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Integration of the Equations of Planetary Motion in Rectangular Coordinates

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Matrix methods for the approximate solution of differential equations are applied to the development of general perturbations in rectangular coordinates. The solution is obtained directly in the form of complementary function and particular integral. The possible role of the arbitrary constants is discussed; these appear unambiguously in the complementary function, the nature of which is known from the properties of Keplerian motion. Formulas for the application of this method to the calculation of special perturbations are also included.

1. Introduction

The normal method of general perturbations in Cartesian coordinates is due to Brouwer (1944). He approaches the deviations from some reference Keplerian orbit by the method of "variation of constants," obtaining relatively simple expressions for the perturbations, but using relatively sophisticated transformations to find them. A slight modification of this method has been given by Gontkovskaya (1958), who uses the true anomaly rather than the time as the independent variable, and whose derivation is more direct. In Brouwer's method the role of the arbitrary constants (introduced in the integrations) is not very clear, and the introduction of their numerical values leads to a fairly comprehensive change in the first-order solution. The choice of constants is equivalent to the choice of a mean orbit; to take advantage of them, the disturbing function must be developed with respect to the mean orbit; but this point is not discussed by Brouwer. (Gontkovskaya does not consider the constants of integration at all.)

The difficulty over the constants and the indirect derivation of the final expressions seem to follow from the use of the method of variation of constants. But there is another approach to the approximate solution of differential equations which involves matrices. Here the first variational equations of the unperturbed system of differential equations are solved and their complete, general solution is embodied in a matrix called the "matrizant" or "fundamental solution matrix." Once this is known, the first-order solution of the perturbed equations can be written down immediately, and from this, the higher-order solutions follow. The form of the solution is the simple one of complementary function (involving the arbitrary constants) and particular integral.

The author first became aware of the possibilities of this method when reading a paper by Myachin (1959) on the accumulation of errors in the numerical integration of equations in celestial mechanics; and the motivation for this work must be credited to him. Applications in numerical work are becoming popular, but so far as the author is aware no use has been made of matrix methods in the field of general perturbations. Since this method is relatively new in celestial mechanics (with references thinly scattered), and it has considerable possibilities, it will be briefly developed and discussed from first principles.

Consider the system of n first-order differential equations

$$dX_i/dt=f_i(X_1,X_2,\cdots,X_n,t),\quad (i=1,2,\cdots,n) \tag{1}$$

relating the n coordinates X_i and the time t. These can be written symbolically in the condensed form

$$\mathbf{X}'=\mathbf{f}(\mathbf{X},t), \tag{1a}$$

where $\mathbf{X}'$ and $\mathbf{f}$ are column matrices; the prime represents differentiation with respect to the time. If a solution, $\mathbf{X}_R(t)$, is known, having initial conditions $\mathbf{X}_R(t_0)=\mathbf{X}_0$, then a "slightly different" solution, $\mathbf{X}_R(t)+\delta\mathbf{X}(t)$, can be found from the first variational equations of the system (1). That is,

$$\begin{bmatrix}\delta X_1'\\ \delta X_2'\\ \cdot\\ \cdot\\ \cdot\\ \delta X_n'\end{bmatrix}=\begin{bmatrix}\partial f_1/\partial X_1 & \partial f_1/\partial X_2 & \cdots & \partial f_1/\partial X_n\\ \partial f_2/\partial X_1 & \partial f_2/\partial X_2 & \cdots & \partial f_2/\partial X_n\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \partial f_n/\partial X_1 & \partial f_n/\partial X_2 & \cdots & \partial f_n/\partial X_n\end{bmatrix}\begin{bmatrix}\delta X_1\\ \delta X_2\\ \cdot\\ \cdot\\ \cdot\\ \delta X_n\end{bmatrix} \tag{2}$$

or

$$\delta\mathbf{X}'=\mathbf{A}\delta\mathbf{X}. \tag{2a}$$

The solution $\mathbf{X}_R(t)$ will be called the "reference orbit." Each of the partial differential coefficients in the n-by-n matrix $\mathbf{A}$ is evaluated along the reference orbit, so that $\mathbf{A}$ is a known function of the time. The squares and products of the δX_i have been neglected; so the solution will be accurate only to the first order of small quantities.

Equations (2) are solved when any set of n linearly independent solutions is known. Let this consist of the separate columns of the matrix with elements $x_{ij}(t)$. Since any linear combination of these columns also gives a solution, the columns of

$$\mathbf{\Omega}\equiv[x_{ij}(t)][x_{ij}(t_0)]^{-1} \tag{3}$$

must all be solutions. The matrix $\mathbf{\Omega}$ has the property that at t_0 it is equal to the identity matrix; this provides the initial conditions necessary to find it by numerical integration. Since it is essentially a function of t_0 as well as t, it will be written as $\mathbf{\Omega}(t_0,t)$. It is called the "matrizant" or "fundamental solution matrix" of the system (2). As each of its columns satisfies (2), it must itself satisfy the equation

$$\mathbf{\Omega}'=\mathbf{A}\mathbf{\Omega}, \tag{4}$$

where

$$\mathbf{\Omega}(t_0,t_0)=\mathbf{I}.$$

If the function $\delta\mathbf{X}(t)$ were to have initial conditions $\delta\mathbf{X}(t_0)=\delta\mathbf{X}_0$, then the solution of (2) would be

$$\delta\mathbf{X}(t)=\mathbf{\Omega}(t_0,t)\delta\mathbf{X}_0. \tag{5}$$

This matrix was developed by Peano and Baker; a description of their methods is given by Ince (1956). Baker develops the matrizant as an infinite series and is concerned with the relation of the matrizant of (2) to that of the system

$$\delta\mathbf{X}'=(\mathbf{A}+\mathbf{B})\delta\mathbf{X}.$$

The results are applied particularly where $\mathbf{A}$ is constant and $\mathbf{B}$ is periodic. This theory is of no concern here since the unperturbed equations of planetary motion give Keplerian motion, and the matrizant of the appropriate system (2) is known analytically.

Consider the equation

$$\delta\mathbf{X}'=\mathbf{A}\delta\mathbf{X}+\mathbf{g}(t), \tag{6}$$

in which a "forcing function" $\mathbf{g}(t)$ has been added to (2). Its solution can be written

$$\delta\mathbf{X}=\mathbf{\Omega}(t_0,t)\delta\mathbf{X}_0+\mathbf{\Omega}(t_0,t)\int_{t_0}^{t}\mathbf{\Omega}^{-1}(t_0,\tau)\mathbf{g}(\tau)d\tau. \tag{7}$$

This is the exact solution of (6), subject to the initial conditions $\delta\mathbf{X}(t_0)=\delta\mathbf{X}_0$. No conditions about orders of magnitude are imposed. The product

$$\mathbf{\Omega}(t_0,t)\mathbf{\Omega}^{-1}(t_0,\tau)$$

is seen to be the Green's function of the system (2), and it is this property of matrizants that is important here.

The first term on the right-hand side of (7) is the complementary function [abbreviated as (CF) in the equations that follow] and this bears the entire burden of the constants of integration. The second term is the particular integral (PI) which gives the perturbations from the reference orbit that are induced by $\mathbf{g}$. Suppose Eqs. (1) to be the unperturbed equations of motion for some physical system, and $\mathbf{X}_0$ to represent the observed conditions at time t_0. If small perturbing forcing functions $\mathbf{g}(t)$ are added, then the first-order solution of the perturbed equations, subject to the observed initial conditions, is

$$\mathbf{X}_R(t)+(\mathrm{PI}).$$

But from (7) this can be written

$$\mathbf{X}_R(t)-(\mathrm{CF})+\delta\mathbf{X},$$

or

$$\mathbf{X}_M(t)+\delta\mathbf{X},$$

where $\mathbf{X}_M(t)$ represents a "mean" orbit. The mean orbit is slightly different from the reference orbit and has initial conditions $\mathbf{X}_M(t_0)=\mathbf{X}_0-\delta\mathbf{X}_0$; $\delta\mathbf{X}_0$ consists of the n arbitrary constants of integration. The whole point of the mean orbit is that $\delta\mathbf{X}_0$ should be chosen to make $\delta\mathbf{X}$ in (7) as small and tractable as possible, over a reasonable time interval. But it is necessary to start to solve the problem using the observed initial conditions $\mathbf{X}_0$, when $\mathbf{X}_R$ is the osculating orbit at the epoch t_0. The arbitrary constants, and therefore the mean orbit, cannot be chosen to any purpose until something is known of the first-order perturbations contained in the particular integral.

The solution (7) can be verified by direct substitution into (6); it can also be recovered by varying the arbitrary "constants" $\delta\mathbf{X}_0$. This might be called the method of "variation of initial conditions." If other constants are to be varied, then the full paraphernalia of coordinate transformations become necessary.

If $\mathbf{\Omega}^{-1}$ is required, it can be found without inverting the matrix, since the inverse itself is the solution of the set of equations

$$d\mathbf{Y}/dt=-\mathbf{Y}\mathbf{A},$$

where $\mathbf{Y}$ is equal to the identity matrix at time t_0; this is called the adjoint equation of (4). Also if the particular integral is required numerically, it can be found most simply by solving Eqs. (6) subject to the initial conditions $\delta\mathbf{X}(t_0)=0$.

Now suppose that in setting up Eqs. (1) we had chosen to ignore certain small terms $\mathbf{g}(\mathbf{X},t)$ involving a small coefficient α. To the first order (neglecting α^2), we can calculate their effects by evaluating the terms along the reference (or osculating) orbit, when they become known functions of the time: $\mathbf{g}[\mathbf{X}_R(t),t]$. We then have equations like (6), the solution of which can be written down. From this first-order solution a mean orbit ought to be found, but for the present the discussion will be continued with the osculating orbit for reference.

To find a more accurate solution, involving α^2, proceed as follows. Let it be

$$\mathbf{X}(t)=\mathbf{X}_R(t)+\delta\mathbf{X}(t)+\delta^2\mathbf{X}(t).$$

If this is substituted into the equations of motion

$$X_i'(t)=f_i(\mathbf{X},t)+g_i(\mathbf{X},t)\quad(i=1,2,\cdots,n), \tag{8}$$

and all terms of order α^3 are ignored, the ith equation becomes

$$\begin{aligned}X_{R,i}'+\delta X_i'+\delta^2X_i'&\\ =f_i[\mathbf{X}_R(t),t]&+\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial X_j}\right)_R(\delta X_j+\delta^2X_j)\\ &+\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial^2 f_i}{\partial X_j\partial X_k}\right)_R\delta X_j\delta X_k\\ &+g_i[\mathbf{X}_R(t),t]+\sum_{j=1}^{n}\left(\frac{\partial g_i}{\partial X_j}\right)_R\delta X_j.\end{aligned} \tag{9}$$

The partial differential coefficients are all evaluated along the reference orbit. From their definitions, the terms of zero and first order cancel and we are left with the second-order terms:

$$\delta^2 X_i' = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial X_j}\right)_R \delta^2 X_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial^2 f_i}{\partial X_j \partial X_k}\right)_R \delta X_j \delta X_k + \sum_{j=1}^{n} \left(\frac{\partial g_i}{\partial X_j}\right)_R \delta X_j. \quad (10)$$

Apart from $\delta^2\mathbf{X}$ all the terms in (10) are known functions of the time, so that the form of (10) when all n equations are combined is identical with that of (6), and the solution can be written down. Apparently there are another n constants of integration $\delta^2\mathbf{X}_0$ available; but in the final solution these will simply be added to the $\delta\mathbf{X}_0$ in the complementary function $\mathbf{\Omega}$ $(\delta\mathbf{X}_0+\delta^2\mathbf{X}_0)$. So no degrees of freedom are added to the solution, although the mean orbit can be made more sophisticated.

The equations for the solutions $\delta^n\mathbf{X}$, of any order, will always be of the same form as (6), although the "forcing functions" become more complicated. To find the nth-order solution, all the lower-order solutions must be known.

The theory given above applies to any system of first-order differential equations. The situation is simpler where planetary motion is concerned. Then the equations of motion are contained in the vector equation

$$\mathbf{r}'' = \nabla R, \quad (11)$$

where $\mathbf{r}$ is the vector with components (x,y,z), or the column matrix trans$[xyz]$, and R is the force function, involving the dominant inverse-square attraction and the disturbing function R_1. Equations (11) can be written in the form

$$\begin{aligned} X_1' &= X_4, \\ X_2' &= X_5, \\ X_3' &= X_6, \\ X_4' &= \partial R/\partial X_1, \\ X_5' &= \partial R/\partial X_2, \\ X_6' &= \partial R/\partial X_3, \end{aligned} \quad (11a)$$

when the theory described above can be applied directly. But it is simpler to work from the second-order equations.

The first variational equations of (11) can be written as

$$\delta\mathbf{r}'' = \mathbf{A}\delta\mathbf{r}, \quad (12)$$

where

$$\mathbf{A} = \begin{bmatrix} \partial^2 R/\partial x^2 & \partial^2 R/\partial x\partial y & \partial^2 R/\partial x\partial z \\ \partial^2 R/\partial y\partial x & \partial^2 R/\partial y^2 & \partial^2 R/\partial y\partial z \\ \partial^2 R/\partial z\partial x & \partial^2 R/\partial z\partial y & \partial^2 R/\partial z^2 \end{bmatrix}$$

evaluated along a reference orbit. To solve (12) introduce two 3-by-3 matrices $\mathbf{U}(t_0,t)$ and $\mathbf{V}(t_0,t)$, where

$$\begin{aligned} &\mathbf{U}'' = \mathbf{AU}, \quad \mathbf{U}(t_0,t_0) = \mathbf{I}, \quad \mathbf{U}'(t_0,t_0) = 0; \\ &\mathbf{V}'' = \mathbf{AV}, \quad \mathbf{V}(t_0,t_0) = 0, \quad \mathbf{V}'(t_0,t_0) = \mathbf{I}. \end{aligned} \quad (13)$$

$\mathbf{U}$ and $\mathbf{V}$ are called the first and second fundamental solution matrices, respectively; their six columns are six linearly independent solutions of (12). Let $\delta\mathbf{r}_0$ and $\delta\mathbf{r}_0'$ be the initial increments in position and velocity to be applied to the reference orbit at time t_0. Then

$$\delta\mathbf{r} = \mathbf{U}(t_0,t)\delta\mathbf{r}_0 + \mathbf{V}(t_0,t)\delta\mathbf{r}_0' \quad (14)$$

at any time t. $\delta\mathbf{r}$ is a solution of (12). Furthermore, the components of $\delta\mathbf{r}_0$ and $\delta\mathbf{r}_0'$ can be interpreted to be six independent, arbitrary constants; so it is clear that (14) is the general solution of (12).

Given analytical expressions for $\mathbf{U}$ and $\mathbf{V}$, we can consider them to be functions of the two variables t_0 and t. Then the "initial" conditions (13) apply whenever the two are equal. Now consider the integral

$$\int_{t_0}^{t} \mathbf{U}(\tau,t)d\tau.$$

It satisfies the differential equation for $\mathbf{V}$, and also the initial conditions for $\mathbf{V}$; so it must be equal to $\mathbf{V}$. Hence

$$\mathbf{U}(t_0,t) = -(\partial/\partial t_0)\mathbf{V}(t_0,t). \quad (15)$$

If Eq. (14) is differentiated with respect to t, it becomes

$$\delta\mathbf{r}' = \frac{\partial\mathbf{U}(t_0,t)}{\partial t}\delta\mathbf{r}_0 + \frac{\partial\mathbf{V}(t_0,t)}{\partial t}\delta\mathbf{r}_0'. \quad (16)$$

Therefore considering (14) and (16) together, and applying (15) we see that the matrizant of (11a) is

$$\mathbf{\Omega} = \begin{bmatrix} -\dfrac{\partial\mathbf{V}}{\partial t_0} & \mathbf{V} \\ -\dfrac{\partial^2\mathbf{V}}{\partial t\partial t_0} & \dfrac{\partial\mathbf{V}}{\partial t} \end{bmatrix}. \quad (17)$$

Given analytical expressions for the components of $\mathbf{V}$, the inverse of $\mathbf{\Omega}$ could be found simply by exchanging t_0 and t. It could also be found from the adjoint equations of the first variational equations of (11a), when it is seen that $\mathbf{\Omega}^{-1}$ is of the form

$$\mathbf{\Omega}^{-1} = \begin{bmatrix} -\dfrac{\partial\mathbf{W}}{\partial t} & \mathbf{W} \\ -\dfrac{\partial^2\mathbf{W}}{\partial t\partial t_0} & \dfrac{\partial\mathbf{W}}{\partial t_0} \end{bmatrix}, \quad (18)$$

where

$$\mathbf{W}'' = \mathbf{WA}, \quad \mathbf{W}(t_0,t_0) = 0, \quad \mathbf{W}'(t_0,t_0) = -\mathbf{I}.$$

Let $\mathbf{W}^T$ denote the transpose of $\mathbf{W}$. Since $\mathbf{A}$ is symmetrical, $(\mathbf{W}'')^T=\mathbf{A}\mathbf{W}^T$, so that, from the initial conditions for $\mathbf{W}$ it is seen that

$$\mathbf{W}=-\mathbf{V}^T. \tag{19}$$

Then

$$\mathbf{V}(t_0,t)=-\mathrm{trans}\mathbf{V}(t,t_0). \tag{20}$$

Also we can find $\partial\mathbf{V}/\partial t$ in (17) by taking the transpose of $-\partial\mathbf{V}/\partial t_0$ and exchanging t_0 and t.

Now let a forcing function

$$\mathbf{h}(t)=\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix}$$

be added to (11). The addition to (11a) would be the column matrix $\mathbf{g}=\mathrm{trans}[0\,0\,0\,h_x\,h_y\,h_z]$. It can be verified (most simply by direct substitution) that the solution of the resulting equations is

$$\delta\mathbf{r}=\mathbf{U}(t_0,t)\delta\mathbf{r}_0+\mathbf{V}(t_0,t)\delta\mathbf{r}_0'+\int_{t_0}^{t}\mathbf{V}(\tau,t)\mathbf{h}(\tau)d\tau. \tag{21}$$

Furthermore, this must be the general solution.

At first sight, (21) does not appear to resemble (7) to any great degree. But if (7) is written in the expanded form

$$\begin{bmatrix} \delta\mathbf{r} \\ \\ \delta\mathbf{r}' \end{bmatrix}=\begin{bmatrix} -\dfrac{\partial\mathbf{V}}{\partial t_0} & \mathbf{V}(t_0,t) \\ -\dfrac{\partial^2\mathbf{V}}{\partial t\partial t_0} & \dfrac{\partial\mathbf{V}}{\partial t} \end{bmatrix}\begin{bmatrix} \delta\mathbf{r}_0 \\ \\ \delta\mathbf{r}_0' \end{bmatrix}+\begin{bmatrix} -\dfrac{\partial\mathbf{V}}{\partial t_0} & \mathbf{V}(t_0,t) \\ -\dfrac{\partial^2\mathbf{V}}{\partial t\partial t_0} & \dfrac{\partial\mathbf{V}}{\partial t} \end{bmatrix}\int_{t_0}^{t}\begin{bmatrix} -\dfrac{\partial\mathbf{W}}{\partial\tau} & \mathbf{W}(t_0,\tau) \\ -\dfrac{\partial^2\mathbf{W}}{\partial t_0\partial\tau} & \dfrac{\partial\mathbf{W}}{\partial t_0} \end{bmatrix}\begin{bmatrix} 0 \\ \\ \mathbf{h}(\tau) \end{bmatrix}d\tau,$$

the first three components are seen to be

$$\delta\mathbf{r}=-\frac{\partial V}{\partial t_0}\delta\mathbf{r}_0+\mathbf{V}\delta\mathbf{r}_0'$$

$$+\int_{t_0}^{t}\left[-\frac{\partial\mathbf{V}(t_0,t)}{\partial t_0}\mathbf{W}(t_0,\tau)+\mathbf{V}(t_0,t)\frac{\partial\mathbf{W}(t_0,\tau)}{\partial t_0}\right]\mathbf{h}(\tau)d\tau,$$

and since this must be identical with (21) for all $\mathbf{h}$, we have the identity

$$\mathbf{V}(\tau,t)=-\frac{\partial\mathbf{V}(t_0,t)}{\partial t_0}\mathbf{W}(t_0,\tau)+\mathbf{V}(t_0,t)\frac{\partial\mathbf{W}(t_0,\tau)}{\partial t_0}.$$

This is contained in the important identity

$$\mathbf{\Omega}(\tau,t)\equiv\mathbf{\Omega}(t_0,t)\mathbf{\Omega}^{-1}(t_0,\tau). \tag{22}$$

If the matrizant and its inverse are to be found numerically, the equations can be solved subject to appropriate initial conditions at some convenient epoch t_0. Then (22) enables the matrizant relating any two times to be found.

If we can solve the equation

$$\delta\mathbf{r}''+\mathbf{B}\delta\mathbf{r}'+\mathbf{A}\delta\mathbf{r}=0 \tag{23}$$

by finding first and second fundamental solution matrices $\mathbf{U}$ and $\mathbf{V}$, where each satisfies (23) and obeys the initial conditions of (13), then the solution, if a forcing function $\mathbf{h}(t)$ is added, is given by (21). Equation (23) would arise if, for instance, the reference orbit were to have a uniform rotation.

In a recent paper Alexeev (1961) has considered the solution for perturbed Keplerian motion in the form

$$\mathbf{r}=\mathbf{r}_0+\int_{t_0}^{t}\mathbf{r}'dt,$$

$$\mathbf{r}'=\mathbf{r}_K'+\int_{t_0}^{t}\frac{\partial\mathbf{V}(\tau,t)}{\partial t}\mathbf{h}(\tau)d\tau. \tag{24}$$

Here $\mathbf{r}_K'$ is the velocity in the osculating Keplerian orbit at t_0, and $\mathbf{V}$ is the second fundamental solution matrix considered above. For a fixed t_0 (24) gives a first-order solution of the kind considered above, and the second equation could be immediately written down using formulas given by Myachin. An alternative formula, more useful when the velocity is not required, is

$$\mathbf{r}=\mathbf{r}_K+\int_{t_0}^{t}\mathbf{V}(\tau,t)\mathbf{h}(\tau)d\tau. \tag{25}$$

The application of these formulas in numerical work is limited by the inability of an osculating orbit to provide an adequate approximation of perturbed motion for many revolutions. But this need not be serious if t_0 is revised periodically. Given a reasonably simple method for calculating $\mathbf{V}$ (or $\partial\mathbf{V}/\partial t$) numerically, (25) or (24) would furnish a method for special perturbations that might be promising in some circumstances. A quicker method than that given by Alexeev for calculating the matrices is given in Sec. 9 at the end of this paper.

2. FORMAL SOLUTION OF THE PLANETARY EQUATIONS

If Eq. (11) involves only inverse-square attraction, with force function R_0, then the motion is Keplerian and the components of $\mathbf{V}$ are known analytically. This paper is concerned with motion that is nearly elliptic, so the reference orbits will be Keplerian ellipses. For a start $\mathbf{r}_R(t)$ will be the osculating ellipse at t_0. The equations for the first-order solution are

$$\delta\mathbf{r}''=\mathbf{A}_R\delta\mathbf{r}+[\nabla R_1]_R, \tag{26}$$

or

$$\delta x''=\sum_{\alpha=x,y,z}\left(\frac{\partial^2 R_0}{\partial x\partial\alpha}\right)_R\delta\alpha+\left(\frac{\partial R_1}{\partial x}\right)_R, \tag{26a}$$

with similar equations for y and z. The solution is given by (21) and is separated as before into the complementary function (CF) and particular integral (PI). Since the reference orbit is osculating at t_0 the complementary function is zero, and to the first order we have

$$\mathbf{r}=\mathbf{r}_R+(\text{PI}). \tag{27}$$

The particular integral will contain purely secular terms which are objectionable. To cope with this situation the complementary function, with six appropriate constants of integration, is introduced to define a mean orbit $\mathbf{r}_M$ such that

$$\mathbf{r}=\mathbf{r}_M+\delta\mathbf{r}, \tag{27a}$$

where

$$\mathbf{r}_M=\mathbf{r}_R-(\text{CF}) \tag{27b}$$

and, from (21),

$$\delta\mathbf{r}=(\text{CF})+(\text{PI}). \tag{27c}$$

The constants in the complementary function are chosen to make $\delta\mathbf{r}$ as tractable as possible. The reason for this is, obviously, to make the solution as accurate as possible; but this requires that the terms that are neglected shall be as small as possible. The expressions given by (27) and (27a) are equally accurate, and juggling the arbitrary constants between them cannot possibly increase the accuracy of that solution. The adequacy of the solution can only be tested by examining the second-order equations.

Let the second-order solution be

$$\mathbf{r}=\mathbf{r}_M+\delta\mathbf{r}+\delta^2\mathbf{r}.$$

Substitution of this into the equation of motion results in the following equation for x:

$$x_M''+\delta x''+\delta^2 x''=\left(\frac{\partial R_0}{\partial x}\right)_M+\sum_{\alpha=x,y,z}\left(\frac{\partial^2 R_0}{\partial x\partial\alpha}\right)_M(\delta\alpha+\delta^2\alpha)$$
$$+\frac{1}{2}\sum_{\alpha,\beta=x,y,z}\sum\left(\frac{\partial^3 R_0}{\partial x\partial\alpha\partial\beta}\right)_M\delta\alpha\delta\beta$$
$$+\left(\frac{\partial R_1}{\partial x}\right)_M+\sum_{\alpha=x,y,z}\left(\frac{\partial^2 R_1}{\partial x\partial\alpha}\right)_M\delta\alpha. \tag{28}$$

Here the partial differential coefficients are evaluated along the mean orbit in order that advantage can be taken of the relatively small $\delta\mathbf{r}$. Certainly the terms of zero order in (28) cancel. But the terms of first order do not; for the first-order equation is (26a) in which the partial differential coefficients are evaluated along the osculating orbit.

Let the complementary function, which by (27b) is equal to $\mathbf{r}_R-\mathbf{r}_M$, have components (ξ,η,ζ). Let the particular integral have components (λ,μ,ν). All of these quantities are likely to contain purely secular terms. Equation (26a) can be written:

$$\delta x''=\sum_{\alpha=x,y,z}\left(\frac{\partial^2 R_0}{\partial x\partial\alpha}\right)_M\delta\alpha+\sum_{\alpha=x,y,z}\sum_{\gamma=\xi,\eta,\zeta}\left(\frac{\partial^3 R_0}{\partial x\partial\alpha\partial\gamma}\right)_M\delta\alpha\delta\gamma$$
$$+\left(\frac{\partial R_1}{\partial x}\right)_M+\sum_{\gamma=\xi,\eta,\zeta}\left(\frac{\partial^2 R_1}{\partial x\partial\gamma}\right)_M\delta\gamma,$$

to the second order. If this is subtracted from (28), we are left with the equation for the second-order quantities:

$$\delta^2 x''=\sum_{\alpha=x,y,z}\left(\frac{\partial^2 R_0}{\partial x\partial\alpha}\right)_M\delta^2\alpha$$
$$+\frac{1}{2}\sum_{\alpha=x,y,z}\sum_{\beta=\lambda,\mu,\nu}\left(\frac{\partial^3 R_0}{\partial x\partial\alpha\partial\beta}\right)_M\delta\alpha\delta\beta$$
$$+\sum_{\beta=\lambda,\mu,\nu}\left(\frac{\partial^2 R_1}{\partial x\partial\beta}\right)_M\delta\beta,$$

since $\lambda=\delta x-\xi$, etc. Therefore, regardless of the choice of mean orbit, the forcing function in the second-order differential equation will contain purely secular terms, so that the second-order solution will contain terms in t^2.

The cure for this difficulty is to solve for the first-order perturbations again, once the mean orbit has been chosen. This means that the disturbing function, and its derivatives, must be developed along the mean orbit. Then

$$\delta\mathbf{r}''=\mathbf{A}_M\delta\mathbf{r}+[\nabla R_1]_M. \tag{29}$$

The difference between this and (26) is of the second order, and should apparently make no difference to the accuracy of the first-order solution. But it relieves the pressure in the second-order equation, for (28) now becomes

$$\delta^2 x''=\sum_{\alpha=x,y,z}\left(\frac{\partial^2 R_0}{\partial x\partial\alpha}\right)_M\delta^2\alpha+\frac{1}{2}\sum_{\alpha,\beta=x,y,z}\sum\left(\frac{\partial^3 R_0}{\partial x\partial\alpha\partial\beta}\right)_M\delta\alpha\delta\beta$$
$$+\sum_{\alpha=x,y,z}\left(\frac{\partial^2 R_1}{\partial x\partial\alpha}\right)_M. \tag{30}$$

The $\delta\alpha$ and $\delta\beta$ occurring in (30) are components of $\delta\mathbf{r}$ of (27c), in which there should, at least, be no secular terms. Therefore we can now, and only now, reap the benefit of the choice of a mean orbit. This applies whether the second-order equations are to be solved or not.

Formal equations for third-order perturbations can be written down easily. If six (second-order) constants are chosen in the solution of (30) to lead to a new mean orbit, the formulas become more complicated if advantage is to be taken of any reduction in the magnitude of $\delta^2\mathbf{r}$. But scarcely any terms would need to be considered anyway, and decisions on this matter would depend on the problem in hand. Therefore third-order perturbations will not be discussed here.

3. COMPONENTS OF U AND V IN ELLIPTIC MOTION

The components of $\mathbf{U}$ and $\mathbf{V}$ can be found most directly by considering the meaning of Eq. (14). That is, we take a reference orbit, and start a neighboring orbit at time t_0 with residuals (with respect to the reference) $\delta\mathbf{r}_0$ and $\delta\mathbf{r}_0'$. Then we follow the history of the residuals at future times. In order to do this it is convenient to group together the formulas necessary to find $\mathbf{r}$ and $\mathbf{r}'$, at time t, given $\mathbf{r}_0$ and $\mathbf{r}_0'$ at time t_0; then increments are applied to everything visible except the time. This has been done by Bower (1932) as the basis of a method for the differential correction of orbits. The expressions for $\mathbf{U}$ and $\mathbf{V}$ are not given explicitly, but intermediate formulas are used to proceed from t_0 to t. The formulas leading to the calculation of $\mathbf{U}$ and $\mathbf{V}$ are given in Sec. 9 at the end of this paper; but although Bower's method is probably the best one for numerical work involving Eq. (25), it does not result in analytical expressions appropriate to general perturbations.

Since it is analytical expressions that are important, it is only necessary to find the components v_{ij} of $\mathbf{V}$. These are found by introducing errors into the velocity at time t_0 and finding their effect on position at time t. Let axes be chosen such that the x axis points toward perihelion and the y axis points toward that point in the orbit for which the true anomaly is 90°. The v_{ij} with respect to these axes can be found fairly easily. For instance, if a velocity increment $\delta x_0'$ is introduced, the other coordinates at t_0 remaining unaltered, then the resulting increments in semimajor axis, eccentricity, eccentric anomaly at t_0, true anomaly at t_0, and eccentric anomaly at t, are given respectively by

$$\delta a = -(2a/nr_0)S_0\delta x_0',$$

$$\delta e = -[(1-e^2)/nr_0]S_0C_0\delta x_0',$$

$$\delta E_0 = (1/enr_0)[2-2eC_0-(1-e^2)C_0^2]\delta x_0',$$

$$\delta v_0 = [(1-e^2)^{\frac{1}{2}}/enr_0][2-eC_0-C_0^2]\delta x_0',$$

$$\delta E = \left\{\frac{3a}{rr_0}(t-t_0)S_0 - \frac{a(1-e^2)}{nrr_0}S_0C_0(S-S_0) + \frac{1}{ern}[2-2eC_0-(1-e^2)C_0^2]\right\}\delta x_0'.$$

(Here S, C, S_0, and C_0 denote $\sin E$, $\cos E$, $\sin E_0$, and $\cos E_0$, respectively.)

The position in the reference orbit at time t is

$$x = a(C-e), \quad y = a(1-e^2)^{\frac{1}{2}}S.$$

To find the displaced position, a, e, and E are varied, and the axes are rotated through δv_0. The displacement in x is given by

$$\delta x = \delta a(C-e) - a\delta e - aS\delta E + a(1-e^2)^{\frac{1}{2}}S\delta v_0.$$

Noting that, in this instance, $\delta x = v_{11}\delta x_0'$, we can find the value of v_{11} by substitution. And so on for the other v_{ij}. Because of the choice of axes, v_{ij} is zero if one of the subscripts is equal to three, and v_{33} is simply the "g" of the f and g series (but expressed in closed form).

The values for the v_{ij} given below were taken from the paper by Myachin. He apparently found them by solving the differential equation $\mathbf{V}'' = \mathbf{AV}$, and he states laconically that "the justice of the equalities is established by means of the direct substitution of their right-hand parts into the recorded equations, and by the verification of the corresponding initial conditions." The present author found it more expeditious to check the results independently, using the methods described above. The expressions are:

$$\begin{aligned}
v_{11} = (a^2/nrr_0)\{&[(1-e^2)SC+eS]C_0^2 \\
&+(1-e^2)[-C^2-eC+2]C_0S_0 \\
&+[(e-e^3)SC+2(1+e^2)S]C_0 \\
&+[-eC^2-2(1+e^2)C+5e]S_0 \\
&\quad+[-2(1-e^2)SC-5eS]-3SS_0(E-E_0)\},
\end{aligned}$$

$$\begin{aligned}
v_{12} = [a^2(1-e^2)^{\frac{1}{2}}/nrr_0]\{&[C^2+eC-2]C_0^2 \\
&+[SC+eS]S_0C_0+[-eC^2+2C-e]C_0 \\
&+[-eSC+2S]S_0 \\
&\quad+[C^2+eC-2]+3SC_0(E-E_0)\},
\end{aligned}$$

$$\begin{aligned}
v_{21} = [a^2(1-e^2)^{\frac{1}{2}}/nrr_0]\{&[-C^2+eC-1]C_0^2 \\
&+[-SC+eS]S_0C_0+[-eC^2-2C-e]C_0 \\
&+[-eSC-2S]S_0 \\
&\quad+[2C^2+eC+2]+3CS_0(E-E_0)\},
\end{aligned} \tag{31}$$

$$\begin{aligned}
v_{22} = (a^2/nrr_0)\{&[SC-eS]C_0^2 \\
&+[-C^2+(e+e^3)C-1]S_0C_0 \\
&+[-(e+e^3)SC+2S]C_0+[eC^2-2C+e]S_0 \\
&\quad+[SC-eS]-3(1-e^2)CC_0(E-E_0)\},
\end{aligned}$$

$$v_{33} = (1/n)\{SC_0+[-C+e]S_0-eS\}.$$

An alternative expression for $\mathbf{V}$, which separates it into two matrices, one a function of t_0 and one a function of t, can be written down using (22). Let T be a time of perihelion passage, then

$$\mathbf{\Omega}(t_0,t) = \mathbf{\Omega}(T,t)\mathbf{\Omega}(t_0,T),$$

since $\mathbf{\Omega}^{-1}(T,t_0)=\mathbf{\Omega}(t_0,T)$. Hence

$$\mathbf{V}(t_0,t)=\left[-\frac{\partial\mathbf{V}(T,t)}{\partial T}\ \mathbf{V}(T,t)\right]\begin{bmatrix}\mathbf{V}(t_0,T)\\ \dfrac{\partial\mathbf{V}(t_0,T)}{\partial T}\end{bmatrix}. \qquad (32)$$

Remembering that $E=0$ at time T, the submatrices can be easily found.

The matrix $\mathbf{U}$ is needed only in the complementary function. The full expressions for the components will not be given here, since it is intended to make out a case for putting $t_0=T$ in the complementary function, when

$$\mathbf{U}(T,t)=-\partial\mathbf{V}(T,t)/\partial T. \qquad (33)$$

If a general choice of axes is to be considered, it is necessary to find the rotation matrix that transforms the special axes considered above into any other set. This is the usual matrix

$$\mathbf{S}=\begin{bmatrix}P_x & Q_x & R_x\\ P_y & Q_y & R_y\\ P_z & Q_z & R_z\end{bmatrix}.$$

Then the new $\mathbf{U}$ and $\mathbf{V}$ are found from the expressions above, and

$$\mathbf{SUS}^T,\quad \mathbf{SVS}^T,$$

where $\mathbf{S}^T$ is the transpose of $\mathbf{S}$.

For work in general perturbations the axes would be fixed firmly in the mean orbit. But for special perturbations, if rectification were necessary, it would probably be better to use fixed axes, when the formulas of this section could be used. In this case S would have to be found, and rectified; this need not be necessary in Bower's method, and the latter is probably preferable.

4. FIRST-ORDER SOLUTION AND THE CONSTANTS OF INTEGRATION

The general approach to the integration of the first-order equations is fairly clear. Suppose that the disturbing function and its derivatives are expressed in terms of the mean anomalies of the planets concerned. The components of $\mathbf{V}$ can be expanded in series based on the mean anomaly, using the standard expansions of elliptic motion. The form given in (32) would be most suitable for this purpose. But two multiplications, and many additions of series would be necessary before the particular integral could be expressed as a Fourier series in the mean anomaly.

Alternatively, the disturbing forces can be expressed in terms of the eccentric anomaly of the disturbed planet. The argument of a typical term is $(pM+qM_1)$, where M and M_1 are the mean anomalies of the perturbed and perturbing planets. Let n and n_1 be their mean motions, then this argument can be written $(p+qn_1/n)M+\text{const}$, and, using Kepler's equation, this is

$$(p+qn_1/n)(E-e\sin E)+\text{const}.$$

The sine or cosine of this can easily be expanded into Fourier series by the use of Bessel functions. A strong case can be made out for such a development, for then the particular integral can be written down almost at once, without any series multiplication.

Although $\mathbf{V}$ can be expressed in terms of the true anomaly, there does not seem to be any reason for doing this.

Not the least important consideration in finding the first-order solution is that of allotting values to the six arbitrary constants. And we reiterate that, according to the results of Sec. 2, no purpose whatever is served by giving the constants nonzero values unless the first-order perturbations are to be reworked, with the disturbing forces evaluated along the mean orbit. Although accuracy in the final solution ought to be the main consideration, expediency is also important. There can be no perfect set of values for the constants. Their choice will be preceded and followed by elaborate calculations, and the more elaborate these may be, the better must be our reasons for the choice.

In order to see what can and what cannot be done with the constants of integration, we will consider the nature of the various terms in the complementary function and the particular integral.

The complementary function describes the difference between two "nearly equal" Keplerian orbits. It contains periodic terms, but only of short period (i.e., the period of the perturbed planet). It contains secular terms that arise solely because the two orbits have different mean motions (or semimajor axes). The expression for δz contains only periodic terms. The expressions for δx and δy contain mixed terms (i.e., trigonometric functions with coefficients proportional to the time), but these would vanish if the mean motions were the same.

It is normal to determine the arbitrary constants by considering the perturbation in true or mean longitude. The perturbation in true longitude is given by

$$r^2\delta\theta=x\delta y-y\delta x,$$

and that part due to the complementary function is

$$\begin{aligned}r^2\delta\theta_1=a\frac{(1-e^2)^{\frac{1}{2}}}{(1-e)^2}&\{-eSC+(2+3e-e^2)S-3E\}\delta x_0\\ &+\frac{a}{1-e}\{-eC^2+2C-1-e+e^2\}\delta y_0\\ &+\frac{a(1-e^2)^{\frac{1}{2}}}{n}\{-eC^2+2C-2+e\}\delta x_0'\\ &+\frac{a}{n}\{-2eSC+(4+3e+e^2)S-3(1+e)E\}\delta y_0'.\end{aligned} \qquad (34)$$

Here, for simplicity, E_0 has been taken to be zero; no essential ingredient is added if a nonzero value is included. The result can be very easily expressed in terms of the mean anomaly.

The value of $r^2\delta\theta_1$ when $E=E_0=0$ is $a(1-e)\delta y_0$. The corresponding quantity in the particular integral is zero. Now the secular part of (34), and possibly the S and C terms can be matched with corresponding terms in the particular integral, but there seems to be no purpose in taking a nonzero initial value for the perturbation in longitude, since it is hard to see how it can ever make the perturbations smaller. Therefore the degree of freedom in allotting the initial value of the longitude in the mean orbit is of no use. There are effectively three useful degrees of freedom in (34), and we might as well put $\delta y_0=0$. (This applies to the case $E_0=0$; in general, $x_0\delta y_0-y_0\delta x_0=0$.)

The particular integral contains a greater variety of terms. *Short-period terms* will normally involve the anomalies of both planets. The interesting ones, from the point of view of finding the constants, are those that involve only the perturbed planet, because only they can be compared with terms in the complementary function. Parts of these terms arise in the x and y coordinates because the osculating orbit has eccentricity and line of apsides slightly different from those of the "best" mean orbit at t_0; and in the z coordinate because the orientation of the plane of the osculating orbit is not quite the same as that of the mean orbit. Therefore the reduction of these terms (actually just those in $\sin E$ and $\cos E$, or $\sin M$ and $\cos M$) will take us to the mean orbit insofar as orientation and shape are concerned. But the best mean orbit at some epoch will not be the same as that for another epoch because all the elements are slowly, secularly varying. This will show in the *mixed secular terms* which cannot be paired with anything in the complementary function. Therefore if the secular change in any element is at all rapid, then the short-period terms will become swamped, and any attempt to remove terms in S and C would be like playing King Canute. This situation might be relieved if the reference orbit were to rotate in space, a possibility that is discussed later in this paper.

The *long-period terms* appear only in the particular integral, and no choice of constants can relieve them. But a possible way of dealing with them is suggested later in this paper.

Purely secular terms arise because the mean motion in the osculating orbit is not the same as the average sidereal motion in the actual orbit. If the sidereal period of the perturbed planet remains fairly steady (this precludes near-resonance) then the adjustment from the osculating to the mean orbit can be made simply by changing the size of the osculating orbit. This adjustment must always be made, for it is the secular terms that hurt most in the second-order equations.

In all the expressions t_0, or E_0, will be involved in moderately unpleasant ways. Clearly the choice of t_0 represents a degree of freedom, but it is apparently not one that can be used constructively. The constants that lead from the osculating to the mean orbit will vary with t_0, but the mean orbit itself should not be a function of t_0, at any rate over a few revolutions. For some t_0 the constants might have minimum values, but this would not improve the mean orbit. The advantage of a good choice of t_0 would occur only when no re-valuation of the first-order perturbations with respect to a mean orbit were to take place. But it is difficult to see how this good t_0 could be chosen without a first-order solution being to hand. Therefore there seems to be complete justification for choosing t_0 arbitrarily, and in particular it can be given its most convenient value. Almost certainly this is a time of perihelion passage, when $E_0=0$.

A possible procedure for the first-order solution might be as follows. Find an orbit from observation, and choose a time of perihelion passage as the epoch t_0. Using this orbit, find the disturbing forces. Solve the first-order integral to find those parts of the particular integral that are secular and those that contain $\sin E$ and $\cos E$. Choose the constants so that these terms in the expressions for the true longitude and δz vanish. From these constants determine a mean orbit. Find the disturbing forces for this mean orbit. Calculate again the first-order perturbations, remembering that the constants in the complementary function are minus those found before (for they go from the mean orbit, instead of to it).

Other ways of finding a mean orbit are possible. Obviously, observation over a sufficiently long time would provide information about the average sidereal period, shape, and orientation of an orbit. This would be the procedure for a major planet. Alternatively, a numerical integration over several revolutions using Encke's method, or the variation of elements (rather than Cowell's method, which would suffer from round-off errors) could be used as a substitute for observation.

The use of the constants of integration to simplify the expression for the perturbation in longitude has obvious advantages when the equations of motion are integrated in polar coordinates. The same method has been applied when rectangular coordinates were used, but arguments in its favor lean heavily on esthetics. For the kind of term removed from the longitude will still appear in δx and δy, and in a form that may be none the better for the choice of constants.

The most important criterion for the mean orbit is that there should be no secular perturbations with respect to it. If simplicity is the second criterion, then the mean orbit could have the same shape and orientation as the osculating orbit, but different size. Then no reorientation of axes, or change of eccentricity would be necessary, so that the redevelopment of the disturbing forces along the mean orbit would be relatively simple. If n is the mean motion in the osculating orbit and $n+\delta n$ the perturbed mean motion, then the constants in the plane of the orbit would be given by

$$\begin{aligned}\delta x_0&=-\tfrac{2}{3}x_0(\delta n/n),\\ \delta y_0&=-\tfrac{2}{3}y_0(\delta n/n),\\ \delta x_0'&=\tfrac{1}{3}x_0'(\delta n/n),\\ \delta y_0'&=\tfrac{1}{3}y_0'(\delta n/n).\end{aligned}\tag{35}$$

The complementary function takes the simple form (for any t_0):

$$\begin{aligned}\delta x&=-\tfrac{2}{3}a(\delta n/n)[C-e+\tfrac{3}{2}(a/r)n(t-t_0)S]\\ \delta y&=-\tfrac{2}{3}a(\delta n/n)[S-\tfrac{3}{2}(a/r)n(t-t_0)C],\end{aligned}\tag{36}$$

in which the secular terms should cancel similar terms in the particular integral. The constants δz_0 and $\delta z_0'$ would be zero.

As was mentioned above, if the disturbing forces are known in terms of the eccentric anomaly of the perturbed body, then the first-order solution can be written down fairly easily. For consider a term, $\cos pE$ in $\partial R_1/\partial x$, say; we can follow through its contribution to δx and δy using nothing more than elementary trigonometry. This appears to be the best approach, but it is possible that some regrouping of the equations or some further substitutions will lead to greater simplicity. An example of a rearrangement concerns the terms involving $(E-E_0)$ in the v_{ij}. We shall consider the contribution to δx of these terms alone. It comes from

$$\int_{t_0}^{t}\left(v_{11}\frac{\partial R_1}{\partial x_0}+v_{12}\frac{\partial R_1}{\partial y_0}\right)dt_0$$

and is

$$\frac{a^2}{nr}\int_{t_0}^{t}\frac{1}{r_0}\left[-3SS_0\frac{\partial R_1}{\partial x_0}+3(1-e^2)^{\frac{1}{2}}SC_0\frac{\partial R_1}{\partial y_0}\right](E-E_0)dt_0,$$

where the variables in the integrand are the quantities with zero subscripts. If we put

$$-aS_0=\left(\frac{dx}{dE}\right)_0,\quad a(1-e^2)^{\frac{1}{2}}C_0=\left(\frac{dy}{dE}\right)_0,\quad\text{and}\quad \frac{na}{r_0}dt_0=dE_0,$$

then it becomes

$$\frac{3S}{n^2r}\int_{E_0}^{E}\left[\frac{\partial R_1}{\partial x_0}\frac{dx_0}{dE_0}+\frac{\partial R_1}{\partial y_0}\frac{dy_0}{dE_0}\right](E-E_0)dE_0.\tag{37}$$

This integral also occurs in the expression for δy. The term in square brackets in (37) could be found during the development of the disturbing function, when it is still expressed in terms of the separate anomalies of the perturbed and perturbing planets; for then the term is $\partial R_1/\partial E_0$. Once this is found, the integral can be worked out quite simply. If this is not done, more calculation is required in the solution, and this calculation will include the subtraction of terms in E^2 that should be equal.

There is an interesting interpretation of (37). From the equations for the variation of the elements we have

$$\frac{dn}{dE_0}=-\frac{3}{na^2}\left[\frac{\partial R_1}{\partial x_0}\frac{dx_0}{dE_0}+\frac{\partial R_1}{\partial y_0}\frac{dy_0}{dE_0}\right].\tag{38}$$

Therefore, apart from some multiplying factor, (37) becomes

$$\int_{E_0}^{E}(E-E_0)\frac{dn}{dE_0}dE_0=-n_0(E-E_0)+\int_{E_0}^{E}ndE_0,$$

where n_0 is the value of n at t_0. The integral on the right-hand side can be written

$$\int\int\frac{dn}{dE_0}dE_0dE_0$$

when it at once becomes familiar as the type of double integral that can occur as a result of the removal of secular terms in the differential equations of perturbations. This is of no importance here, since the double integral, as such, would not be evaluated; but it shows that the integral can be introduced if it is wanted. n has no secular perturbation, so that no integral above contains terms in $(E-E_0)^2$. But the separate expressions

$$\int_{E_0}^{E}\frac{\partial R_1}{\partial x_0}\frac{dx_0}{dE_0}(E-E_0)dE_0\quad\text{and}\quad\int_{E_0}^{E}\frac{\partial R_1}{\partial y_0}\frac{dy_0}{dE_0}(E-E_0)dE_0\tag{39}$$

do contain such terms, although they must cancel in the sum.

In most perturbation theories a substitution is made in order that secular terms disappear from the differential equations. The reason for this is that the equations themselves cry out for some such change, and the substitution is a manipulation in the calculus that slightly changes the meaning of a variable. To get back to the variable that is physically more useful, the double integral appears. That there is nothing sacrosanct about the duplicity of the integral has been well demonstrated by Herrick (1951), who compared two methods for calculating the perturbed mean anomaly, one with a double integral, and one without.

This point is being labored because the method of the present paper must at first be suspect because traditional devices are not used. In fact the double integral can be introduced, if enough ingenuity is used in the algebra, but it is not needed. If the same basic equations are solved by different methods and the answers cast in the same form, then those answers should be the same. Therefore, double integral or no, a first-order solution obtained by the method of this paper should agree with one found by Brouwer's method (barring differences of the second order). But it does seem that some devices, such as the evaluation of the integral (37) rather than the separate integrals of (39), might be worthwhile.

5. SECOND-ORDER SOLUTION

The basic equations for the second-order solution have already been given. The main problem is the formation of the appropriate forcing function, involving the multiplication of three series; but this seems to be a complaint of any second-order approximation. Usually, short-period terms do not appear in second-order solutions. These terms can be removed before the series multiplication takes place. They can be avoided in the integration by putting $E=2k\pi$, where k is an integer; then the particular integral is

$$\int_0^{2k\pi} \mathbf{V}(E_0,2k\pi)\mathbf{f}\,dE_0,$$

where $\mathbf{f}$ is the forcing function. In this form $\mathbf{V}$ is much simpler. After integration, $2k\pi$ can be replaced by the eccentric (or true) anomaly.

6. IMPROVEMENT OF THE SOLUTION

In this section we discuss the possibility of finding, by numerical means, corrections to be applied to expressions of general perturbations. The problem can be stated as follows. "We have a system of differential equations that can be written down rigorously. We also have an analytical expression that nearly satisfies the equations. How can this be corrected?" It would be possible to set up higher-order perturbation equations and to solve them numerically instead of analytically; but it is the setting up of these equations and not their solution that is so tedious. It is better to take the approximate solution at its face value, and not to inquire how it was found. Then, since the corrections to be applied are small, it is possible to go back to first-order theory.

The exact equations of motion are

$$\mathbf{r}''=\nabla R. \tag{40}$$

Let $\mathbf{r}_0(t)$ be the approximate analytical solution, and let $\delta\mathbf{r}$ be the correction to be applied to it, such that $[\mathbf{r}_0(t)+\delta\mathbf{r}]$ satisfies Eq. (40). (Owing to a shortage of symbols for small quantities the notation in this section differs in detail from that of the others.) Ignoring the square of $\delta\mathbf{r}$, we find

$$\frac{d^2\mathbf{r}_0}{dt^2}+\frac{d^2\delta\mathbf{r}}{dt^2}=[\nabla R]_0+\mathbf{A}_0\delta\mathbf{r}$$

or

$$\frac{d^2\delta\mathbf{r}}{dt^2}=\mathbf{A}_0\delta\mathbf{r}+\left\{[\nabla R]_0-\frac{d^2\mathbf{r}_0}{dt^2}\right\}. \tag{41}$$

Here $\mathbf{A}_0$ is the matrix occurring in Sec. 1, Eq. (12), *et seq.* It is evaluated along the orbit $\mathbf{r}_0(t)$. We can write $\mathbf{A}_0=\mathbf{A}+\mathbf{B}$, where $\mathbf{A}$ is the same matrix evaluated along the mean orbit, and the components of $\mathbf{B}$ are much smaller than those of $\mathbf{A}$. Then since $\delta\mathbf{r}$ will be very small, we are apparently justified in neglecting $\mathbf{B}$.

The term in curly brackets on the right-hand side of (41) represents failure to solve Eq. (40) correctly. It is found by substitution, and is therefore a known function of the time. But this means that it can be treated as a forcing function to the homogeneous system

$$d^2\delta\mathbf{r}/dt^2=\mathbf{A}\delta\mathbf{r},$$

so that

$$\delta\mathbf{r}=\int_{t_0}^{t}\mathbf{V}(\tau,t)\Re(\tau)d\tau, \tag{42}$$

where

$$\Re(t)=[\nabla R]_0-d^2\mathbf{r}_0/dt^2,$$

is the remainder when the approximate solution is substituted into the equations of motion. Then $\delta\mathbf{r}$ can be found by the evaluation of a definite single integral in each of the three coordinates. Constants of integration can be introduced if corrections to the initial conditions are necessary. If $\Re(t)$ were to be calculated over several revolutions, inspection of $\delta\mathbf{r}$ would show if any essential terms had been omitted from $\mathbf{r}_0(t)$, and these terms could then be empirically added to the solution.

The presentation given above is weak for two main reasons. We are finding perturbations from a mean orbit, but the expressions for these perturbations will contain more decimal places than the mean orbit. This is the great strength of the method of differential corrections, and follows from the fact that the numbers of significant figures in $\mathbf{r}_0$ and $\delta\mathbf{r}$ are essentially the same. In fact it may require several revolutions before the errors in the solution become noticeable in Eq. (40). Even so, a spurious $\Re(t)$ will appear, due to random roundoff errors in the calculation of position in the mean orbit. Therefore an alternative approach will be described; this is based on Encke's method of special perturbations.

Suppose the analytical correction to the mean orbit to be $\Delta\mathbf{r}$, so that

$$\mathbf{r}_0=\mathbf{r}_M+\Delta\mathbf{r}.$$

The correct solution is

$$\begin{aligned}\mathbf{r}&=\mathbf{r}_0+\delta\mathbf{r}\\&=\mathbf{r}_M+\Delta\mathbf{r}+\delta\mathbf{r}\\&=\mathbf{r}_M+\boldsymbol{\rho},\end{aligned} \tag{43}$$

where $\boldsymbol{\rho}$ rigorously satisfies Encke's equations:

$$\boldsymbol{\rho}''=(\mu/r_M^3)\{qf(\boldsymbol{\rho}+\mathbf{r}_M)-\boldsymbol{\rho}\}+\nabla R_1, \tag{44}$$

where

$$q=(1/r_M^2)\{\mathbf{r}_M\cdot\boldsymbol{\varrho}-\tfrac{1}{2}\rho^2\}$$

and

$$qf=1-(1+2q)^{-\frac{3}{2}}.$$

By substituting $\Delta\mathbf{r}$ into (44) we could form the remainder $\mathfrak{R}(t)$ equal to the right-hand side minus the left-hand side. Now let $\boldsymbol{\varrho}=\Delta\mathbf{r}+\delta\mathbf{r}$ be substituted into (44). The magnitude of $\delta\mathbf{r}$ is small compared with that of $\Delta\mathbf{r}$, and if the square of $\delta\mathbf{r}$ is ignored it is possible to show that an equation of the form

$$\delta\mathbf{r}''=(\mathbf{A}+\mathbf{C})\delta\mathbf{r}+\mathfrak{R}(t)$$

results. Here $\mathbf{A}$ is the matrix referred to above, evaluated along the mean orbit, and the components of $\mathbf{C}$, while extremely involved, are much smaller than those of $\mathbf{A}$. Therefore we are justified, with a high degree of accuracy, in substituting $\Delta\mathbf{r}$ into Encke's equations, and using the remainder in Eq. (42) to find the correction to $\Delta\mathbf{r}$.

7. REDUCTION OF LONG-PERIOD AND MIXED SECULAR TERMS

Long-period terms affect the longitude most violently, and their effects might be mitigated by perturbing the time. Consider such a term with period $2\pi/\beta$; β is small and β^2 will be neglected. Let

$$t+\alpha\cos\beta t=\tau, \tag{45}$$

so that, neglecting β^2,

$$t=\tau-\alpha\cos\beta\tau$$

and

$$dt=d\tau(1+\alpha\beta\sin\beta\tau).$$

Therefore

$$\frac{d^2\mathbf{r}}{dt^2}=\frac{d^2\mathbf{r}}{d\tau^2}-2\alpha\beta\sin\beta\tau\frac{d^2\mathbf{r}}{d\tau^2}.$$

The equation of motion becomes

$$\frac{d^2\mathbf{r}}{d\tau^2}+\mu\frac{\mathbf{r}}{r^3}=\nabla R_1+2\alpha\beta\sin\beta\tau\frac{d^2\mathbf{r}}{d\tau^2}.$$

Since β is small, the term $d^2\mathbf{r}/d\tau^2$ on the right-hand side can be put equal to $d^2\mathbf{r}/dt^2$, evaluated along the mean orbit; then the equation of motion becomes

$$\frac{d^2\mathbf{r}}{d\tau^2}+\mu\frac{\mathbf{r}}{r^3}=\nabla R_1-2\mu\alpha\beta\sin\beta\tau\frac{\mathbf{r}}{r^3}. \tag{46}$$

The extra term in the forcing function results in the addition of

$$-2\mu\alpha\beta\int_{E_0}^{E}\frac{\sin\beta\tau}{r^3}\mathbf{Vr}dE_0$$

to the particular integral. This term can be integrated without much trouble. It contains a term in $\cos\beta\tau$, and with a suitable choice of α, this can be made to cancel the offending long-period term. This is the principle behind the method.

The introduction of the perturbed time will modify the disturbing function. To see the effect of this, consider a term $\cos(pt+q)$. This becomes

$$\cos(p\tau-p\alpha\cos\beta\tau+q),$$

which can be expanded in a Fourier series, using Bessel functions. So the perturbing forces become more complicated, but the parts producing the long-period term are not affected, and the additional terms will not lead to further long-period terms.

No trouble is passed on to the second-order equations, of the kind that followed the use of the osculating instead of the mean orbit, considered in Sec. 2. But the second-order equations are unpleasant, partly because of the confusion in orders of magnitude between the perturbing mass and β. But if a second-order solution were going to be carried out, the suggestion just made would be pointless, for its aim is to improve the first-order expressions.

Mixed terms that arise from secular perturbations in the elements Ω, i, and $\tilde{\omega}$, can be removed if the reference orbit rotates in a suitable way. The equation of motion with respect to axes rotating with angular velocity $\boldsymbol{\omega}$ is

$$\frac{d^2\mathbf{r}}{dt^2}+\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\mu\frac{\mathbf{r}}{r^3}=\nabla R_1.$$

The matrizant of the first variational equations, with R_1 ignored, can be found fairly easily from first principles. Alternatively ω^2 can be neglected, and the equation written

$$\frac{d^2\mathbf{r}}{dt^2}+\mu\frac{\mathbf{r}}{r^3}=\nabla R_1-\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt}, \tag{47}$$

where $d\mathbf{r}/dt$ on the right-hand side is evaluated along the mean orbit. This term gives rise to an addition to the particular integral that is readily calculated; it includes mixed secular terms, and the three numerical components of $\boldsymbol{\omega}$ must be chosen so that these terms cancel, as much as possible, with similar terms in the remainder of the particular integral.

The dilemma here concerns the expression of the disturbing forces with respect to the rotating coordinates. This presents no difficulty in some problems of satellite motion, but may not be practicable in planetary theory.

8. CALCULATION OF UNKNOWN FORCES

Comments have already been made on the use of Eq. (21) in work on special perturbations. It can also be used for the inverse problem. Suppose that some motion is observed so completely that three coordinates can be plotted against the time. The force $\mathbf{F}$ causing this motion can be calculated by the following formula, given by Deberdeev (1960):

$$\mathbf{r}(t)=\mathbf{r}_0+\mathbf{r}_0'(t-t_0)+\int_{t_0}^{t}(t-\tau)\mathbf{F}(\tau)d\tau. \qquad (48)$$

Deberdeev shows how (48) can be treated as a set of integral equations and solved numerically.

This formula is based on perturbations from motion in a straight line with constant speed. The matrix $\mathbf{V}$ is simply $\mathbf{I}(t-t_0)$, and $\mathbf{U}$ is the identity matrix. If motion is subject to a dominant inverse-square-law field of force, then residuals from a mean orbit can be measured, and the appropriate equation for the perturbing force $\delta\mathbf{F}$ would be

$$\delta\mathbf{r}(t)=\mathbf{U}\delta\mathbf{r}_0+\mathbf{V}\delta\mathbf{r}_0'+\int_{t_0}^{t}\mathbf{V}(\tau,t)\delta\mathbf{F}(\tau)d\tau. \qquad (49)$$

Deberdeev's equation is a zero-order expression that is used to calculate first-order quantities; it is better to work entirely in the first order.

If a reference orbit were calculated subject to a dominant inverse-square-law field of force and to small perturbing forces, then residuals from this reference orbit could be used in Eq. (49) to find additional perturbing forces. Furthermore, the matrices for Keplerian motion could be used, since the additions for the perturbed motion would be relatively small. Equation (49) can also be used to predict effects of neglected forces, and of uncertainties in physical parameters.

9. FORMULAS FOR THE NUMERICAL CALCULATION OF THE MATRICES

Formulas are given here for the calculation of the matrices $\mathbf{V}$, $\mathbf{U}=-\partial\mathbf{V}/\partial t_0$, and $\mathbf{W}=\partial\mathbf{V}/\partial t$. [$\mathbf{W}$ is the matrix that Alexeev (1961) calls $\mathbf{A}$.] Notation here is based on that of Bower (1932), and the procedure is substantially his.

a, e, the two times t and t_0, and the mean (or eccentric) anomaly at some epoch are all assumed known; hence the eccentric anomalies at t and t_0 can be found. It is assumed that Cartesian coordinates for any time can be calculated, either from a knowledge of the rotation matrix $\mathbf{S}$ (see Sec. 3) or from position and velocity at some epoch, by use of the f and g functions. The order of calculation would be derived from the following formulas (depending on what was required):

$$\tau=k(t-t_0),$$
$$\Delta E=E-E_0,$$
$$r=a(1-e\cos E),$$
$$r_0=a(1-e\cos E_0),$$
$$F=a(1-\cos\Delta E),$$
$$f=1-F/r_0,$$
$$f_0=1-F/r,$$
$$G=a^{\frac{1}{2}}\sin\Delta E,$$
$$H=a^{\frac{1}{2}}e\sin E,$$
$$H_0=a^{\frac{1}{2}}e\sin E_0,$$
$$J=a^{\frac{3}{2}}\Delta E-aG,$$
$$g=\tau-J,$$
$$L=(1/r)(3J+2FH+Gr_0),$$
$$L_0=(1/r_0)(-3J+2FH_0-Gr),$$
$$(2M)=(a/r_0)(GL-2F),$$
$$(2N)=a(-3J+FL),$$
$$(3)=FG/rr_0,$$
$$(2)=(2N)/r_0^3+(3),$$
$$(1)=(1/r_0^3)[G^2r_0/r+(2M)+F],$$
$$(4)=F^2/r,$$
$$(2M)_0=(a/r)(-GL_0-2F),$$
$$(2N)_0=a(3J+FL_0),$$
$$(2)_0=(2N)_0/r^3-(3),$$
$$(1)_0=(1/r^3)[G^2r/r_0+(2M)_0+F],$$
$$(4)_0=F^2/r_0,$$

$$\mathbf{\Phi}=\begin{bmatrix}x & x'\\ y & y'\\ z & z'\end{bmatrix},$$

$$\mathbf{\Phi}_0=\begin{bmatrix}x_0 & x_0'\\ y_0 & y_0'\\ z_0 & z_0'\end{bmatrix},$$

$$\mathbf{\Phi}^T=\text{the transpose of }\mathbf{\Phi},$$

$$\mathbf{U}=f\mathbf{I}+\mathbf{\Phi}_0\begin{bmatrix}(1) & (3)\\ (2) & (4)\end{bmatrix}\mathbf{\Phi}_0^T,$$

$$\mathbf{V}=g\mathbf{I}+\mathbf{\Phi}_0\begin{bmatrix}(3) & (2M)\\ (4) & (2N)\end{bmatrix}\mathbf{\Phi}_0^T,$$

$$\mathbf{W}=f_0\mathbf{I}+\mathbf{\Phi}\begin{bmatrix}(1)_0 & (2)_0\\ -(3) & (4)_0\end{bmatrix}\mathbf{\Phi}^T.$$

The matrix $\mathbf{W}$, and also $\partial\mathbf{U}/\partial t$, can be found alternatively by selecting the formulas necessary to calculate $\mathbf{U}$ and $\mathbf{V}$ and differentiating them with respect to t.

It is possible to write down series expansions in powers of the time. For instance

$$\mathbf{V}(t_0,t)=\mathbf{V}(t_0,t_0)+(t-t_0)\left(\frac{\partial\mathbf{V}}{\partial t}\right)_{t=t_0}+\tfrac{1}{2}(t-t_0)^2\left(\frac{\partial^2\mathbf{V}}{\partial t^2}\right)_{t=t_0}+\cdots.$$

But from the equation for $\mathbf{V}$, and the initial conditions, we find

$$\mathbf{V}(t_0,t_0)=0,\quad \left(\frac{\partial\mathbf{V}}{\partial t}\right)_{t=t_0}=\mathbf{I},$$

$$\left(\frac{\partial^2\mathbf{V}}{\partial t^2}\right)_{t=t_0}=0,\quad \left(\frac{\partial^3\mathbf{V}}{\partial t^3}\right)_{t=t_0}=\mathbf{A}_0,\ \text{etc.}$$

so that

$$\mathbf{V}(t_0,t)=\mathbf{I}(t-t_0)+\frac{1}{3!}\mathbf{A}_0(t-t_0)^3+\frac{1}{4!}2\mathbf{A}_0'(t-t_0)^4+\frac{1}{5!}(3\mathbf{A}_0''+\mathbf{A}_0\mathbf{A}_0)(t-t_0)^5+\frac{1}{6!}(4\mathbf{A}_0'''+4\mathbf{A}_0'\mathbf{A}_0+2\mathbf{A}_0\mathbf{A}_0')(t-t_0)^6+\cdots,$$

where $\mathbf{A}_0$ and its derivatives can be found from

$$\mathbf{A}=-\frac{k^2}{r^3}\mathbf{I}+\frac{3k^2}{r^5}\begin{bmatrix}x\\y\\z\end{bmatrix}[x\ y\ z].$$

For integration with respect to t_0 this series is not suitable; but $\mathbf{V}(t_0,t)$ can be expanded in a Taylor series about time t, if it is remembered that

$$\mathbf{V}(t,t)=0,\quad \left(\frac{\partial\mathbf{V}}{\partial t_0}\right)_{t_0=t}=-\mathbf{I},\quad\text{and}\quad \frac{\partial^2\mathbf{V}}{\partial t_0^2}=\mathbf{VA}.$$

The series is

$$\mathbf{V}(t_0,t)=-\mathbf{I}(t_0-t)-\frac{1}{3!}\mathbf{A}(t_0-t)^3-\frac{1}{4!}2\mathbf{A}'(t_0-t)^4-\frac{1}{5!}(3\mathbf{A}''+\mathbf{AA})(t_0-t)^5+\cdots.$$

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