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Preliminary Study of Orbits of Interest for Moon Probes

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Ideal orbits of the space vehicle for moon probes have been studied preliminarily under the approximation implied in the restricted three-body problem by the method of successive approximation carried out on the IBM 7090 digital computer.

I. INTRODUCTION

An ideal orbit of a space vehicle for a moon probe is one which passes around the moon and afterwards approaches the earth at a short distance, so that whatever information it has received during its passage around the moon can be transmitted back to the earth when it is again near the earth. We can divide such orbits into two kinds: (1) orbits which enclose the earth and the moon and which pass both of them at short distances for a number of times, and (2) orbits which first pass around the moon closely and then return to the same geographical point at which the moon-probing vehicle is launched. We shall study here only orbits of the first kind, which lead us to the problem of searching for orbits in the well-known problem of three bodies in celestial mechanics.

With the advent of electronic computers of high speed, we can tackle this problem numerically. This is especially true for any problem such as the present one, where the time scale involved is not astronomically long but is rather humanly short.

Several papers (Egorov 1958; Message 1959; Newton 1959) have appeared recently in which the periodic orbits in the restricted three-body problem have been derived by means of numerical computations. In these papers references to earlier work in this field can also be found. In the present investigation, we emphasize the practical aspect that the orbits may be used for actual moon probes. Consequently, mathematical rigor is not demanded here. In fact, we discuss the accuracy of numerical approximations more in line with the engineering feasibility than with its usual sense understood in celestial mechanics. Similarly, the words, "stable" and "unstable" are used here with respect to a time scale of a few years or at most a few tens of years but not more than 100 years. Also, the word "stable" is associated with finite deviations (arising, for instance, from the limitation of accuracy in the launching of the space vehicle) that will be encountered in practical applications instead of infinitesimal deviations generally understood in analytical treatments.

II. STARTING CONDITION OF NUMERICAL EXPERIMENTS

In order to find the desirable orbits numerically, we must have some starting conditions from which these orbits can be obtained by successive approximation. As the starting condition, we employ the orbits which enclose both the earth and the moon, which have periods commensurable with the period of the moon, and which pass at relatively short distances from the earth as well as from the moon. The effect of the moon is neglected in the treatment of the starting condition. Therefore, in this section we consider simply the orbits of the vehicle in the gravitational field of the earth alone.

Let the semimajor axes of the orbits of the moon and of the space vehicle around the earth be 1 and a, and their periods P_0 and P, respectively. If the two periods P_0 and P have a ratio given by

$$P/P_0=n/m, \tag{1}$$

where both m and n are integers, the vehicle will repeatedly reach the moon and come back to the neighborhood of the earth. Assume that the encounter of the vehicle with the moon takes place at the apogee of its orbit, which is at a distance of α from the earth. Obviously α is given by

$$(1+e)a=\alpha, \tag{2}$$

where e represents the eccentricity of the orbit. In order to make the vehicle a moon probe, α must be of the order of unity.

It follows from (1) that

$$a=(1-\mu)^{\frac{1}{3}}(n/m)^{\frac{2}{3}}, \tag{3}$$

where μ is the fraction of the mass of the moon in the earth–moon system and is equal to 0.01215. Thus, Eq. (3) determines the semimajor axis a of the required orbit of the vehicle in terms of two integers m and n. Once a is determined, e can be obtained from (2) provided that α is given. In other words, for each pair of values m and n there corresponds to an one-parameter family of initial orbits.

Table I gives the values of a for different pairs of integers m and n that are relative prime. It is apparent that large values of n are not desirable because it takes too long to have an encounter between the vehicle and the moon. If n cannot be large, m must be also small; otherwise the orbit will be too small to reach the moon.

The limitation of initial orbits can be most profitably discussed in terms of α. If α is very near to unity, the space vehicle will be strongly perturbed by the moon or will even collide with its surface, and consequently it would not come back to the neighborhood of the earth. On the other hand, if α is considerably different from

TABLE I. Values of a for different combinations of m and n.

$m \backslash n$	1	2	3	4	5	6
1	0.9959	1.5809	2.0716	2.5096	2.9121	3.2885
2	0.6274	···	1.3050	···	1.8345	···
3	0.4788	0.7600	···	1.2065	1.4000	···
4	0.3952	···	0.8221	···	1.1557	···
5	0.3406	0.5401	0.7085	0.8583	···	1.1247
6	0.3016	···	···	···	0.8819	···
7	0.2722	0.4320	0.5661	0.6858	0.7958	0.8987
8	0.2490	···	0.5179	···	0.7280	···
9	0.2302	0.3654	···	0.5800	0.6731	···
10	0.2146	···	0.4463	···	···	···
11	0.2014	0.3196	0.4188	0.5074	0.5888	0.6649
12	0.1900	···	···	···	0.5556	···

unity, the purpose of probing the moon will be lost because the vehicle will be too far away from the moon during the encounter. We may tentatively set the desired values of α between 1.08 and 1.20. This is, of course, only for α of the initial orbits. The value of α corresponding to the final orbit will be different from that of the starting one.

Now α is furthermore limited by the condition that the orbit of the vehicle should be an ellipse; i.e., e must be less than 1. If follows from (2) that

$$\alpha < 2a. \tag{4}$$

If α must be greater than 1.08, we can immediately eliminate those entries in Table I where a <0.54. On the other hand, e must be greater than zero, hence

$$\alpha > a. \tag{5}$$

If we insist on $\alpha < 1.2$, so that the vehicle can reach points near the moon, we must exclude those values of a in Table I which are greater than 1.2.

Practical considerations such as the launching of the vehicle and later communications with it, require that $a(1-e)$ be not too large. If it is necessary to restrict $a(1-e)$ to be less than a certain value, say γ, then

$$a < \tfrac{1}{2}(\alpha+\gamma). \tag{6}$$

If we take 1.2 as the largest value for α which is still meaningful for the moon probe and take $\gamma=0.5$, a must be less than 0.85 according to (6). Therefore we may eliminate all cases with a >0.85 in Table I.

After the values of a which are either greater than 0.85 or smaller than 0.54 have been excluded as possible semimajor axes of starting orbits for a moon probe, only a few cases are left in Table I that are suitable. In the present paper we shall consider the case $n/m=\frac{2}{3}$, the case $n/m=\frac{1}{2}$ having been studied by Message and by Newton.

III. PROCEDURE FOR DERIVING THE DESIRED ORBITS

A starting orbit proposed in the previous section does not represent the true orbit of the vehicle for a moon probe because of lunar perturbation. In this preliminary study it will be assumed that the moon's orbit is circular. This reduces the perturbation calculation to the integration of the differential equations of the restricted three-body problem. The result thus derived does not give exactly the required orbit for the moon probe in the sun–earth–moon system. However, it provides a basis for treating by successive approximations the more realistic problem that includes the solar attractions.

Following the usual notation (e.g., Moulton 1914) we use a coordinate system rotating with the moon and

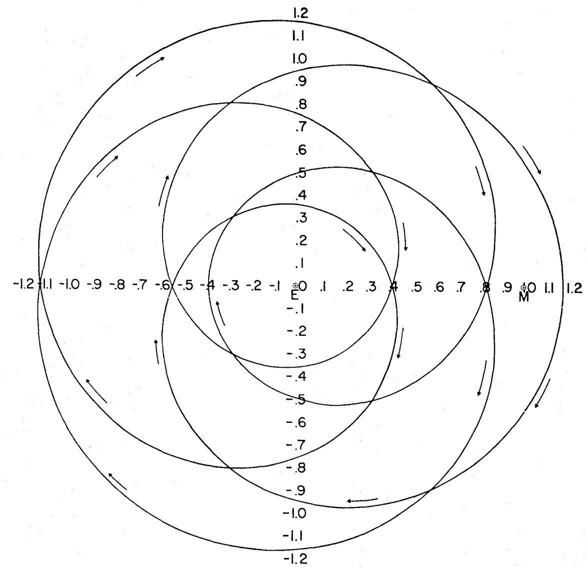

FIG. 1. A retrograde orbit which makes periodic encounters with the moon. The orbit is drawn in the frame of reference rotating with the earth E and the moon M.

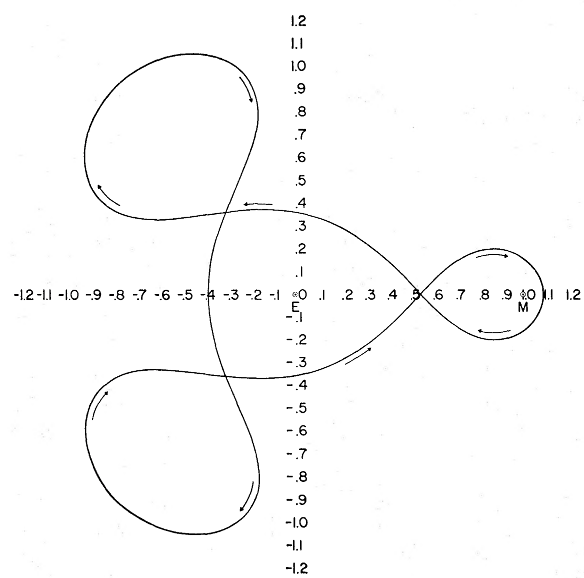

FIG. 2. A direct orbit which makes periodic encounters with the moon. The orbit is drawn in the frame of reference rotating with the earth E and the moon M.

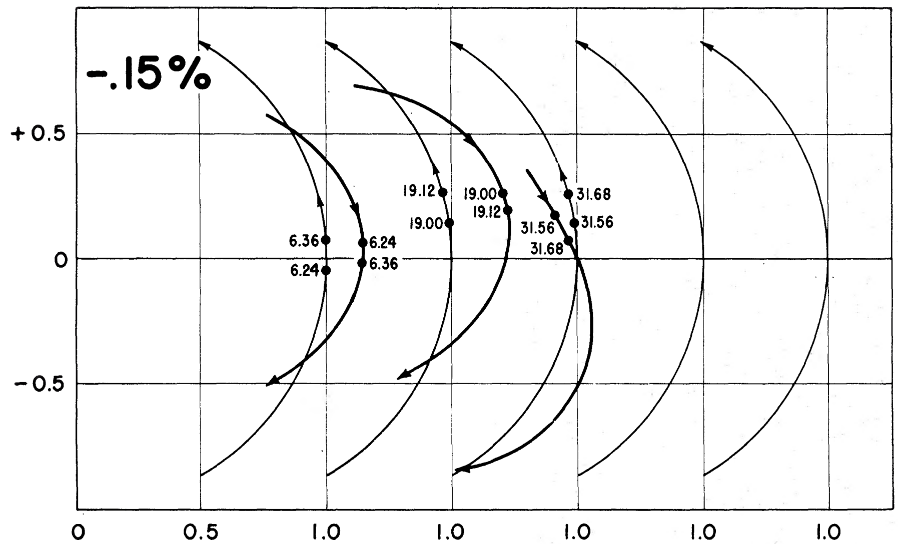

FIG. 3(a).

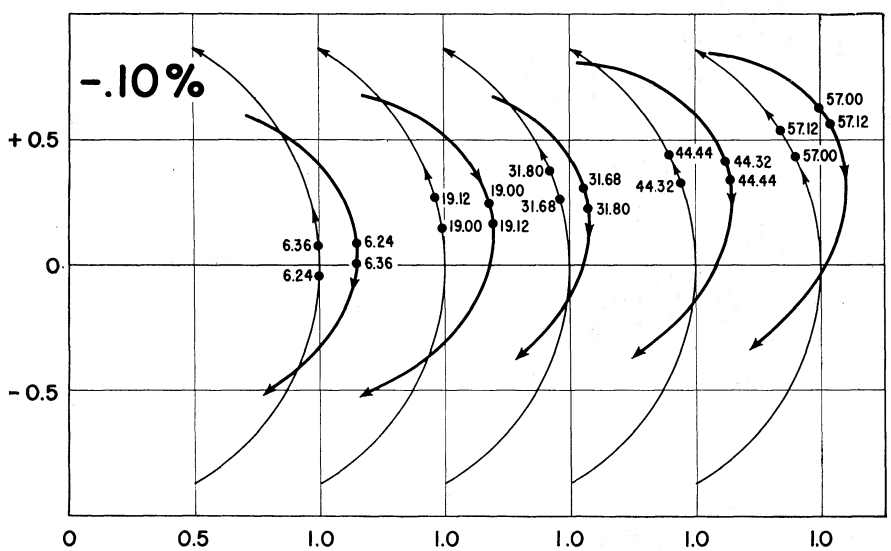

FIG. 3. The effect of a small change in the initial conditions on the stability of the retrograde orbit. Plotted here are the motions of the moon and of the third body. The numerical values in the diagram denote times of passage of the moon and of the third body at various points on their respective orbits near encounters. The percentage deviation of the initial velocity from the correct value is marked at the upper left corner in each diagram. Except for the case of -0.15%, all show stability of the orbits.

FIG. 3(b).

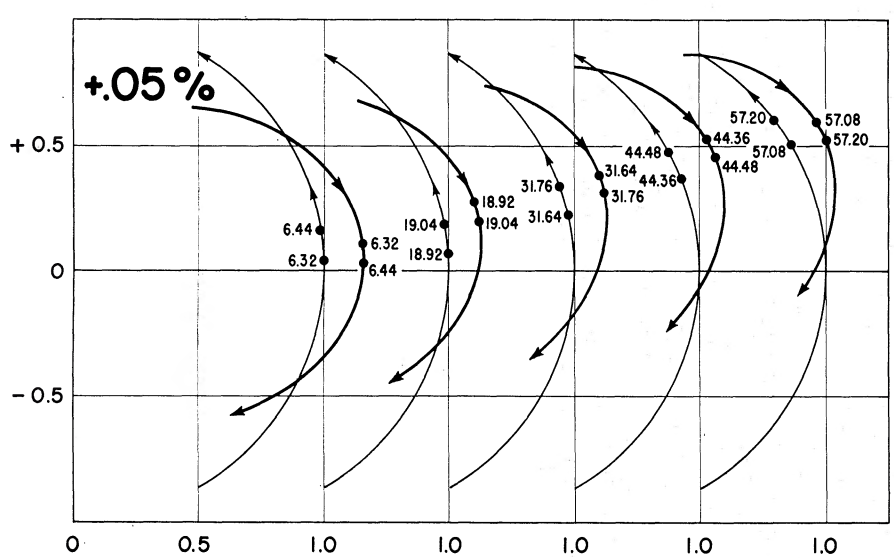

FIG. 3(c).

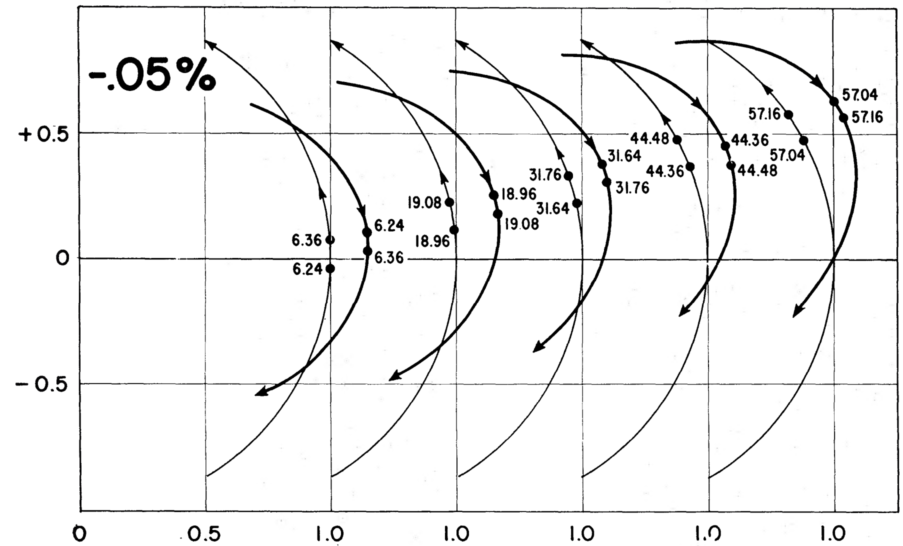

Fig. 3(d).

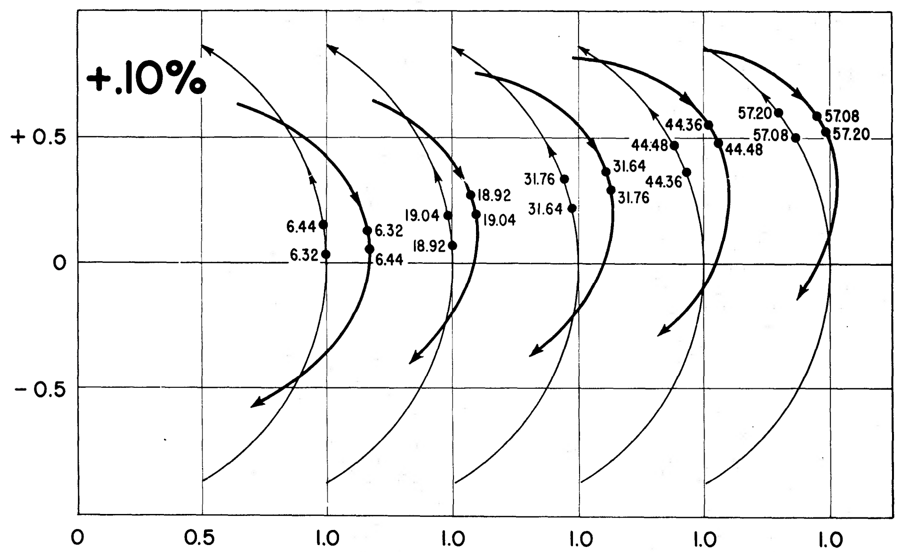

Fig. 3(e).

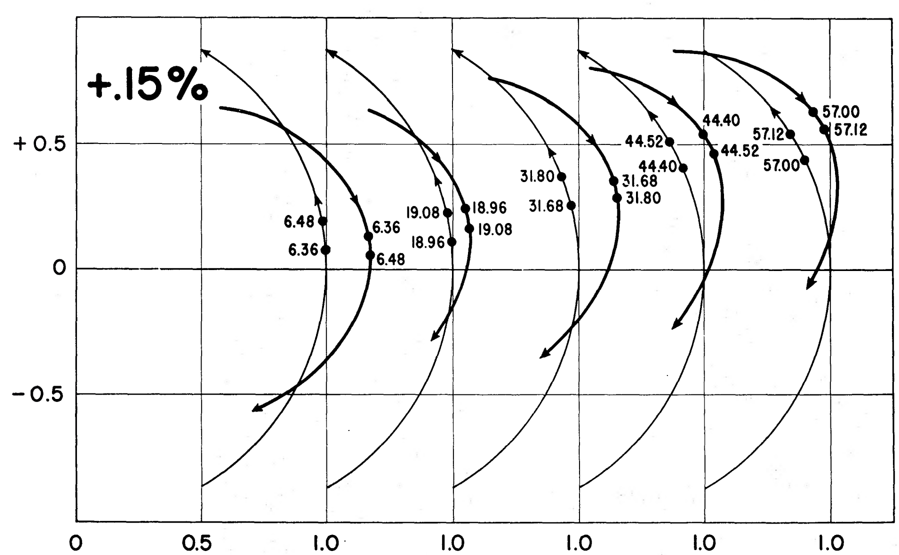

Fig. 3(f).

with its origin located at the center of mass of the earth–moon system and with the separation between the two as the unit of length. The total mass of the system will be taken as unity. Thus, the mass of the earth is $1-\mu$, and that of the moon μ, μ being equal to 0.01215. With this system of units, the period of the moon around the earth is 2π. If we now confine the third body, i.e., the space vehicle, to the orbital plane of the moon around the earth, the equations of its motion assume the form

$$\frac{d^2x}{dt^2}-2\frac{dy}{dt}=x-(1-\mu)\frac{x-x_1}{r_1^3}-\mu\frac{x-x_2}{r_2^3} \tag{7}$$

and

$$\frac{d^2y}{dt^2}+2\frac{dx}{dt}=y-(1-\mu)\frac{y}{r_1^3}-\mu\frac{y}{r_2^3}, \tag{8}$$

where the various symbols have the obvious meanings.

The space vehicle for the moon probe is supposed to be launched at the perigee, whose coordinates are

$$x=-a(1-e)-\mu, \qquad y=0, \tag{9}$$

and whose distance from the earth is $a(1-e)$. The initial velocity that is necessary in order to make the vehicle to enter the starting orbit is given approximately by

$$\frac{dx}{dt}=0, \qquad \frac{dy}{dt}=\pm\left[\frac{1+e}{a(1-e)}\right]^{\frac{1}{2}}+a(1-e). \tag{10}$$

The minus sign in the second of Eqs. (10) applies to the injection of a vehicle revolving in the same sense as the moon around the earth (direct orbits), while the plus sign applies to vehicles revolving in the opposite sense as the moon (retrograde orbits). It is obvious that the perturbing effect of the moon on the vehicle is greater in the first case than in the second case, because when the vehicle is rotating in the same sense as the moon, their encounter will last longer than when the two are rotating in the opposite sense. Therefore we would expect that it will be easier to find desired orbits of retrograde motion than those of direct motion. Indeed, as we will see immediately, the orbits rotating in the same sense are unstable.

With the four initial conditions as given by (9) and (10), we can proceed to integrate (7) and (8) and are thereby able to find by successive approximation the required orbits. The integrations were carried out on the IBM 7090 digital computer at Goddard by Clarence Wade, Jr. The Runge–Kutta method was used with $\Delta t=0.01$.

We have studied the case $\alpha=1.14$ which is in the middle of the range of interest from 1.08 to 1.20. By the method of successive approximation we obtain a set of correct initial conditions:

$$x=-0.39215,\ y=0,\ dx/dt=0,\ dy/dt=2.35164, \tag{11}$$

with which a final integration has been performed up to $t=240$. During this time interval of nearly 3 years (38.20 sidereal months), we have seen 19 encounters with the moon. The orbit repeats itself every time after each encounter. This indicates that the orbit is a closed one for all practical purposes. It is shown in Fig. 1. The final value of α is about 1.16, which is larger than the initial value of 1.14.

As a passing remark we should mention that it took the IBM 7090 computer about $\frac{1}{2}$ hr to compute, with double precision, the path of the third body from $t=0$ to $t=240$ with $\Delta t=0.01$.

The desired initial conditions corresponding to an initial α of 1.14 but with negative injection, i.e., with the minus sign in the second of Eqs. (10), can be similarly obtained. They are

$$x=-0.39215,\ y=0,\ dx/dt=0,\ dy/dt=-1.61025, \tag{12}$$

with which the final integration was carried out. The result is not quite satisfactory. After three passages around the moon which are identical within the accuracy of plotted figures, the third body deviates greatly after the fourth passage and never reaches the other side of the moon again before $t=60$. The orbit before the fourth encounter with the moon is illustrated in Fig. 2.

It can be seen from the figure that the final value of α is about 1.07, which is much smaller than the initial value of 1.14. The strong perturbation by the moon at the time of close encounter is the reason why a closed orbit is so difficult to obtain in this case. However, the orbit has the advantage that it gives the probing vehicle a longer period of time to look at the other side of the moon than the retrograde orbit can provide.

From the time interval between two consecutive encounters we can easily see that the direct orbit of the third body is strongly perturbed by the moon even before the fourth encounter because its line of apsides has rotated by an appreciable angle after each encounter. If the successive encounters only make the line of apsides rotate without any effect on other elements of the orbit of the third body, the orbit will be a stable one. Actually, the perturbation does cause the changes in other orbital elements. These changes destroy the synchronization of the motions of the moon and the third body.

An appreciable degree of rotation of the line of apsides of the orbit of the third body also raises the difficulty of using them for practical moon probes. Since the moon is not actually revolving in a circular orbit as is assumed here but in an elliptical orbit, the change in the separation between the moon and the earth would destroy the regularity of encounters even at the first time. Therefore, whatever merits the direct orbit has, it should not be used for a moon probe with the intention of circling the vehicle around both the earth and the moon for a long period of time.

In both direct and retrograde orbits, the line of apsides

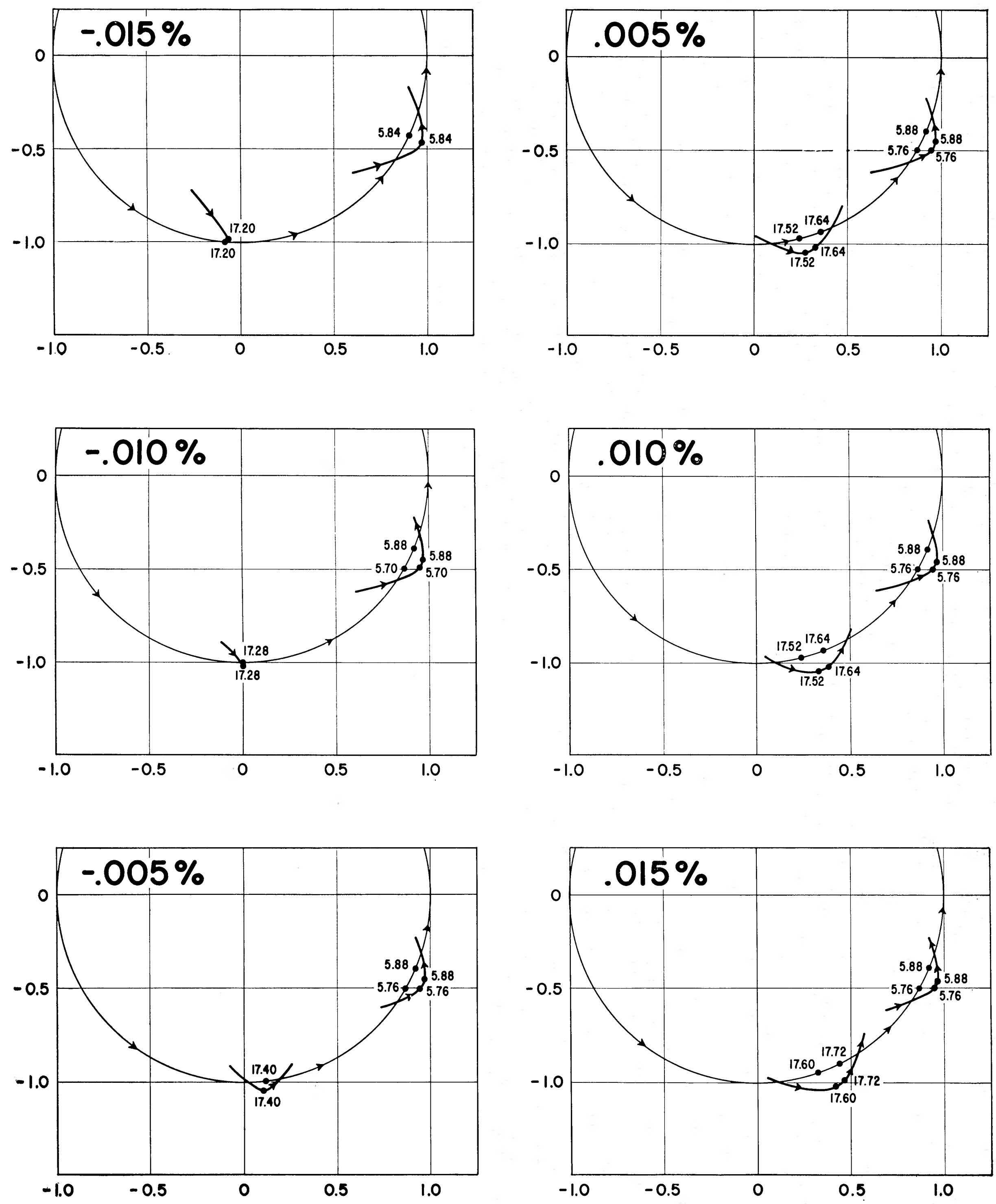

Fig. 4. The effect of a small change in the initial conditions on the stability of the direct orbit. The percentage deviation from the correct initial velocity is in each case 10 times less than those given in Fig. 3. Even with such a small deviation, no more than two encounters can be obtained, a fact indicating the instability of the orbits.

has a retrograde motion, as a result of encounters with the moon, as we have mentioned. Consequently, the time interval between two consecutive encounters is slightly more than two sidereal months in the case of retrograde orbits and less than two sidereal months in the case of direct orbits.

IV. STABILITY

The possibility of finding the desired orbits in the actual system of the earth, the moon, and the sun depends ultimately upon the tolerance in the initial conditions such that the orbits obtained under the approximation of the restricted three-body problem will not be easily destroyed. In order to examine this tolerance, we have integrated six more cases for positive injection with dy/dt at $t=0$ to deviate from the correct value of 2.35164 by ± 0.05, ± 0.10 $\pm 0.15\%$, but with no change in other initial conditions. For each case we have integrated the equations of motion up to $t=60$. Figure 3 illustrates the behavior of the resulting orbits in the stationary frame of reference. Only the portion of the orbit during the encounter with the moon is drawn. Each diagram in the figure has marked at the upper left corner the percentage deviation from the correct value of dy/dt. Five encounters are shown in each case. The dots mark the position of the moon and the third body at the labeled times during encounters.

From Fig. 3, we notice that synchronization of the motion of the moon and of the third body is completely destroyed in the case of -0.15% after the third encounter, which takes place on the wrong side of the moon, while in the other five cases, the regularity is maintained during the computed time. The orbits in these five cases undergo oscillations in the distance of the encounter with the moon. This appears to indicate the stability of the orbit under a small change in initial conditions. We are encouraged by this property to suggest that an orbit encircling both the earth and the moon for a period of a few encounters should be further studied under the realistic configuration of the earth–moon–sun system.

Similarly, we have integrated the equations for six more cases in connection with negative injection. Their initial conditions follow Eqs. (12) except with dy/dt at $t=0$ being ± 0.005, ± 0.010. $\pm 0.015\%$ from the correct value of -1.61025. Note that the percentage changes are only one-tenth of those considered for the retrograde orbit. We have illustrated the results in Fig. 4. As with the case of retrograde orbits, only that portion of the path which encounters the moon is plotted. Similarly the time and the positions of both the moon and the third body during each encounter are marked in the diagram. All orbits in the figure make only two encounters with the moon before they are perturbed out of synchronization. This shows that the orbit is not stable under a slight change in the initial conditions.

Although the distances of encounter with the moon in these cases are too short to derive a general conclusion, we expect that this instability will persist even when the third body encounters the moon at greater distances.

An interesting result which may have an important bearing on the practical application is the fact, as can be seen from Figs. 3 and 4, that a slightly larger magnitude of the injecting velocity than the correct one is less damaging to the regularity of encounters than a correspondingly smaller magnitude of the injecting velocity.

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