

A RIGOROUS EXPOSITION OF THE BOUSSINESQ APPROXIMATIONS APPLICABLE TO A THIN LAYER OF FLUID

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ABSTRACT

The Boussinesq system of hydrodynamic equations applicable to a thin layer of fluid is rigorously developed from the Navier-Stokes equations and thermodynamic equation. The Boussinesq system is shown to be valid if two independent, non-dimensional parameters ϵ_1 and ϵ_2 are sufficiently small. If α' is the coefficient of volume expansion; κ_0 , the thermal diffusivity; C_0 , the specific heat at constant pressure; Θ' , a characteristic temperature difference; and L' a characteristic length; then $\epsilon_1 \equiv \alpha'\Theta'$ and $\epsilon_2 \equiv \kappa_0^2/C_0 L'^2 \Theta'$. By expanding the general energetic equations in a power series in ϵ_1 and ϵ_2 , the Boussinesq energetics are completely clarified.

I. INTRODUCTION

It is the aim of this paper to develop systematically the Boussinesq¹ (1903) system of hydrodynamic equations applicable to a thin layer of fluid. The Boussinesq system is obtained by making the following approximations: (a) neglect the variation of density except as it modifies gravity; (b) require zero velocity divergence; and (c) calculate the substantial temperature change from the diffusion of heat only. The Boussinesq system is significant because it is considerably simpler than the general equations (to be defined in Sec. II). Indeed, many problems in thermal convection would be analytically intractable without the use of the Boussinesq approximations.

The basic plan of this paper is as follows: given the set of dimensional parameters that characterize a general class of thermal convection problems, we utilize the Buckingham Pi theorem to construct a set of non-dimensional parameters. In our problem this leads to four non-dimensional parameters: ϵ_1 , ϵ_2 , R , and σ , where R is the Rayleigh number and σ is the Prandtl number. (ϵ_1 and ϵ_2 are defined in Sec. IV.) Let ξ be any dependent variable and write

$$\xi = \xi^{0,0} + \xi^{1,0} \epsilon_1 + \xi^{0,1} \epsilon_2 + \xi^{1,1} \epsilon_1 \epsilon_2 + \dots$$

The expansions are inserted into the non-dimensional governing equations. We require the equations to be satisfied for all sufficiently small but finite values of ϵ_1 and ϵ_2 . This leads to sets of ordered equations. In the Boussinesq problem, the simplified system occurs to the zeroth order in ϵ_1 and ϵ_2 . Since the Boussinesq system occurs uniformly to the same order in ϵ_1 and ϵ_2 , we may say the Boussinesq system is uniformly consistent.

As a side product of the above method we may easily write down higher-order systems. Now in some problems these higher-order systems are of little interest. However, it can be shown that the Boussinesq energy transformations and entropy production depend on the higher-order systems because of the peculiar simplification of the thermodynamic energy equation. In Section V we show how our procedure completely clarifies the Boussinesq energetics. We shall leave the entropy problem to a later work.

Let us contrast our procedure with that of Spiegel and Veronis (1960). These authors treated a fairly deep layer of ideal gas. The method used was an order-of-magnitude analysis in which the neglected terms were compared with the significant terms in thermal convection. Neither the energetics nor the entropy production was examined. Fi-

¹ Although the approximations examined in this paper are named after Boussinesq, they were used earlier by Oberbeck (1888) in his study of the atmospheric Hadley Regime.

nally, the authors did not show that the simplified equations are uniformly consistent in the sense of this paper.

The writer believes that the intuitive approach of Spiegel and Veronis (1960) and the rigorous approach of this paper are mutually complementary. Spiegel and Veronis aimed for physical insight, while we aim for mathematical rigor. Both approaches help to clarify the nature of the Boussinesq approximations.

II. THE "GENERAL" EQUATIONS

In this Section, we present the equations that govern a wide class of thermal convection problems. A primed quantity will denote a dimensional quantity, while a non-primed quantity will denote a non-dimensional quantity.

Let there be established an x'_1, x'_2, x'_3 Cartesian co-ordinate system in which gravity acts in the negative x'_3 direction. Then

$$\rho'_0(1 - \underline{\alpha'}\theta') \frac{du'_i}{dt'} = -\frac{\partial p'}{\partial x'_i} + \rho'_0 g' \alpha' \theta' k'_i + F'_i, \quad (2.1)$$

$$-\alpha' \frac{d\theta'}{dt'} + (1 - \alpha' \theta') \chi' = 0, \quad (2.2)$$

$$\rho'_0(1 - \underline{\alpha'}\theta') C'_{\varpi'} \frac{d\theta'}{dt'} = \frac{\partial}{\partial x'_j} K' \frac{\partial \theta'}{\partial x'_j} + \Phi + \frac{T' \alpha'}{1 - \alpha' \theta'} \left(\frac{dp'}{dt'} - \rho'_0 g' u'_3 \right), \quad (2.3)$$

where

$$F'_i \equiv \frac{\partial}{\partial x'_j} \tau'_{i,j}, \quad \tau'_{i,j} \equiv \mu'_2 \chi' \delta_{ij} + \mu'_1 \left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i} \right),$$

$$\Phi' \equiv \tau'_{i,j} \frac{\partial u'_i}{\partial x'_j}, \quad \chi' \equiv \frac{\partial u'_i}{\partial x'_i}, \quad \frac{d}{dt'} \equiv \frac{\partial}{\partial t'} + u'_i \frac{\partial}{\partial x'_i}.$$

In these equations and definitions, equation (2.1) is the momentum-conservation principle, (2.2) is the mass-conservation principle, and (2.3) is the energy-conservation principle; u'_i is the x'_i component of velocity; ϖ' , is the thermodynamic pressure, t' , the time; g' , the acceleration of gravity; χ' , the velocity divergence; ρ' , the density; $C'_{\varpi'}$, the specific heat at constant pressure; Φ' , the viscous dissipation function; and T' , the Kelvin temperature. Also α' is the coefficient of volume expansion; K' , the thermal conductivity; k'_i is a vector having the components (0, 0, 1); μ'_1 and μ'_2 are the two dynamic viscosity coefficients; and $\tau'_{i,j}$ is the viscous stress tensor under the Stokes hypothesis.

In writing the governing equations, we have postulated the equation of state

$$\rho' = \rho'_0(1 - \alpha' \theta'), \quad (2.4)$$

where

$$\theta' \equiv T' - T'_0.$$

As can be seen, we have neglected the isothermal compressibility, β' . Here p' is defined by

$$p' \equiv \varpi' + \rho'_0 g' x'_3.$$

It can be shown from classical thermodynamics that $C'_{\varpi'}$ and e' , the specific internal energy, are given by

$$C'_{\varpi'} = C'(\theta') - \frac{2\alpha'^2 T' \varpi'}{\rho'_0(1 - \alpha' \theta')^3}, \quad (2.5)$$

$$e' = \int C'(\theta') d\theta' - \frac{T' a' \varpi'}{\rho'_0 (1 - a' \theta')^2}. \quad (2.6)$$

These general equations may be applied to the Bénard problem. In Figure 1, a fluid is contained between two infinite flat plates separated by the vertical distance D' . The lower plate is warmer than the upper plate by the amount $\Delta T'$. Gravity acts as shown. The problem is to determine the fluid motion.

It can easily be shown that the isothermal compressibility, β' , has a negligible influence if the following parameter (ϵ) is small:

$$\epsilon \equiv \frac{\beta' \rho'_0 g' D'}{a' \Delta T'},$$

where ϵ measures the ratio of the variation of density from pressure variations to the variation of density from temperature variations. The depth of the fluid is to be small so that ϵ will be small.

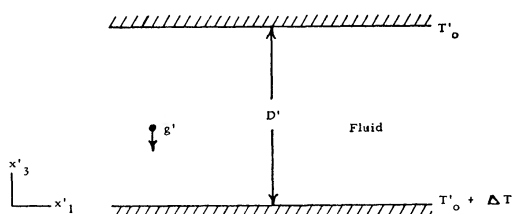


FIG. 1.—Schematic picture of the external geometry in the Bénard problem

III. THE BOUSSINESQ SIMPLIFIED SYSTEM

As indicated in Section I, the Boussinesq approximations consist of the neglect of the underlined terms in equations (2.1) and (2.3) plus the assumption of zero velocity divergence. In addition, we replace $C'_{\varpi'}$ by $C'(\theta')$ and take K' , μ'_1 , and μ'_2 to be functions solely of θ' . The Boussinesq simplified system thus is

$$\rho'_0 \frac{du'_i}{dt'} = - \frac{\partial p'}{\partial x'_i} + \rho'_0 g' a' \theta' k'_i + F'_i, \quad (3.1)$$

$$\chi' = 0, \quad (3.2)$$

$$\rho'_0 C' \frac{d\theta'}{dt'} = \frac{\partial}{\partial x'_j} K' \frac{\partial \theta}{\partial x'_j}. \quad (3.3)$$

This system has the dependent variables u'_i , θ' , and p' . The three equations, therefore, form a mathematically closed system.

IV. A RIGOROUS EXPOSITION OF THE BOUSSINESQ SYSTEM

Let C'_0 , K'_0 , and μ'_0 be the values of $C'(\theta')$, and $K'(\theta')$, and $\mu'_1(\theta')$ at $\theta' = 0$. Define

$$C_{\varpi'} \equiv \frac{C'_{\varpi'}}{C'_0}, \quad C \equiv \frac{C'}{C'_0}, \quad \mu_1 \equiv \frac{\mu'_1}{\mu'_0}, \quad \mu_2 \equiv \frac{\mu'_2}{\mu'_0}, \quad K \equiv \frac{K'}{K'_0}, \quad \nu'_0 \equiv \frac{\mu'_0}{\rho'_0}, \quad \kappa'_0 \equiv \frac{K'_0}{\rho'_0 C'_0},$$

where ν'_0 is the kinematic viscosity coefficient and κ'_0 is the thermal diffusivity. The accompanying table shows the variation for water of K , C , and μ_1 with $T'_0 = 25^\circ$ C. Of the

	24° C	25° C	26° C
$K \dots$	0 9983	1	1 0034
$C \dots \dots$	1 002	1	0 998
$\mu_1 \dots$	1 02	1	0 98

three variables considered, μ_1 has the greatest variation with temperature. The variation and even the sign of μ_2 are in doubt.

Let L' , V' , Θ' , and P' be scaling units for length, velocity, temperature, and pressure, respectively. New non-dimensional variables are defined as follows:

$$x_i \equiv \frac{x'_i}{L'}, \quad u_i \equiv \frac{u'_i}{V'}, \quad \theta \equiv \frac{\theta'}{\Theta'}, \quad p \equiv \frac{p'}{P'}.$$

We have taken

$$V' \equiv \kappa'_0 L'^{-1}, \quad P' \equiv \rho'_0 V'^2,$$

implying that time is scaled with L'/V' . The form of V' was deliberately selected by the considerations of Section I.

It should be noted that our choice of scaling units leads to eight independent parameters; namely, L' , ρ'_0 , g' , ν'_0 , κ'_0 , Θ' , α' , and C'_0 . There are four fundamental units: mass, length, time, and temperature. Therefore, by the Buckingham Pi theorem, there should be four independent, non-dimensional parameters which can be constructed from the dimensional set of parameters. Equations (2.1), (2.2), and (2.3) transform to

$$(1 - \epsilon_1 \theta) \frac{du_i}{dt} = - \frac{\partial p}{\partial x_i} + g \theta k_i + \sigma F_i, \quad (4.1)$$

$$- \epsilon_1 \frac{d\theta}{dt} + (1 - \epsilon_1 \theta) \chi = 0, \quad (4.2)$$

$$(1 - \epsilon_1 \theta) C_\infty \frac{d\theta}{dt} = \frac{\partial}{\partial x_j} K \frac{\partial \theta}{\partial x_j} + \epsilon_2 \left[\sigma \Phi + \frac{T}{1 - \epsilon_1 \theta} \left(\epsilon_1 \frac{dp}{dt} - g u_3 \right) \right], \quad (4.3)$$

where

$$F_i \equiv \frac{\partial}{\partial x_j} \tau_{i,j}, \quad \tau_{i,j} \equiv \mu_2 \chi \delta_{i,j} + \mu_1 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \Phi \equiv \tau_{i,j} \frac{\partial u_i}{\partial x_j},$$

and

$$\left. \begin{aligned} \epsilon_1 &\equiv \alpha' \Theta', \\ \epsilon_2 &\equiv \frac{\kappa'_0{}^2}{C'_0 L'^2 \Theta'}, \\ \sigma &\equiv \frac{\nu'_0}{\kappa'_0}, \\ g &\equiv R \sigma, \end{aligned} \right\} \text{four independent, non-dimensional parameters}$$

where σ is the Prandtl number and R is the Rayleigh number and

$$R \equiv \frac{g' \alpha' \Theta' L'^3}{\nu'_0 \kappa'_0}.$$

Let ξ be any one of the variables u , θ , or p and write

$$\xi = \xi^{0,0} + \xi^{1,0} \epsilon_1 + \xi^{0,1} \epsilon_2 + \xi^{1,1} \epsilon_1 \epsilon_2 + \dots$$

We insert the expansions of u_i , θ , and p into the system (4.1), (4.2), and (4.3) and require that the equations be satisfied for all finite values of ϵ_1 and ϵ_2 near zero.

Entering the formal expansions, we obtain

$$\frac{\partial u_i^{0,0}}{\partial t} + u_j^{0,0} \frac{\partial}{\partial x_j} u_i^{0,0} = - \frac{\partial p^{0,0}}{\partial x_i} + g \theta^{0,0} k_i + \sigma F_i^{0,0}, \quad (4.4)$$

$$\chi^{0,0} = 0, \quad (4.5)$$

$$C^* \left(\frac{\partial \theta^{0,0}}{\partial t} + u_j^{0,0} \frac{\partial}{\partial x_j} \theta^{0,0} \right) = \frac{\partial}{\partial x_j} K^* \frac{\partial \theta^{0,0}}{\partial x_j}, \quad (4.6)$$

where

$$F_i^{0,0} = \frac{\partial}{\partial x_j} \tau_{i,j}^{0,0}, \quad \tau_{i,j}^{0,0} = \mu_1^* \left(\frac{\partial u_i^{0,0}}{\partial x_j} + \frac{\partial u_j^{0,0}}{\partial x_i} \right),$$

$$\mu_1^* \equiv \mu_1(\theta^{0,0}), \quad K^* \equiv K(\theta^{0,0}), \quad \text{and} \quad C^* \equiv C(\theta^{0,0}).$$

Note that $\tau_{i,j}^{0,0}$ contains no reference to μ_2 , since $\chi^{0,0} = 0$.

Although higher-order systems are easily written down, we shall only note that from equation (4.2) we obtain

$$\chi^{1,0} = \frac{\partial \theta^{0,0}}{\partial t} + u_j^{0,0} \frac{\partial}{\partial x_j} \theta^{0,0},$$

and from equation (4.6) we conclude that

$$\chi^{1,0} = \frac{1}{C^*} \left(\frac{\partial}{\partial x_j} K^* \frac{\partial \theta^{0,0}}{\partial x_j} \right). \quad (4.7)$$

To apply the results of this Section to the Bénard problem of Figure 1, we should take

$$L' = D' \quad \text{and} \quad \Theta' = \Delta T'.$$

To estimate ϵ_1 and ϵ_2 we take, for water in C.G.S. units and $^\circ\text{C}$,

$$L' \sim 1, \quad \alpha' \sim 10^{-4}, \quad \Theta' \sim 1, \quad g' \sim 10^3, \quad C'_0 \sim 10^7, \quad \nu'_0 \sim 10^{-2}, \quad \kappa'_0 \sim 10^{-2}.$$

Then

$$R \sim 10^3, \quad \sigma \sim 1, \quad g \sim 10^3,$$

and

$$\epsilon_1 \sim 10^{-4}, \quad \epsilon_2 \sim 10^{-11}.$$

The above calculation suggests that in the Bénard problem, the Boussinesq approximations in the thermodynamic equation have greater validity than those in the mechanical equations.

V. THE BOUSSINESQ ENERGETICS

In this section we examine the energetics of the system (2.1), (2.2), and (2.3). It will be shown that the Boussinesq system (4.4), (4.5), and (4.6) does not by itself provide sufficient information to comprehend clearly the Boussinesq energetic transformations.

We imagine that the fluid is bound in a rigid vessel. On the boundary walls we impose either the no-slip or no-stress condition, so that no mechanical energy is transferred through the walls.

Let $\delta v'$ be an infinitesimal fluid volume, $\delta s'$ be an infinitesimal area of the bounding wall, $\underline{n}'_i$ be the outward unit normal vector to the bounding wall, and n' be distance in the direction of $\underline{n}'_i$. Also let $\underline{K}'$ be the total kinetic energy, $\underline{P}'$ be the total potential energy, and U' be the total internal energy. Thus

$$\underline{K}' \equiv \int_{v'} \frac{1}{2} \rho' u_i'^2 \delta v', \quad \underline{P}' \equiv \int_{v'} \rho' g' x_3' \delta v', \quad U' \equiv \int_{v'} \rho' e' \delta v'.$$

The system (2.1)–(2.3), after some manipulation, yields the following:

$$\frac{D}{Dt'} \underline{K}' = A' + B' - C', \quad (5.1)$$

$$\frac{D}{Dt'} \underline{P}' = -A' + D', \quad (5.2)$$

$$\frac{D}{Dt'} U' = E' - B' - D' + C'. \quad (5.3)$$

Also,

$$A' \equiv \rho'_0 g' \alpha' \int_{v'} u_3' \theta' \delta v'$$

is the rate of conversion of potential to kinetic energy and

$$B' \equiv \int_{v'} p' \chi' \delta v'$$

is the pressure work term and represents the rate of conversion of internal to kinetic energy,

$$C' \equiv \int_{v'} \Phi' \delta v'$$

is the rate of dissipation of kinetic energy by viscosity. Since this process must be irreversible and $\mu'_1 > 0$, we must have $\mu'_2 \geq \frac{2}{3}\mu'_1$ to render Φ' positive definite; C' is always a positive rate of conversion of kinetic to internal energy;

$$D' \equiv -\rho'_0 g' \int_{v'} x_3' \chi' \delta v'$$

is the rate of working of the hydrostatic pressure with $\rho' = \rho'_0$ through the divergence field and is a rate of conversion of internal to potential energy; and

$$E' \equiv \int_{s'} K' \frac{\partial \theta'}{\partial n} \delta v'$$

is the conductive heat flux to the fluid through the bounding walls. It is a rate of increase of internal energy.

The Boussinesq energetics are obtained by non-dimensionalizing, as in Section IV,

equations (5.1)–(5.3) and expanding the resulting equations in a power series in ϵ_1 and ϵ_2 . The results are

$$\frac{D}{Dt} \underline{K}^{0,0} = A^{0,0} + B^{0,0} - C^{0,0}, \quad (5.4)$$

$$\frac{D}{Dt} \underline{P}^{0,0} = -A^{0,0} + D^{0,0}, \quad (5.5)$$

$$\frac{D}{Dt} U^{0,0} = E^{0,0} - B^{0,0} - D^{0,0} + C^{0,0}, \quad (5.6)$$

where

$$A^{0,0} = g \int_v u_3^{0,0} \theta^{0,0} \delta v, \quad B^{0,0} = 0, \quad C^{0,0} = \sigma \int_v \Phi^{0,0} \delta v, \\ D^{0,0} = -g \int_v x_3 \chi^{1,0} \delta v, \quad E^{0,0} = \int_s K^* \frac{\partial \theta^{0,0}}{\partial n} \delta s, \quad \text{and} \quad \Phi^{0,0} = \tau_{ij}^{0,0} \frac{\partial u_j^{0,0}}{\partial x_i}.$$

The reference to $\chi^{1,0}$ and $\theta^{0,1}$ should be noted in the Boussinesq energetics. These variables are examples of the implied links to higher-order systems suggested in Section I.

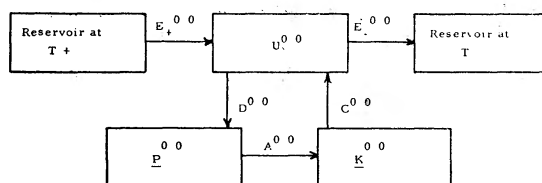


FIG. 2.—The energy cycle in the establishment of thermal convection, using the Boussinesq system of equations.

To illustrate the above equations, suppose that we maintain a reservoir at temperature T^+ adjacent to one part of the bounding wall and another reservoir at lower temperature T^- adjacent to another part of the bounding wall. Let s_+ be that part of the wall where $\partial \theta^{0,1}/\partial n > 0$ and s_- be that part of the wall where $\partial \theta^{0,1}/\partial n < 0$. Define

$$E_+^{0,0} \equiv \int_{s_+} K^* \frac{\partial \theta^{0,1}}{\partial n} \delta s, \quad E_-^{0,0} \equiv - \int_{s_-} K^* \frac{\partial \theta^{0,1}}{\partial n} \delta s.$$

The establishment of thermal convection using the Boussinesq system will take place in the following steps (see Fig. 2):

- Heat is conducted to the fluid at the rate $E_+^{0,0}$ and is converted to internal energy.
- The heating must be so arranged that the fluid is heated at low values of x_3 and cooled at high values of x_3 . Under this condition $D^{0,0}$ is positive.
- With a positive correlation of $u_3^{0,0}$ and $\theta^{0,0}$ the thermal convection itself tends to degrade the potential energy, so that $A^{0,0}$ is positive and potential energy is converted to kinetic energy.
- Since $C^{0,0}$ is positive (even irrespective of μ_2^* , since $\chi^{0,0} = 0$), kinetic energy is converted to internal energy at the rate $C^{0,0}$.
- Finally, heat is withdrawn from the system at the rate $E_-^{0,0}$, thus lowering the internal energy. For steady state

$$D^{0,0} = A^{0,0} = C^{0,0} \quad \text{and} \quad E_+^{0,0} = E_-^{0,0}.$$

The above results are a major aspect of this paper. As indicated before, they cannot be rigorously deduced from the Boussinesq system alone.

VI. CONCLUSIONS

In Section IV we formally derived the Boussinesq simplified system. We have demonstrated that if the series ϵ_1 and ϵ_2 converge, the simplified system will more accurately approximate the general equations as ϵ_1 and ϵ_2 tend to zero. We have shown that the Boussinesq system is uniformly consistent.

By expanding the general energy equations, we have clarified the zeroth-order energy transformations and shown the role of $\chi^{1,0}$ and $\theta^{0,1}$ in completing the Boussinesq energetics. The role of $\chi^{1,0}$ can probably be deduced from a less rigorous procedure, but it is doubtful whether the role of $\theta^{0,1}$ in the internal energy equation could have been foreseen.

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