ON SUPER-POTENTIALS IN THE THEORY OF NEWTONIAN GRAVITATION

S. CHANDRASEKHAR AND NORMAN R. LEBOVITZ
University of Chicago and Massachusetts Institute of Technology
Received September 1, 1961

ABSTRACT

The character of the gravitational equilibrium of bodies in rotation and with prevalent magnetic fields depends on the tensor potential,

$$\mathfrak{B}_{ij}=G\int_V \rho(x')\frac{(x_i-x_i')(x_j-x_j')}{|x-x'|^3}\,dx',$$

and the associated tensors,

$$\mathfrak{W}_{ij}=-\frac{1}{2}\int_V \rho\,\mathfrak{B}_{ij}\,dx \qquad \text{and} \qquad \mathfrak{W}_{pq;ij}=\int_V \rho\,x_p\frac{\partial \mathfrak{B}_{ij}}{\partial x_q}\,dx\,.$$

This paper is devoted to a consideration of these fundamental tensors It is shown, in particular, that the tensor potential can be expressed in the form

$$\mathfrak{B}_{ij}=\mathfrak{B}\,\delta_{ij}+\frac{\partial^2\chi}{\partial x_i\partial x_j},$$

where $\mathfrak{B}$ is the gravitational potential as usually defined and χ is a *super-potential* determined by the equation

$$\nabla^2\chi=-2\mathfrak{B}\,.$$

I. INTRODUCTION

The development of the virial theorem in its general tensor form (Chandrasekhar 1960, 1961*a*) requires the consideration of the symmetric tensor potential

$$\mathfrak{B}_{ij}(x)=G\int_V \rho(x')\frac{(x_i-x_i')(x_j-x_j')}{|x-x'|^3}\,dx', \tag{1}$$

where, for the sake of brevity, we have written $dx'=dx_1\,dx_2\,dx_3$ and abridged three integral signs into one. In equation (1) the integration is effected over the entire volume V occupied by the fluid, and G is the constant of gravitation. The gravitational potential,

$$\mathfrak{B}(x)=G\int_V \frac{\rho(x')}{|x-x'|}\,dx', \tag{2}$$

as usually defined, is the contracted form of $\mathfrak{B}_{ij}$; thus

$$\mathfrak{B}=\mathfrak{B}_{ii}\,. \tag{3}$$

Associated with the tensor potential $\mathfrak{B}_{ij}$ is the potential energy tensor,

$$\mathfrak{W}_{ij}=-\tfrac{1}{2}G\int_V\int_V \rho(x)\,\rho(x')\frac{(x_i-x_i')(x_j-x_j')}{|x-x'|^3}\,dx\,dx'; \tag{4}$$

and this tensor represents a similar generalization of the usual definition of the gravitational potential energy:

$$\mathfrak{W} = \mathfrak{W}_{ii} = -\tfrac{1}{2}G\int_V\int_V \frac{\rho(x)\rho(x')}{|x-x'|}\,dx\,dx' . \tag{5}$$

A further quantity which occurs when treating small oscillations about equilibrium is the super-matrix (cf. Chandraskehar 1960 and 1961*a*, § 118)

$$\mathfrak{W}_{pq;ij} = \int_V \rho(x)\, x_p \frac{\partial \mathfrak{B}_{ij}}{\partial x_q}\,dx . \tag{6}$$

It appears that a knowledge of the tensor potential $\mathfrak{B}_{ij}$ and the associated tensors $\mathfrak{W}_{ij}$ and $\mathfrak{W}_{pq;ij}$ is essential to an understanding of the character of the gravitational equilibrium of bodies in rotation and with magnetic field. Thus many features of the equilibrium of rotating masses which appear unexpected and obscure can be readily interpreted and understood in terms of the tensors $\mathfrak{W}_{ij}$ and $\mathfrak{W}_{pq;ij}$. For these reasons, and also for their own interest, this paper will be devoted to a consideration of these fundamental tensors in gravitational potential theory.

II. THE TENSOR POTENTIAL

First, we observe that, in terms of the vector (cf. Lebovitz 1961, eqs. [48] and [49]),

$$\mathfrak{D}_i = G\int_V \rho(x')\frac{x_i'}{|x-x'|}\,dx' , \tag{7}$$

we can express $\mathfrak{B}_{ij}$ in the manner

$$\mathfrak{B}_{ij} = -x_i \frac{\partial \mathfrak{B}}{\partial x_j} + \frac{\partial \mathfrak{D}_i}{\partial x_j} . \tag{8}$$

Since $\mathfrak{B}_{ij}$ is symmetric in its indices,

$$-x_i \frac{\partial \mathfrak{B}}{\partial x_j} + \frac{\partial \mathfrak{D}_i}{\partial x_j} = -x_j \frac{\partial \mathfrak{B}}{\partial x_i} + \frac{\partial \mathfrak{D}_j}{\partial x_i} , \tag{9}$$

or

$$\frac{\partial \mathfrak{D}_i}{\partial x_j} - \frac{\partial \mathfrak{D}_j}{\partial x_i} = x_i \frac{\partial \mathfrak{B}}{\partial x_j} - x_j \frac{\partial \mathfrak{B}}{\partial x_i} . \tag{10}$$

Alternatively, we can write

$$\text{curl}\ \mathfrak{D} = \text{grad}\ \mathfrak{B} \times x = \text{curl}\,(\mathfrak{B}x) . \tag{11}$$

Consequently, $\mathfrak{D}$ can differ from $\mathfrak{B}x$ only by the gradient of a scalar function; and, denoting this scalar function by χ, we can write

$$\mathfrak{D}_i = \mathfrak{B}x_i + \frac{\partial \chi}{\partial x_i} . \tag{12}$$

The equation governing χ can be obtained by substituting the foregoing expression for $\mathfrak{D}$ in the contracted version of equation (8), namely,

$$\mathfrak{B} = -x_i \frac{\partial \mathfrak{B}}{\partial x_i} + \frac{\partial \mathfrak{D}_i}{\partial x_i} ; \tag{13}$$

we find

$$\mathfrak{B} = -x_i \frac{\partial \mathfrak{B}}{\partial x_i} + \left(x_i \frac{\partial \mathfrak{B}}{\partial x_i} + 3\mathfrak{B} + \nabla^2 \chi \right). \tag{14}$$

Hence

$$\nabla^2 \chi = -2\mathfrak{B} . \tag{15}$$

Taking the Laplacian of equation (15) and making use of Poisson's equation

$$\nabla^2 \mathfrak{B} = -4\pi G \rho \tag{16}$$

we obtain

$$\nabla^4 \chi = 8\pi G \rho . \tag{17}$$

It would thus be appropriate to call χ the *super-potential* of the gravitational field. In its term the tensor potential $\mathfrak{B}_{ij}$ is given by

$$\mathfrak{B}_{ij} = \mathfrak{B}\,\delta_{ij} + \frac{\partial^2 \chi}{\partial x_i \partial x_j} . \tag{18}$$

There is an alternative way of exhibiting the relationship between $\mathfrak{B}$ and χ that is instructive. Define

$$\chi = -G \int_V \rho(x') \, | x - x' | \, dx' . \tag{19}$$

By differentiating this equation with respect to x_i and making use of equations (2) and (7) (which *define* $\mathfrak{B}$ and $\mathfrak{D}_i$), we obtain

$$\frac{\partial \chi}{\partial x_i} = -G \int_V \rho(x') \frac{x_i - x_i'}{| x - x' |} \, dx' = -x_i \mathfrak{B} + \mathfrak{D}_i , \tag{20}$$

in agreement with equation (12). And, by a further differentiation, we find

$$\begin{aligned} \frac{\partial^2 \chi}{\partial x_j \partial x_i} &= -G \int_V \rho(x') \left[\frac{\delta_{ij}}{| x - x' |} - \frac{(x_i - x_i')(x_j - x_j)}{| x - x' |^3} \right] dx' \\ &= -\mathfrak{B}\,\delta_{ij} + \mathfrak{B}_{ij} , \end{aligned} \tag{21}$$

in agreement with equation (18).

The very existence of equation (19) shows that the entire gravitational problem is reduced, in principle at least, to the determination of the single function χ: $\mathfrak{B}$ is determined from χ by means of equation (18) (or [21]). And, moreover, equations (2) and (19) show precisely in what sense χ is to be regarded as a super-potential.

III. THE POTENTIAL ENERGY TENSOR

It follows from the definitions of the respective quantities that (cf. eqs. [1] and [4])

$$\mathfrak{W}_{ij} = -\frac{1}{2} \int_V \rho \mathfrak{B}_{ij} dx . \tag{22}$$

Also, it can be readily shown that (cf. Chandrasekhar 1960, eq. [10])

$$\mathfrak{W}_{ij} = \int_V \rho x_i \frac{\partial \mathfrak{B}}{\partial x_j} \, dx = \int_V \rho x_j \frac{\partial \mathfrak{B}}{\partial x_i} \, dx . \tag{23}$$

The contraction of equations (22) and (23) leads to known results in potential theory; thus

$$\mathfrak{W} = -\frac{1}{2}\int_V \rho\mathfrak{V}dx = \int_V \rho x_i \frac{\partial \mathfrak{V}}{\partial x_i}dx\,. \qquad (24)$$

IV. THE SUPER-MATRIX $\mathfrak{W}_{pq;ij}$

As defined, the tensor

$$\mathfrak{W}_{pq;ij} = \int_V \rho x_p \frac{\partial \mathfrak{V}_{ij}}{\partial x_q}dx \qquad (25)$$

is clearly symmetric in its second pair of indices; and its contraction with respect to this pair gives

$$\mathfrak{W}_{pq;ii} = \int_V \rho x_p \frac{\partial \mathfrak{V}}{\partial x_q}dx = \mathfrak{W}_{pq}\,, \qquad (26)$$

which *is* symmetric in p and q. However, as we shall presently see, the uncontracted tensor is not, generally, symmetric in its first pair of indices.

By an integration by parts, we obtain from equation (25) the formula

$$\mathfrak{W}_{pq;ij} = -\int_V x_p \frac{\partial \rho}{\partial x_q}\mathfrak{V}_{ij}dx - \delta_{pq}\int_V \rho\mathfrak{V}_{ij}dx\,, \qquad (27)$$

on the assumption that the density vanishes on the boundary of V (or that it vanishes at infinity with sufficient rapidity). Making use of equation (22), we can rewrite equation (27) in the form

$$\mathfrak{W}_{pq;ij} = -\int_V x_p \frac{\partial \rho}{\partial x_q}\mathfrak{V}_{ij}dx + 2\,\delta_{pq}\mathfrak{W}_{ij}\,. \qquad (28)$$

Now substituting for $\mathfrak{V}_{ij}$ in equation (25) its explicit expression given in equation (1), we obtain

$$\begin{aligned}\mathfrak{W}_{pq;ij} &= G\int_V dx\,\rho(x)\,x_p\frac{\partial}{\partial x_q}\int_V dx'\rho(x')\frac{(x_i-x_i')(x_j-x_j')}{|x-x'|^3}\\ &= \delta_{qi}G\int_V\int_V dx\,dx'\rho(x)\,\rho(x')\frac{x_p(x_j-x_j')}{|x-x'|^3}\\ &\qquad + \delta_{qj}G\int_V\int_V dx\,dx'\rho(x)\,\rho(x')\frac{x_p(x_i-x_i')}{|x-x'|^3}\\ &\qquad - 3G\int_V\int_V dx\,dx'\rho(x)\,\rho(x')\frac{x_p(x_q-x_q')(x_i-x_i')(x_j-x_j')}{|x-x'|^5}.\end{aligned} \qquad (29)$$

By averaging equation (29) and the equation resulting from it by interchanging the primed and the unprimed variables of integration, we obtain

$$\begin{aligned}\mathfrak{W}_{pq;ij} &= -\mathfrak{W}_{pi}\delta_{qj} - \mathfrak{W}_{pj}\delta_{qi}\\ &\qquad -\tfrac{3}{2}G\int_V\int_V \rho(x)\,\rho(x')\frac{(x_p-x_p')(x_q-x_q')(x_i-x_i')(x_j-x_j')}{|x-x'|^5}dx\,dx',\end{aligned} \qquad (30)$$

where it should be noted that the last term on the right-hand side is completely symmetric in all four indices.

Several elementary identities can be deduced from equation (30). Thus

$$\mathfrak{W}_{pp;ii} = \mathfrak{W}_{ii;pp} \quad \text{(summation convention suspended)} \tag{31}$$

and

$$\mathfrak{W}_{ij;ij} - \mathfrak{W}_{ji;ij} = \mathfrak{W}_{jj} - \mathfrak{W}_{ii} \quad \text{(summation convention suspended)}\,. \tag{32}$$

Next, by contracting equation (30) with respect to two selected indices out of the four, we obtain (in turn)

$$\mathfrak{W}_{pq;ii} = \mathfrak{W}_{pq}\,, \qquad \mathfrak{W}_{pp;ij} = \mathfrak{W}_{ij}\,, \qquad \mathfrak{W}_{pq;iq} = -\mathfrak{W}_{pi}\,,$$

and

$$\mathfrak{W}_{pq;pj} = 2\mathfrak{W}_{qj} - \mathfrak{W}\,\delta_{qj}\,. \tag{33}$$

An identity of a different sort follows from inserting in the definition of $\mathfrak{W}_{pq;ij}$ the representation of $\mathfrak{B}_{ij}$ in terms of the super-potential χ. Thus, when $i \neq j$ and with the summation convention suspended, we have

$$\begin{aligned}\mathfrak{W}_{ij;ij} &= \int_V \rho\, x_i \frac{\partial \mathfrak{B}_{ij}}{\partial x_j}\, dx \\ &= \int_V \rho\, x_i \frac{\partial}{\partial x_j}\left(\frac{\partial^2 \chi}{\partial x_i \partial x_j}\right) dx \\ &= \int_V \rho\, x_i \frac{\partial}{\partial x_i}\left(\frac{\partial^2 \chi}{\partial x_j^2}\right) dx \\ &= \int_V \rho\, x_i \frac{\partial}{\partial x_i}\left(\mathfrak{B}_{jj} - \mathfrak{B}\right) dx\,.\end{aligned} \tag{34}$$

Thus

$$\mathfrak{W}_{ij;ij} = \mathfrak{W}_{ii;jj} - \mathfrak{W}_{ii} \quad \text{(summation convention suspended)}\,. \tag{35}$$

Relation (32) is now seen to be a consequence of relations (31) and (35).

V. THE FUNDAMENTAL TENSORS FOR SYSTEMS WITH TRIPLANAR SYMMETRY

We shall say that a configuration has *triplanar symmetry* if its density distribution, with respect to a suitably chosen system of Cartesian co-ordinates, is an even function of the co-ordinates, i.e., if a co-ordinate system exists such that

$$\rho(-x_1, x_2, x_3) \equiv \rho(x_1, -x_2, x_3) \equiv \rho(x_1, x_2, -x_3) \equiv \rho(x_1, x_2, x_3)\,. \tag{36}$$

When the density distribution has this symmetry, the moment of inertia tensor, I_{ij}, is clearly diagonal:

$$I_{ij} = \int_V \rho\, x_i x_j\, dx = 0 \qquad \text{if } i \neq j\,. \tag{37}$$

From equations (2) and (19) defining $\mathfrak{B}$ and χ, it follows that the potential and the super-potential have the same symmetry as ρ with respect to reflection. Therefore, if ρ is an even function of the co-ordinates, then so are $\mathfrak{B}$ and χ. From equation (18) we may now conclude that under these circumstances

$$\mathfrak{B}_{ij} \text{ is odd in } x_i \text{ and } x_j \text{ if } i \neq j\,, \text{ and} \tag{38}$$

$$\text{is even in all three co-ordinates if } i = j\,.$$

Considering, now, the potential energy tensor, we infer from the symmetry properties of $\mathfrak{B}_{ij}$ and the formula

$$\mathfrak{W}_{ij} = -\frac{1}{2}\int_V \rho\, \mathfrak{B}_{ij}\, dx \tag{39}$$

that

$$\mathfrak{W}_{ij} = 0 \qquad \text{if } i \neq j \, . \tag{40}$$

The potential energy tensor and the moment of inertia tensor can therefore be brought to the diagonal form simultaneously if the object has triplanar symmetry.

Turning our attention next to the super-matrix,

$$\mathfrak{W}_{pq;ij} = \int_V \rho x_p \frac{\partial \mathfrak{B}_{ij}}{\partial x_q} dx \, , \tag{41}$$

we observe that when $i = j$, the integrand is odd in two of the co-ordinates, x_p and x_q, if $p \neq q$, while it is even in all three co-ordinates if $p = q$. Therefore,

$$\mathfrak{W}_{pq;ij} = 0 \qquad \text{if} \qquad p \neq q \qquad \text{and} \qquad i = j \, . \tag{42}$$

On the other hand, if $i \neq j$, then the integrand is odd in x_i and x_j if $p = q$; the integral will therefore vanish under these circumstances. Similarly, if $p \neq q$ and one of them is not equal to *either* i *or* j, then the integrand will again be odd in two of the three co-ordinates and the integral will again vanish. The only circumstance under which $\mathfrak{W}_{pq;ij}$ will not vanish identically is when the integrand is even in all three co-ordinates; and, when $i \neq j$, this can happen only in two cases, namely, when the pair of indices (p, q) coincides with the pair (i, j) or (j, i). Thus

$$\mathfrak{W}_{pq;ij} \neq 0 \text{ when } i \neq j \text{ only when } p = i \text{ and } q = j \, , \quad \text{or } p = j \text{ and } q = i \, . \tag{43}$$

VI. THE FUNDAMENTAL TENSORS FOR SYSTEMS WHICH ARE IN ADDITION AXIALLY SYMMETRIC ABOUT x_3

We shall now suppose that the systems considered in Section V are, in addition, axially symmetric about x_3. Then, in cylindrical polar co-ordinates, $\varpi (= \sqrt{[x_1^2 + x_2^2]})$, $z \; (= x_3)$, and ϕ (= the azimuthal angle), the density distribution is of the form

$$\rho \equiv \rho(\varpi, z) \qquad \text{and} \qquad \rho(\varpi, z) \equiv \rho(\varpi, -z) \, . \tag{44}$$

From equations (2) and (19) it follows that $\mathfrak{B}$ and χ have the same symmetries as ρ. However, not all components of the tensor potential, $\mathfrak{B}_{ij}$, share these symmetries. Indeed, as we shall see presently, with the exception of $\mathfrak{B}_{33}$, all of them explicitly depend on x_1 and/or x_2. The nature of these dependences can be readily established. Thus, considering $\mathfrak{B}_{11}$, for example, we have

$$\mathfrak{B}_{11} = \mathfrak{B} + \frac{\partial^2 \chi}{\partial x_1^2} = \mathfrak{B} + \frac{\partial}{\partial x_1} \left(\frac{x_1}{\varpi} \frac{\partial \chi}{\partial \varpi} \right) \tag{45}$$

or

$$\mathfrak{B}_{11} = \mathfrak{B} + \frac{1}{\varpi} \frac{\partial \chi}{\partial \varpi} + \frac{x_1^2}{\varpi} \frac{\partial}{\partial \varpi} \left(\frac{1}{\varpi} \frac{\partial \chi}{\partial \varpi} \right) . \tag{46}$$

Similarly,

$$\mathfrak{B}_{22} = \mathfrak{B} + \frac{1}{\varpi} \frac{\partial \chi}{\partial \varpi} + \frac{x_2^2}{\varpi} \frac{\partial}{\partial \varpi} \left(\frac{1}{\varpi} \frac{\partial \chi}{\partial \varpi} \right) , \tag{47}$$

$$\mathfrak{B}_{12} = \frac{x_1 x_2}{\varpi} \frac{\partial}{\partial \varpi} \left(\frac{1}{\varpi} \frac{\partial \chi}{\partial \varpi} \right) . \tag{48}$$

$$\mathfrak{V}_{33}=\mathfrak{V}+\frac{\partial^2\chi}{\partial z^2}, \tag{49}$$

$$\mathfrak{V}_{13}=\frac{x_1}{\varpi}\frac{\partial^2\chi}{\partial\varpi\partial z}, \qquad \text{and} \qquad \mathfrak{V}_{32}=\frac{x_2}{\varpi}\frac{\partial^2\chi}{\partial\varpi\partial z}. \tag{50}$$

Considering the diagonal elements of $\mathfrak{W}_{ij}$ (which are the only non-vanishing ones in view of the triplanar symmetry), we have

$$\mathfrak{W}_{11}=\int_V \rho x_1\frac{\partial\mathfrak{V}}{\partial x_1}dx=\iiint_V \rho x_1^2\frac{\partial\mathfrak{V}}{\partial\varpi}d\varpi dz d\phi$$
$$=\int_0^\infty\int_{-\infty}^{+\infty}\int_0^{2\pi}\rho\varpi^2\cos^2\phi\frac{\partial\mathfrak{V}}{\partial\varpi}d\varpi dz d\phi; \tag{51}$$
[1]

or, effecting the integration over ϕ, we have

$$\mathfrak{W}_{11}=\pi\int_0^\infty\int_{-\infty}^{+\infty}\rho\varpi^2\frac{\partial\mathfrak{V}}{\partial\varpi}d\varpi dz. \tag{52}$$

Clearly, we shall obtain this same expression for $\mathfrak{W}_{22}$. Thus, as one might have expected,

$$\mathfrak{W}_{11}=\mathfrak{W}_{22}. \tag{53}$$
[2]

However,

$$\mathfrak{W}_{33}=2\pi\int_0^\infty\int_{-\infty}^{+\infty}\rho\varpi z\frac{\partial\mathfrak{V}}{\partial z}d\varpi dz \tag{54}$$

will, in general, be different from $\mathfrak{W}_{11}$ (or $\mathfrak{W}_{22}$).

From the equality of $\mathfrak{W}_{11}$ and $\mathfrak{W}_{22}$ it follows from equation (32) that

$$\mathfrak{W}_{12;12}=\mathfrak{W}_{21;12}. \tag{55}$$

Considering, now, the non-vanishing elements of $\mathfrak{W}_{pq;ij}$ systematically, we first observe that the equations

$$\mathfrak{W}_{11;11}=\int_V\rho x_1\frac{\partial}{\partial x_1}\left(\mathfrak{V}+\frac{\partial^2\chi}{\partial x_1^2}\right)dx \tag{56}$$

and

$$\mathfrak{W}_{22;22}=\int_V\rho x_2\frac{\partial}{\partial x_2}\left(\mathfrak{V}+\frac{\partial^2\chi}{\partial x_2^2}\right)dx, \tag{57}$$

after integrations over ϕ, lead to identical expressions. Therefore,

$$\mathfrak{W}_{11;11}=\mathfrak{W}_{22;22}. \tag{58}$$

Similarly, we can readily verify that

$$\mathfrak{W}_{22;33}=\mathfrak{W}_{11;33}, \qquad \mathfrak{W}_{13;13}=\mathfrak{W}_{23;23}, \qquad \mathfrak{W}_{31;13}=\mathfrak{W}_{32;23}. \tag{59}$$

Also, we have the relations (cf. eqs. [33] and [35])

$$\mathfrak{W}_{13;13}=\mathfrak{W}_{33;11}-\mathfrak{W}_{11}=\mathfrak{W}_{11;33}-\mathfrak{W}_{11}=-(\mathfrak{W}_{11;22}+\mathfrak{W}_{11;11}) \tag{60}$$

and

$$\mathfrak{W}_{31;13}=\mathfrak{W}_{11;33}-\mathfrak{W}_{33}=-(\mathfrak{W}_{22;33}+\mathfrak{W}_{33;33}). \tag{61}$$

[1] We are writing the limits (0, ∞) and (−∞, +∞) for ϖ and z only formally; we are not implying that the system necessarily extends to infinity.

[2] It is clear that, under these same circumstances, $I_{11}=I_{22}$.

A less obvious identity (in case of axial symmetry) is the following:

$$\mathfrak{W}_{12;12} = \tfrac{1}{2}(\mathfrak{W}_{11;11} - \mathfrak{W}_{11;22}) . \tag{62}$$

This can be established as follows. Considering, first, $\mathfrak{W}_{12;12}$, we have

$$\begin{aligned}
\mathfrak{W}_{12;12} &= \int_V \rho x_1 \frac{\partial \mathfrak{V}_{12}}{\partial x_2} dx \\
&= -\int_V x_1 \frac{\partial \rho}{\partial x_2} \mathfrak{V}_{12} dx \\
&= -\int_V \frac{x_1^2 x_2^2}{\varpi^2} \frac{\partial \rho}{\partial \varpi} \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \chi}{\partial \varpi}\right) dx \\
&= -2\pi \int_0^\infty \int_{-\infty}^{+\infty} \langle \cos^2\phi \sin^2\phi \rangle \varpi^3 \frac{\partial \rho}{\partial \varpi} \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \chi}{\partial \varpi}\right) d\varpi\, dz \\
&= -\tfrac{1}{4}\pi \int_0^\infty \int_{-\infty}^{+\infty} \varpi^3 \frac{\partial \rho}{\partial \varpi} \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \chi}{\partial \varpi}\right) d\varpi\, dz .
\end{aligned} \tag{63}$$

Similarly,

$$\begin{aligned}
\mathfrak{W}_{11;11} - \mathfrak{W}_{11;22} &= \int_V \rho x_1 \frac{\partial}{\partial x_1}\left(\frac{\partial^2 \chi}{\partial x_1^2} - \frac{\partial^2 \chi}{\partial x_2^2}\right) dx \\
&= -\int_V x_1 \frac{\partial \rho}{\partial x_1}\left(\frac{\partial^2 \chi}{\partial x_1^2} - \frac{\partial^2 \chi}{\partial x_2^2}\right) dx - \int_V \rho \left(\frac{\partial^2 \chi}{\partial x_1^2} - \frac{\partial^2 \chi}{\partial x_2^2}\right) dx \\
&= -\int_V \frac{x_1^2}{\varpi} \frac{\partial \rho}{\partial \varpi}\left[\frac{\partial}{\partial x_1}\left(\frac{x_1}{\varpi}\frac{\partial \chi}{\partial \varpi}\right) - \frac{\partial}{\partial x_2}\left(\frac{x_2}{\varpi}\frac{\partial \chi}{\partial \varpi}\right)\right] dx \\
&= -\int_V \frac{1}{\varpi^2} \frac{\partial \rho}{\partial \varpi} \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \chi}{\partial \varpi}\right)(x_1^4 - x_1^2 x_2^2)\, dx \\
&= -2\pi \int_0^\infty \int_{-\infty}^{+\infty} \varpi^3 \frac{\partial \rho}{\partial \varpi} \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \chi}{\partial \varpi}\right) \\
&\qquad \times (\langle \cos^4\phi \rangle - \langle \cos^2\phi \sin^2\phi \rangle)\, d\varpi\, dz \\
&= -\tfrac{1}{2}\pi \int_0^\infty \int_{-\infty}^{+\infty} \varpi^3 \frac{\partial \rho}{\partial \varpi} \frac{\partial}{\partial \varpi}\left(\frac{1}{\varpi}\frac{\partial \chi}{\partial \varpi}\right) d\varpi\, dz .
\end{aligned} \tag{64}$$

From a comparison of the end results of equations (63) and (64), we obtain relation (62).

Summarizing the results of the preceding analysis, we can express the non-vanishing elements of $\mathfrak{W}_{pq;ij}$ in terms of four of them as follows:

$$\begin{aligned}
&\mathfrak{W}_{11;11} = \mathfrak{W}_{22;22} = A \text{ (say)}, \\
&\mathfrak{W}_{11;22} = \mathfrak{W}_{22;11} = B \text{ (say)},
\end{aligned} \tag{65}$$

$$\mathfrak{W}_{11;33} = \mathfrak{W}_{22;33} = \mathfrak{W}_{33;11} = \mathfrak{W}_{33;22} = C \text{ (say)}, \tag{66}$$

$$\mathfrak{W}_{33;33} = D \text{ (say)}, \tag{67}$$

$$\mathfrak{W}_{12;12} = \mathfrak{W}_{21;12} = \tfrac{1}{2}(\mathfrak{W}_{11;11} - \mathfrak{W}_{22;11}) = \tfrac{1}{2}(A - B), \tag{68}$$

$$\begin{aligned}
&\mathfrak{W}_{13;13} = \mathfrak{W}_{23;23} = -\mathfrak{W}_{11} + \mathfrak{W}_{11;33} = C - \mathfrak{W}_{11} , \\
&\mathfrak{W}_{31;13} = \mathfrak{W}_{32;23} = \mathfrak{W}_{11;33} - \mathfrak{W}_{33} = C - \mathfrak{W}_{33} .
\end{aligned} \tag{69}$$

Among the four constants A, B, C, and E there are three relations. These follow from the general relations (cf. eqs. [33])

$$\mathfrak{W}_{11;qq}=\mathfrak{W}_{11;11}+\mathfrak{W}_{11;22}+\mathfrak{W}_{11;33}=\mathfrak{W}_{11}\,,$$
$$\mathfrak{W}_{1q;1q}=\mathfrak{W}_{11;11}+\mathfrak{W}_{12;12}+\mathfrak{W}_{13;13}=-\mathfrak{W}_{11}\,, \tag{70}$$

and

$$\mathfrak{W}_{qq;33}=\mathfrak{W}_{11;33}+\mathfrak{W}_{22;33}+\mathfrak{W}_{33;33}=\mathfrak{W}_{33}\,.$$

Expressing the various elements in the foregoing relations in accordance with equations (65)–(69), we find

$$A+B+C=\mathfrak{W}_{11}\,, \tag{71}$$

$$2C+D=\mathfrak{W}_{33}\,, \tag{72}$$

and

$$3A-B+2C=0\,. \tag{73}$$

In case the configuration has spherical symmetry, the number of independent elements are further drastically reduced; for in this case we must clearly have

$$A=D \quad \text{and} \quad B=C\,; \tag{74}$$

and, moreover,

$$\mathfrak{W}_{11}=\mathfrak{W}_{33}=\tfrac{1}{3}\mathfrak{W}\,. \tag{75}$$

From equations (71)–(73) it now follows that

$$A=D=-\tfrac{1}{15}\mathfrak{W} \quad \text{and} \quad B=C=\tfrac{1}{5}\mathfrak{W} \tag{76}$$

(in case of spherical symmetry).

VII. ON THE OCCURRENCE OF A POINT OF BIFURCATION IN THE SEQUENCE OF ROTATING EQUILIBRIUM CONFIGURATIONS

It is well known that in the equilibrium sequence of rotating incompressible fluid masses a point of bifurcation occurs at which the Jacobi ellipsoids branch off from the Maclaurin spheroids (cf. Jeans 1929). The origin of such a point of bifurcation has remained obscure (see, however, Lebovitz 1961, § IV, and Chandrasekhar 1961*b*); at least, the question whether a similar point of bifurcation can occur among rotating equilibrium configurations of compressible fluid masses has never been satisfactorily answered. However, we shall show that, in terms of the tensor $\mathfrak{W}_{pq;ij}$, we can give a general criterion for the occurrence of a point of bifurcation.

On the assumption that the rotating configuration has triplanar symmetry with respect to a co-ordinate system in which one of the axes (say x_3) coincides with the direction of $\boldsymbol{\Omega}$, it readily follows from the virial theorem that (cf. Chandrasekhar 1961*a*, eq. [18])

$$\mathfrak{W}_{11}+\Omega^2 I_{11}=\mathfrak{W}_{22}+\Omega^2 I_{22}=\mathfrak{W}_{33}=-\int_V p\,d\boldsymbol{x}\,. \tag{77}$$

Making use of the general relations (cf. eqs. [31] and [35])

$$\mathfrak{W}_{11}=\mathfrak{W}_{11;22}-\mathfrak{W}_{12;12} \quad \text{and} \quad \mathfrak{W}_{22}=\mathfrak{W}_{11;22}-\mathfrak{W}_{21;12}\,, \tag{78}$$

we can reduce the first pair of equalities in equation (77) to

$$-\mathfrak{W}_{12;12}+\Omega^2 I_{11}=-\mathfrak{W}_{21;12}+\Omega^2 I_{22}=\mathfrak{W}_{33}-\mathfrak{W}_{11;22}\,. \tag{79}$$

There are two obvious ways in which the equalities in (79) can be satisfied: *either* by requiring that

$$\mathfrak{W}_{12;12} = \mathfrak{W}_{21;12} \quad \text{and} \quad I_{11} = I_{22} \tag{80}$$

(i.e., by requiring that the object have axial symmetry, satisfying the first equality in [79] *identically* for all Ω^2, and determining Ω^2 by the second equality) *or* by requiring that

$$\Omega^2 I_{11} = \mathfrak{W}_{12;12}\,, \quad \Omega^2 I_{22} = \mathfrak{W}_{21;12}\,, \quad \text{and} \quad \mathfrak{W}_{33} = \mathfrak{W}_{11;22}\,. \tag{81}$$

The equalities in (79) can be satisfied in this latter manner only when Ω^2 exceeds a certain critical value: for, when $\Omega^2 \to 0$, $\mathfrak{W}_{12;12}$ tends to a finite positive value, namely, $-2\mathfrak{W}/15$ (cf. eqs. [68] and [76]), and it will not be possible to satisfy the first two equalities in equations (81). Therefore, for $\Omega^2 \to 0$, the configurations must be axisymmetric about the direction of $\boldsymbol{\Omega}$. However, as we proceed along the sequence of axially symmetric configurations and Ω^2 increases, a point *may* be reached where

$$\Omega^2 I_{11} = \Omega^2 I_{22} = \mathfrak{W}_{12;12} = \mathfrak{W}_{21;12}\,. \tag{82}$$

At this point it will become possible to satisfy conditions (81) for the *first time;* and, for values of Ω^2 larger than that required to satisfy equations (82), the possibility is open for satisfying the necessary conditions with unequal values of I_{11} and I_{22}; this is what must happen at a point of bifurcation; and this is what happens at the point where the Jacobi ellipsoids branch off from the Maclaurin spheroids.

Since $\mathfrak{W}_{33} = \mathfrak{W}_{11;22}$ at the point of bifurcation, this means that, in the notation of Section VI, at this point,

$$B = \mathfrak{W}_{33}\,. \tag{83}$$

This condition combined with equations (71)–(73) leads to the specific results

$$A = 3\mathfrak{W}_{33} - 2\mathfrak{W}_{11}\,, \quad B = \mathfrak{W}_{33}\,, \quad C = 3\mathfrak{W}_{11} - 4\mathfrak{W}_{33}\,, \quad \text{and} \quad D = -6\mathfrak{W}_{11} + 9\mathfrak{W}_{33}\,. \tag{84}$$

The research reported in this paper has in part been supported by the Office of Naval Research under Contract Nonr-2121(20) with the Enrico Fermi Institute for Nuclear Studies, University of Chicago. The work of the second author was supported in part by the United States Air Force under contract No. AF-49(638)-42, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

REFERENCES

Chandrasekhar, S. 1960, *J. Math. Anal. and Appl.*, **1**, 240.
———. 1961*a*, *Hydrodynamic and Hydromagnetic Stability* (Oxford: Clarendon Press), chap xiii, §§ 117 and 118.
———. 1961*b*, *Ap. J.*, **134**, 662.
Jeans, J. H. 1929, *Astronomy and Cosmogony* (Cambridge: Cambridge University Press), chaps viii and ix
Lebovitz, N. R. 1961, *Ap J.*, **134**, 500.