin(x_1^2 + y_1^2) = q \,, \tag{7}$$

for $|q| < 1$, that represents an infinite number of circles around the origin. For a given y_1 we have an infinite number of solutions for x_1; however, only two solutions at most belong to the same circle. In the following, by solutions we mean solutions that can be reached by a continuous displacement on the hypersurface (5).

We give now the following definition. If the solutions (6) have at least one accumulation point at a finite distance, the integral I_k is called *non-isolating*. This is, for example, the case when there is an infinite number of solutions in a finite domain. Then, according to the theorem of Bolzano-Weierstrass, the values of x_1 have at least one accumulation point at a finite distance. We may take as an example the function

$$I_k = \sin^{-1} x_1 - q \sin^{-1} x_2 = c_k \,, \tag{8}$$

where q = irrational. Then

$$x_1 = \sin(c_k + q \sin^{-1} x_2) \,, \tag{9}$$

that is, for a given value of x_2 we have an infinite number of values of x_1 that are densely distributed between -1 and $+1$.

In other cases, however, we may have an infinite number of solutions x_i that are well separated from each other; for example, the function

$$x_1 = \sin^{-1} y_1 + c_k \tag{10}$$

can represent an isolating integral, although for every value of y_1 we have an infinite number of values for x_1.

Any integral of the form of equation (5) is called *isolating* in a certain region Ω_k of the phase space if equation (5) can be solved with respect to every variable x_i or y_i and gives a finite number of solutions in any part of Ω_k of finite dimensions, for any set of values of the other variables inside Ω_k and a certain set of values of c_k: $c_k' \leq c_k \leq c_k''$.

The energy H is an isolating integral because it is an analytic, single-valued function of the variables (x_i, y_i); that is, it can be expressed in the neighborhood of any point P of Ω by a covergent power series. If this series begins with terms of degree l in some variable, then if equation (2) is solved for this variable, it gives at most l solutions in the neighborhood of P. Therefore, any accumulation point of the solutions of equation (2) is a singular point of H, that is, it does not belong to Ω. As Ω is a closed set, it does not contain an infinite number of solutions in any finite part of it.

The same is true for any other integral (5) if it is single-valued in a closed set Ω_k. In many cases, however, I_k has singular points, such that by making a loop around them we return to the same point with a different value for I_k.

In many cases the energy hypersurfaces (2) do not extend to infinity. Then it is sufficient to study the behavior of the integrals I_k inside the finite region Ω. More generally, it is sufficient to study an integral in that part of the phase space that is defined by a range of values of the isolating integrals already known.

In the case of equation (8) the points ($x_1 = \pm 1$, x_2 = arbitrary), (x_1 = arbitrary, $x_2 = \pm 1$) are singular. If, however, Ω is part of the space ($-1 < x_1 < +1$, $-1 < x_2 < +1$), then the different solutions (9) are not connected, that is, they cannot be reached by a continuous motion on the hypersurface (8). Therefore, equation (8) may then represent an isolating integral in Ω. But if Ω includes the square ($-1 \leq x_1 \leq +1$, $-1 \leq x_2 \leq +1$), then, by changing x_2 continuously, so that it goes through the points $x_2 = \pm 1$, we can reach all the values of $\sin^{-1} x_2$ and find an infinity of values for x_1.

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If I_k is analytic inside Ω, it is also single-valued, if Ω is simply connected (with no singular points inside it); because then by following a closed path in Ω we cannot return with another value for I_k. But if Ω is not simply connected, for example, if Ω is a torus, the analyticity of I_k in Ω is not sufficient. However, it would be sufficient to prove the analyticity of I_k in a simply connected region, including Ω. For example, if I_k is given as a series convergent in all the points P of Ω, then it is convergent also for all points whose coordinates are absolutely smaller than the corresponding coordinates of the points P. In this case I_k is evidently isolating.

Until now we have considered integrals of the form (5) which are isolating for a certain range of values of the constant c_k. It may be, however, that there is no such range of values of c_k, although for specific values of c_k the hypersurface (5) is isolating. In this case we have *a set of isolating integral surfaces.*

Cases when integral surfaces exist but do not form necessarily a continuous set have been discussed by Kolmogoroff (1954), Arnold (1961, 1962), and Moser (1962, 1963; see also 1955). They found under certain assumptions invariant tori around a periodic orbit in very general two-dimensional problems; namely, in the three-dimensional space defined by a constant value of the energy constant there are invariant two-dimensional surfaces surrounding a periodic orbit. These tori are isolating integral surfaces of two dimensions; they can be considered as the sections of a set of integral hypersurfaces of three dimensions with a given energy hypersurface.

If the set of these tori is continuous, then we have an isolating integral of motion besides the energy. We call these cases "integrable." In many cases, however, the set of isolating integral surfaces is nowhere continuous, but it has a measure greater than zero; the exceptional points, which do not belong to the integral surfaces, form a Cantor set of very small measure. We call these cases "quasi-integrable."

In any Hamiltonian problem where we can, by a convenient transformation of variables, separate the variables so that we can integrate the equations for each pair of conjugate variables, we have an integrable case that gives as many isolating integrals as there are degrees of freedom. Such a case is also called "separable." It can be proved that, if H is given in the form of a power series in the variables, then we can find a separable Hamiltonian $\bar{H}$ coinciding with H up to the terms of any given degree (Siegel 1956; Contopoulos 1963). We will prove now this basic theorem in the case of two degrees of freedom by using a variation of the method of von Zeipel (1916), which is used extensively in celestial mechanics.

In the case of two degrees of freedom the Hamiltonian may be given in the form

$$H = H_2 + H_3 + H_4 + \ldots , \tag{11}$$

where

$$H_2 = \tfrac{1}{2}(s_1 x_1^2 + y_1^2 + s_2 x_2^2 + y_2^2) , \tag{12}$$

and H_k is a homogeneous polynomial of degree k in x_1, x_2.

We perform then a transformation of variables

$$x_i' = \frac{\partial S}{\partial y_i'} , \qquad y_i = \frac{\partial S}{\partial x_i} , \tag{13}$$

by means of a determining function

$$S = S(x_i, y_i') = x_1 y_1' + x_2 y_2' + S_1 + S_2 + \ldots S_{\nu-2} , \tag{14}$$

where S_k is of degree $k + 2$ in x_i, y_i', so that in the new form of the Hamiltonian

$$H' = H_2' + H_3' + H_4' + \ldots \tag{15}$$

the terms H_k', for $k \leq \nu$, are functions of

$$\Phi_i = \tfrac{1}{2}(s_i x_i'^2 + y_i'^2) . \tag{16}$$

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As $\partial S/\partial y_i'$, $\partial S/\partial x_i$ are polynomials in x_i, y_i', the system (13) can be solved to give x_i, y_i as functions of x_j', y_j'

$$x_i = x_i(x_j', y_j') , \qquad y_i = y_i(x_j', y_j') , \qquad (i = 1, 2; j = 1, 2) \tag{17}$$

or x_i', y_i' as functions of x_j, y_j

$$x_i' = x_i'(x_j, y_j) , \qquad y_i' = y_i'(x_j, y_j) , \qquad (i = 1, 2; j = 1, 2) . \tag{18}$$

In fact, the system (13) can be written in the form

$$\begin{aligned} x_i' - x_i - \frac{\partial S_1}{\partial y_i'} - \ldots - \frac{\partial S_{\nu-2}}{\partial y_i'} &= F_{1i}(x_j, y_j, x_j', y_j') = 0 \\ y_i - y_i' - \frac{\partial S_1}{\partial x_i} - \ldots - \frac{\partial S_{\nu-2}}{\partial x_i} &= F_{2i}(x_j, y_j, x_j', y_j') = 0 , \end{aligned} \qquad (i = 1, 2; \; j = 1, 2) \tag{19}$$

and, according to the general implicit function theorem (see, e.g., Goursat 1956), this system has a solution for x_i, y_i or for x_i', y_i' in the form of convergent series in the neighborhood of the origin, because the Jacobians

$$\frac{D(F_{11}, F_{12}, F_{21}, F_{22})}{D(x_j', y_j' x_2', y_2')} , \qquad \frac{D(F_{11}, F_{12}, F_{21}, F_{22})}{D(x_1, y_1, x_2, y_2)}$$

are equal to 1 for $x_i = y_i = x_i' = y_i' = 0$, $(i = 1, 2)$.

If H_k and S_{k-2} are of the form

$$H_k = \epsilon^{k-2} H_k , \; S_{k-2} = \epsilon^{k-2} S_{k-2} , \tag{20}$$

then the above solutions can be given as convergent series in an arbitrarily large region around the origin, if ϵ is taken sufficiently small.

In the general case we write

$$\begin{aligned} x_i &= x_i' - \frac{\partial S_1}{\partial y_i'} - \frac{\partial S_2}{\partial y_i'} - \sum_{j=1}^{2} \frac{\partial^2 S_1}{\partial y_i' \partial x_j'} \left(- \frac{\partial S_1}{\partial y_j'} - \ldots \right) - \ldots \\ y_i &= y_i' + \frac{\partial S_1}{\partial x_i} + \frac{\partial S_2}{\partial x_i'} + \sum_{j=1}^{2} \frac{\partial^2 S_1}{\partial x_i' \partial x_j'} \left(- \frac{\partial S_1}{\partial y_j'} - \ldots \right) - \ldots , \end{aligned} \qquad (i = 1, 2) \tag{21}$$

where the derivatives are calculated for $x_1 = x_1'$, $x_2 = x_2'$.

If we introduce these values in the Hamiltonian, we find

$$\begin{aligned} & H\left(x_i' - \frac{\partial S_1}{\partial y_i'} - \ldots , \; y_i' + \frac{\partial S_1}{\partial x_i'} + \ldots \right) = H_2 + H_3 + H_4 + \ldots \\ & + \sum_{i=1}^{2} \left(\frac{\partial H_2}{\partial y_i'} \frac{\partial S_1}{\partial x_i'} - \frac{\partial H_2}{\partial x_i'} \frac{\partial S_1}{\partial y_i'} \right) + \sum_{i=1}^{2} \left(\frac{\partial H_3}{\partial y_i'} \frac{\partial S_1}{\partial x_i'} - \frac{\partial H_3}{\partial x_i'} \frac{\partial S_1}{\partial y_i'} \right) \\ & + \sum_{i=1}^{2} \left\{ \frac{\partial H_2}{\partial y_i'} \left(\frac{\partial S_2}{\partial x_i'} - \sum_{j=1}^{2} \frac{\partial^2 S_1}{\partial x_i' \partial x_j'} \frac{\partial S_1}{\partial y_i'} \right) - \frac{\partial H_2}{\partial x_i'} \left(\frac{\partial S_2}{\partial y_i'} - \sum_{j=1}^{2} \frac{\partial^2 S_1}{\partial y_i' \partial x_j'} \frac{\partial S_1}{\partial x_j'} \right) \right\} \\ & + \sum_{i=1}^{2} \sum_{j=1}^{2} \left\{ \tfrac{1}{2} \frac{\partial^2 H_2}{\partial x_i' \partial x_j'} \frac{\partial S_1}{\partial y_i'} \frac{\partial S_1}{\partial y_j'} - \tfrac{1}{2} \frac{\partial^2 H_2}{\partial x_i' \partial y_i'} \frac{\partial S_1}{\partial y_i'} \frac{\partial S_1}{\partial x_j'} + \tfrac{1}{2} \frac{\partial^2 H_2}{\partial y_i' \partial y_j'} \frac{\partial S_1}{\partial x_i'} \frac{\partial S_1}{\partial x_j'} \right\} \\ & + \ldots = H_2' + H_3' + H_4' + \ldots , \end{aligned} \tag{22}$$

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where H_k and all the derivatives are calculated for $x_1 = x_1'$, $y_1 = y_1'$, $x_2 = x_2'$, $y_2 = y_2'$. Hence

$$H_2 = H_2' , \tag{23}$$

$$\sum_{i=1}^{2}\left(y_i' \frac{\partial S_1}{\partial x_i'} - s_i x_i' \frac{\partial S_1}{\partial y_i'}\right) = H_3' - H_3 = Q_1 , \tag{24}$$

and in general

$$\sum_{i=1}^{2}\left(y_i' \frac{\partial S_k}{\partial x_i'} - s_i x_i' \frac{\partial S_k}{\partial y_i'}\right) = Q_k = H_{k+2}' - H_{k+2} + P_k , \tag{25}$$

where P_k is a polynomial of degree $k + 2$ in x_i', y_i', which is known, after H_2', $S_1, \ldots$ H_{k+1}', S_{k-1} are defined.

The corresponding system of equation (25) is

$$dt = \frac{dx_i'}{y_i'} = \frac{dy_i'}{-s_i x_i'} = \frac{dS_k}{Q_k}, \tag{26}$$

where t is an auxiliary variable.

One solution of equation (25) is then

$$S_k = \int Q_k dt , \tag{27}$$

where x_i', y_i' in Q_k have been replaced by the solutions of the system (26), namely,

$$x_i = \frac{\sqrt{2\Phi_i}}{\sqrt{s_i}} \sin \sqrt{s_i}(t - t_i), \qquad y_i = \sqrt{2\Phi_i} \cos \sqrt{s_i}(t - t_i) , \tag{28}$$

where $\Phi_i = \frac{1}{2}(s_i x_i'^2 + y_i'^2)$ and t_1, t_2 are constants.

In order to avoid the appearance of secular terms in (27), that is, terms where t is out of the trigonometric functions, we take H_k' at every step equal to the secular part of $(H_{k+2} - P_k)$; then Q_k contains only trigonometric terms of the form

$$\begin{matrix}\sin \\ \cos\end{matrix}(m\sqrt{s_1} \pm n\sqrt{s_2})t + \text{const.}$$

where m, n are integers. In the general case where $\sqrt{s_1}/\sqrt{s_2}$ is irrational, the integration (27) gives again trigonometric terms of the same form; then by using the relations (28) we can write S_k as a polynomial in x_i', y_i' (Contopoulos 1960).

It can be seen that H_k' does not contain odd powers of $\sqrt{2\Phi_i}$, because terms of this kind in $(H_{k+2} - P_k)$ do not give rise to secular terms. Therefore, H' is a polynomial (or a series) in Φ_1, Φ_2; in particular $H_3' = H_5' = \ldots = 0$.

If H and H' are truncated after the terms of degree ν, we find a new Hamiltonian

$$\bar{H}' = \bar{H}'(\Phi_1, \Phi_2) = \bar{H}(x_1, y_1, x_2, y_2) \tag{29}$$

which is expressed as a series of x_i, y_i, by using relations (18), convergent whenever the series (18) converge. This Hamiltonian coincides with the initial Hamiltonian H up to the terms of degree ν, and has two isolating integrals

$$\Phi_i = \tfrac{1}{2}(s_i x_i'^2 + y_i'^2) = \Phi_i(x_j, y_j) \qquad (i = 1, 2; j = 1, 2) \tag{30}$$

in the form of series of x_j, y_j, that converge in the same region as the series (18); in fact, because of the relations (13), the new variables x_i', y_i' are canonical, hence

$$\frac{dx_i'}{dt} = \frac{\partial \bar{H}'}{\partial y_i'} = \frac{\partial \bar{H}'}{\partial(\Phi_i)} y_i' , \qquad \frac{dy_i'}{dt} = -\frac{\partial \bar{H}'}{\partial x_i'} = -\frac{\partial \bar{H}'}{\partial(\Phi_i)} s_i x_i' , \tag{31}$$

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and therefore

$$\frac{d\Phi_i}{dt} = s_i x_i' \frac{dx_i'}{dt} + y_i' \frac{dy_i'}{dt} = 0\,, \tag{32}$$

that is Φ_i are two integrals of motion of the system with Hamiltonian $\bar{H}$.

The system (31) can be written

$$\frac{\partial \bar{H}}{\partial(\Phi_i)}\, dt = \frac{dx_i}{y_i'} = \frac{dy_i'}{-s_i x_i'}\,, \tag{33}$$

and its solution is given by the formulae (28) if t is replaced by $t\,(\partial \bar{H}'/\partial \Phi_i)$. Therefore the Hamiltonian $\bar{H}$ is separable.

The initial Hamiltonian H may or may not be separable. The fact that it is "near" $\bar{H}$ does not mean that it has as many isolating integrals as $\bar{H}$. This is the case only if the integrals are stable (see Sec. III). But in problems concerning actual stellar systems the Hamiltonian is not completely known; only the first few terms are known approximately, and the higher-order terms are not known at all. Therefore we may choose a separable Hamiltonian to represent at least a part of any stellar system around an equilibrium point. This Hamiltonian has convergent integrals for sufficiently small values of the variables, or for sufficiently small perturbations. The smallness of the variables can be secured if the energy is sufficiently small (being zero at the equilibrium point). For large values of the variables and large perturbations, the new Hamiltonian $\bar{H}$ may not have convergent integrals, or the difference between H and $\bar{H}$ may be great. For this reason it may be impossible to represent a given stellar system by a unique separable Hamiltonian.

II. QUASI-ISOLATING AND ERGODIC INTEGRALS

Sometimes the term "quasi-isolating integral" has been used rather loosely. We will give the following definition for it.

An integral of the form of equation (5) is called "quasi-isolating" if it is non-isolating in that part Ω' of the phase space that is defined by the values of the known integrals, but does not go through the neighborhood of every point of Ω'; that is, if equation (5) is solved for some variable, say x_1, it gives eventually an infinite number of solutions that have accumulation points in Ω'; however, these solutions are not densely distributed everywhere in Ω'.

Finally, we call an integral $I_k(x_i, y_i) = c_k$ "ergodic" in Ω' if it represents a hypersurface that goes through the neighborhood of every point of Ω'.

The results of Kolmogoroff, Arnold, and Moser show that the quasi-isolating integrals are quite common. In fact, if there is no invariant torus between two given tori around a periodic orbit, an orbit whose initial point is between these two tori does not lie on an isolating integral surface; but the orbit cannot go outside the tube defined by the two tori, that is, it is not ergodic on the whole energy-hypersurface of three dimensions (which is the only isolating integral that is known to exist in this case).

The quasi-isolating integrals give a set of isolating integral surfaces whose measure is slightly smaller than the total available measure. Therefore, these quasi-isolating integrals do not differ very much from isolating ones.

The foregoing results, however, have been found only for orbits near periodic orbits. They do not give information for a system at large. On the other hand, there are many cases where numerical calculations indicate the existence of isolating integrals, or at least quasi-isolating integrals, that are very nearly isolating for very long times (Contopoulos 1958, 1960, 1963; Torgård and Ollongren 1960; Ollongren 1962; Barbanis 1962). In fact, in these cases no appreciable change of the empirically found integral hypersurfaces is apparent for astronomically important times, of the order of 10^{10} years.

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III. STABILITY

We come now to a distinction between "stable," "quasi-stable," and "unstable" integrals.

An isolating integral is said to be stable if it remains isolating after a perturbation is applied. If it becomes quasi-isolating it is called quasi-stable, and if it becomes ergodic in the part of the phase space defined by the values of the other isolating integrals, it is called unstable. Similarly a quasi-isolating integral is called unstable if it becomes ergodic after a perturbation is applied.

It is well known that a time-independent perturbation of a potential field leaves the energy integral isolating. This is because the perturbation ΔH is of the same form as H itself, that is, analytic and single-valued in a certain region of the phase space. We say that the energy integral is stable against perturbations of this kind (time-independent, etc.).

On the other hand if an axially symmetric potential field becomes slightly asymmetric, the angular momentum with respect to the previous axis of symmetry is not conserved. Figure 1 represents an orbit in the slightly asymmetric field

$$U = -\tfrac{1}{2}[A(x^2 + y^2) - 2bxy^2] , \tag{34}$$

with $A = 0.1$, $b = 0.05$ and initial conditions $x_0 = 0.27659$, $y_0 = 0$, $X_0 = \dot{x}_0 = 0$, $Y_0 = \dot{y}_0 = 0.08746$. The initial conditions correspond to a circular orbit in the unperturbed field ($b = 0$). The value of the last term of U is of the order of 1/10 of the main term.

Equation (34) represents the field on the plane of symmetry of a galaxy that is also symmetric with respect to the xz-plane but not with respect to the z-axis. If A is given in $(10^7 \text{ years})^{-2}$, b in $(10^7 \text{ years})^{-2}\,\text{kpc}^{-1}$, x_0 in kpc and Y_0 in kpc $(10^7 \text{ years})^{-1}$, then the period of one revolution is about 20×10^7 years.

The motion is initially almost periodic for a few revolutions. The orbit is like a circle slightly compressed in the left; however, it becomes gradually more and more elongated until it becomes almost a straight line when $t \simeq 420 \times 10^7$ years. Then the motion reverses its direction of rotation and at about $t = 800 \times 10^7$ years it describes approximately the initial part of the orbit, but in the opposite direction. When $t \simeq 1220 \times 10^7$ years the orbit becomes again almost rectilinear, and when $t \simeq 1600 \times 10^7$ years it describes again approximately the initial nearly circular orbit in the same direction.

Therefore, the angular momentum changes completely, taking all the values between the initial value C and approximately $-C$, in a period of 16×10^9 years. This period is longer if the perturbation b is smaller. A more detailed description of this and similar cases will be given in another paper.

Another example of an unstable integral is the integral (8) for rational values of q. This is isolating, but a small change of q that makes it irrational transforms the integral (8) into an ergodic one.

The third integral in an axially symmetric field (Contopoulos 1960) and the two integrals besides the energy in asymmetric systems that are "near" a three-axial homogeneous ellipsoid (Contopoulos 1963) seem to be stable, or at least quasi-stable; a small perturbation in the form of an additional term in the potential function does not make the integrals ergodic.

In this connection it must be noted that the perturbation technique used in the earlier papers to derive new integrals in the form of power series is applicable only if the corresponding isolating integrals of the unperturbed case are stable (or at least quasi-stable) for long time intervals.

The stability or instability of an integral can be considered only with respect to a particular perturbation or to particular types of perturbations. For example angular momentum is stable with respect to every perturbation that preserves axial symmetry.

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But energy and presumably the other two integrals of the asymmetric systems above have a much more general kind of stability. However, energy ceases to be an integral in the case of a time-dependent perturbation.

Lynden-Bell (1962*a*, *b*) has given isolating integrals of the form $I[x_i, y_i, \zeta(\lambda)]$ where ζ is an arbitrary function of a given function $\lambda = \lambda(x_i)$, corresponding to a class of potential functions $U[x_i, \zeta(\lambda)]$. Such integrals he called "local," and he gave a list of them in the axially symmetric case in a Euclidean space of three dimensions. He also studied potentials with two arbitrary functions $\zeta(\lambda)$ and $\eta(\mu)$ that admit further isolating integrals besides the energy.

In our terminology these integrals are stable with respect to perturbations that do not change the *form* of the potential function $U[x_i, \zeta(\lambda)]$ or $U[x_i, \zeta(\lambda), \eta(\mu)]$, although ζ and η change. However, as we have seen, there is a much more general class of potential functions that has isolating integrals; these are "near" any potential function given in the form of a series. The stability of these integrals is not yet known; but the numerical

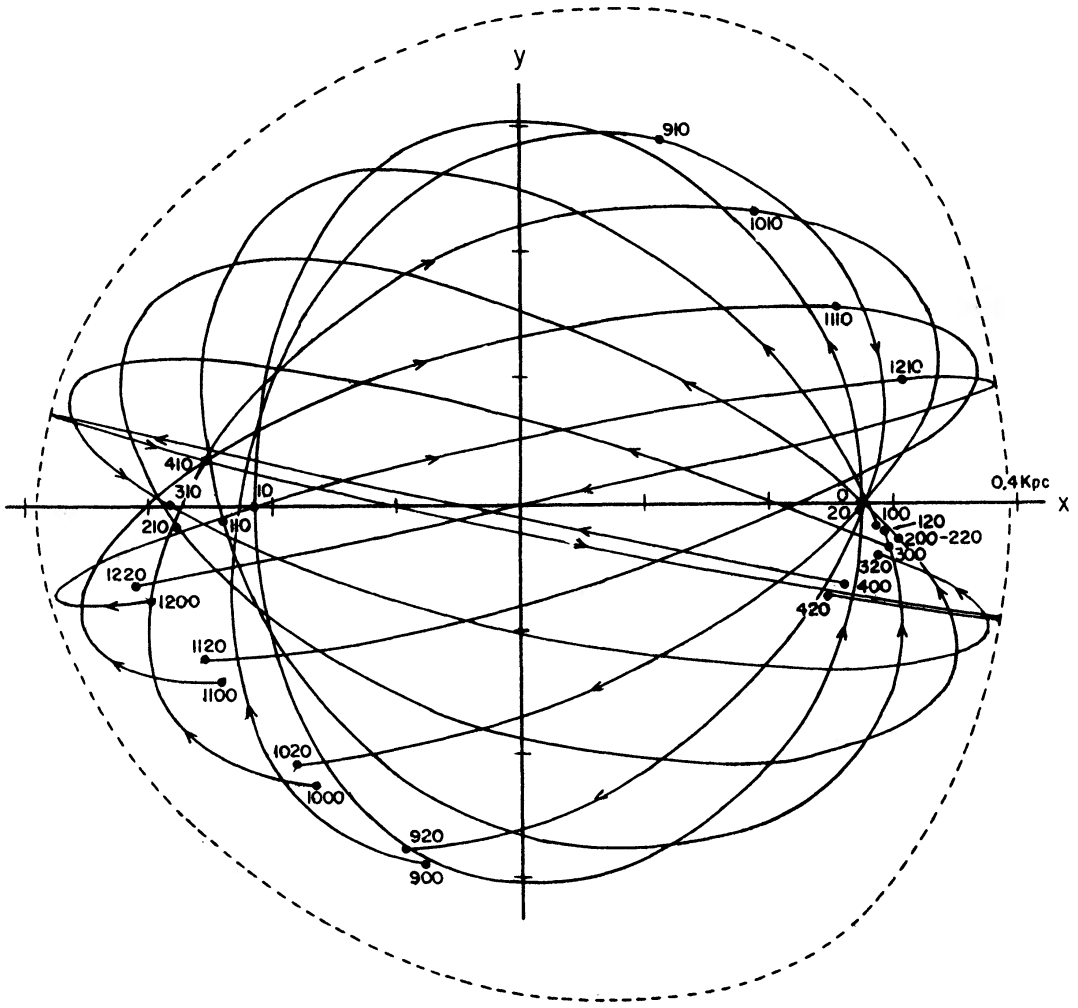


FIG. 1.—An orbit in the potential field. $U = -0.05\,[x^2 + y^2 - xy^2]$ kpc^2/(10^7 years)^2 with initial conditions $x_0 = 0.27659$ kpc, $y_0 = 0$, $X_0 = \dot{x}_0 = 0$, $Y_0 = \dot{y}_0 = 0.08746$ kpc/10^7 years. The dashed line represents the curve of zero velocity. It is evident that the orbit does not fill the whole space inside the curve of zero velocity. This indicates the possible existence of another isolating (or quasi-isolating) integral.

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evidence available indicates that these are stable or at least quasi-stable for very long times. On the other hand, the stability of the local integrals with respect to a change of ζ and η does not mean that a stellar system has the same number of local integrals as it evolves gradually; because the change of U is not due necessarily only to a change of ζ and η.

If we consider the potential function of a stellar system to be of the form

$$U = \sum_{i,\ j,\ k} A_{ijk} x^i y^j z^k , \tag{35}$$

a smooth evolution of the system will only change the coefficients A_{ijk} gradually. Therefore, a system that has initially three isolating or quasi-isolating integrals will have the same number of isolating or quasi-isolating integrals as it gradually evolves, during times of the order of 10^{10} years at least.

A special case is worth mentioning in this connection. The third integral in the case of an axially symmetric field given previously (Contopoulos 1960), which is the same as the second integral, besides the energy, in a two-dimensional field, ceases to be valid whenever the ratio of the frequencies $\sqrt{P}/\sqrt{Q}$ of the unperturbed oscillations along the two axes becomes rational. However, as it will be shown in another paper, one can find an integral in the form of a formal series expansion even in the rational cases. Its form is not very different from that of the general irrational case, if $\sqrt{P}/\sqrt{Q}$ is not equal to 1/1 or 2/1. When $\sqrt{P}/\sqrt{Q} = 1/1$ or 2/1 the form of the integral changes considerably, but there is a gradual transition to these "resonance" cases as $\sqrt{P}/\sqrt{Q}$ tends to 1/1 or 2/1. It seems that in all these cases the integrals are isolating or at least quasi-isolating. The numerical evidence shows that they are very nearly isolating for very long times, at least of the order of some 10^{10} years (see, e.g., Fig. 1).

If the perturbations applied to a Hamiltonian are large enough, the isolating integrals may become quasi-isolating or even ergodic. Such cases have been discussed by Hénon (private communication). Other cases where such a behavior is observed have been considered lately by Contopoulos and Woltjer and will be discussed in another paper.

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