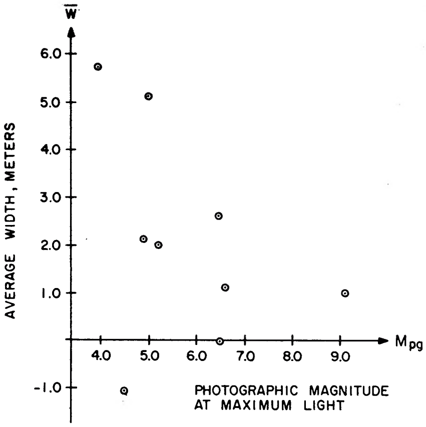

FIG. 4. Width–magnitude relationship.

the light curve. This indicates that the width of the meteor depends upon the degree of fragmentation that occurs.

The bright meteors appear to have larger widths than the fainter meteors. This effect is shown in Fig. 4, where the width of the brightest point of each trail is plotted against its magnitude. The width of trails in this photographic sample is 1 m, though two show widths of 6 m, presumably due to fragmentation.

ACKNOWLEDGMENTS

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Some Properties of Rosette Configurations of Gravitating Bodies in Homographic Equilibrium

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A particular case of the multibody problem which admits of simple solutions has been re-examined. It concerns hypothetical plane arrays of an even number of mutually gravitating bodies revolving about their common barycenter in dynamic equilibrium while maintaining the shape of their configuration (though not necessarily its size). The common property of this family of arrays is the existence of mirror symmetry about the radius vector from the barycenter to each of the massive bodies. Dynamic equilibrium is possible for limited ranges of rhombic configurations and of arrays in the shape of hexagonal and octagonal (and higher polygonal) regular rosettes. Such arrangements comprise alternating heavier bodies, all alike, and an equal number of lighter bodies, also all alike, in regularly alternating fashion. Relationships between permissible radius ratios and mass ratios exist such that, while the mass ratio may assume any positive value, the compatible radii can differ only within narrow limits. A sun body may or may not be located in the barycenter. The orbits are all circles, or, in general, conic sections of equal eccentricity with the barycenter at one focus. The influences of the mass and radius ratios upon the angular velocity of the orbital motion and upon permissible oscillations in and out of the mean orbital plane are delineated.

IN view of the appealing simplicity and periodicity of the relative motion of two astral bodies orbiting in permanent Keplerian ellipses about their common center of gravity, many have been intrigued by the quest for the discovery of conditions under which more than two bodies under mutual gravitational attraction can revolve in permanent dynamic equilibrium or move in periodic orbits. For three bodies of arbitrary mass values, Lagrange, in 1772, published his famous solution of the five libration conditions under which the three bodies may lie either on one straight line or on the three corners of an equilateral triangle. For more than three bodies the situation becomes more complex. It has been proven that four or more massive bodies of arbitrary mass values without any symmetry between them cannot be so arranged that they would remain in the same configuration permanently or perform periodic oscillations about it.

However, theoretically, it would be possible to arrange four or more bodies in such a dynamic equilibrium configuration, provided that some conditions of symmetry are fulfilled. For such plane configurations of equilibrium the resultants of the gravitational attractions of any one of the bodies by all the others must be directed radially towards the common center of gravity, and the orbital angular velocity for which

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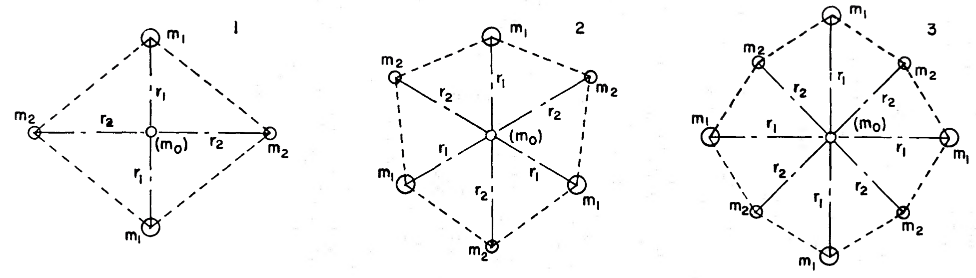

FIGS. 1–3. Rhombic configuration, hexagonal rosette, and octagonal rosette.

the resultant is balanced by the centrifugal force must be the same for all at all times. A possible (though neither sufficient nor necessary) condition is that, in the orbital plane, there is complete symmetry of all mutual gravity forces with respect to the radius vector of the position of every one of the massive bodies from the mass center of the system [1,2] (Numbers in square brackets refer to bibilographic notes at the end of the article).

Symmetry about each radius vector obviously exists for the well-known case of equal masses disposed at the corners of a regular polygon, with or without a sun in the center. The constant angular velocity Ω, at which such a system could rotate in dynamic equilibrium, is obviously defined by

$$\Omega^2=(G/r^3)(m_0+kM),$$

where G is the gravitational constant, r is the distance of each "planet" from the center, m_0 is the mass of the sun, and $M=nm$ is the total mass of all the n planets of mass m each. The constant k is, according to the geometry of a regular n-sided polygon,

$$k=\frac{1}{2n}\left[\frac{1}{2}+\sum_{i=1}^{i=\frac{1}{2}(n-2)}\cos\frac{i\pi}{u}\right]\quad(n \text{ even}),$$

$$k=\frac{1}{2n}\sum_{i=1}^{i=\frac{1}{2}(n-1)}\cos\frac{i\pi}{n}\quad(n \text{ odd}).$$

Table I gives the values of k for the configurations up to the nonagon.

So much for the trivial case of the completely regular polygon, which is by no means the only configuration which exhibits symmetry about each radius vector. Such symmetry is also possessed by a peculiar family of geometrical configurations which may be described as "rosettes." In these an even number of "planets" of two (or more) kinds, one (or some) heavier than the other, but all of each set of equal mass, are placed at the corners of two (or more) interdigitated regular polygons so that the lighter and heavier ones alternate (or follow each other in a cyclic manner) [1]. The question arises: What ratios between the radii are compatible with what ratios between the masses?

TABLE I. Numerical values of k in the formula for Ω^2.

n	2	3	4	5	6	7	8	9
k	0.125	0.192	0.239	0.275	0.305	0.330	0.351	0.369

Let us denote by r_1 and $r_2=\rho r_1$ the radii of two coplanar planet rings, by m_1 and $m_2=\rho m_1$ the respective masses of the two types of planets. The forces acting on any one of the bodies, when they travel in circular orbits at the angular velocity Ω, are readily derived from Newton's law as applied to the four-, six-, and eight-sided rosettes illustrated in Figs. 1, 2, and 3. The radial gravitation component acting on a planet of mass m_1, for instance, is balanced by its centrifugal force. The expressions of this equilibrium, in terms of force (per unit mass) read:

for the rhombus

$$G\left[\frac{m_1}{(2r_1)^2}+\frac{2m_2r_1}{(r_1^2+r_2^2)^{\frac{3}{2}}}\right]=\Omega^2r_1;\qquad(1)$$

for the hexagonal rosette

$$G\left[\frac{m_1}{\sqrt{3}r_1^2}+\frac{m_2}{(r_1+r_2)^2}+\frac{m_2(2r_1-r_2)}{(r_1^2-r_1r_2+r_2^2)^{\frac{3}{2}}}\right]=\Omega^2r_1;\qquad(2)$$

for the octagonal rosette

$$G\left\{\frac{m_1}{r_1^2}\left(\frac{1}{4}+\frac{1}{\sqrt{2}}\right)+m_2\left[\frac{2r_1-\sqrt{2}r_2}{(r_1^2-\sqrt{2}r_1r_2+r_2^2)^{\frac{3}{2}}}\right.\right.$$
$$\left.\left.+\frac{2r_1+\sqrt{2}r_2}{(r_1^2+\sqrt{2}r_1r_2+r_2^2)^{\frac{3}{2}}}\right]\right\}=\Omega^2r_1;\qquad(3)$$

and the procedure can readily be extended to any rosette of more corners. The forces acting on the other massive planets m_2 can similarly be written down by interchanging the subscripts 1 and 2.

If it is now postulated that the orbital angular velocity Ω be the same for both sets, the equation pairs can be combined under elimination of the gravitation constant G and Ω. This results in an equation between the mass ratio μ, and the radius ratio ρ, which has the

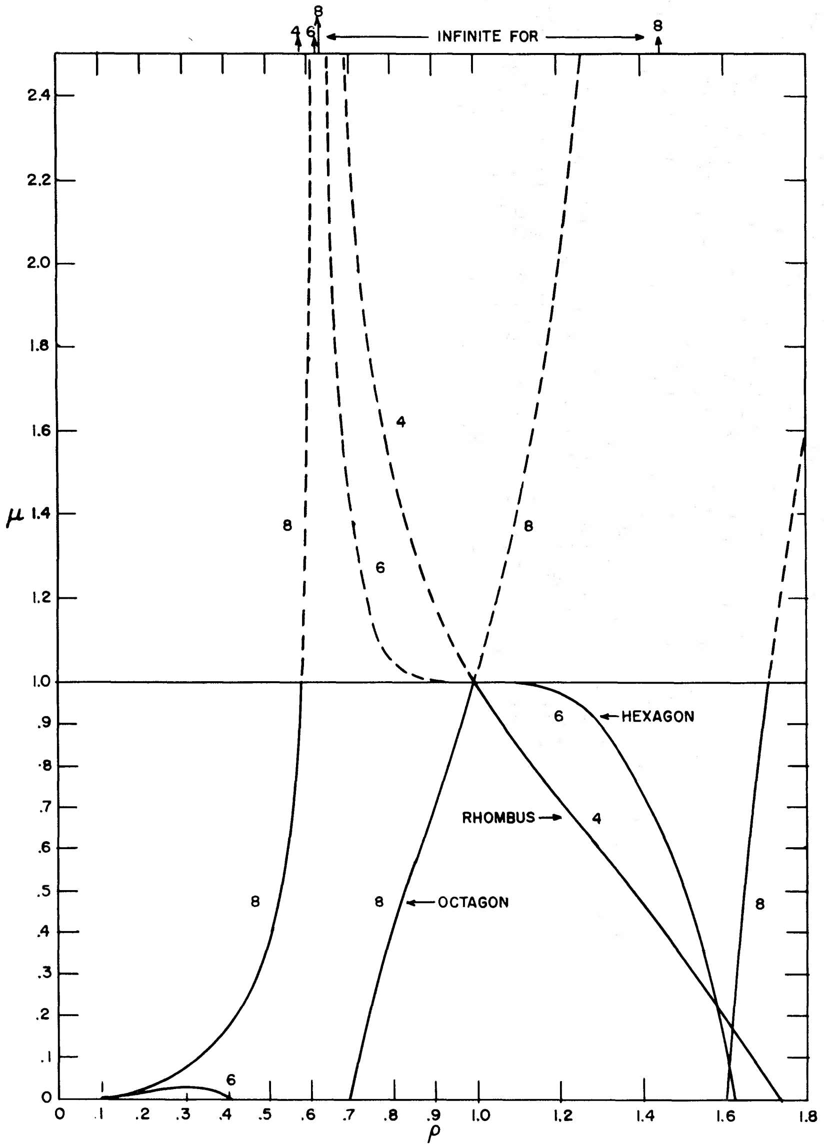

FIG. 4. Relation between mass ratio μ and radius ratio ρ. Case of interdigitated regular polygons.

following solutions for μ in terms of ρ;

for the rhombus [3]

$$\mu=\frac{8-(1+\rho^2)^{\frac{3}{2}}}{8-(1+1/\rho^2)^{\frac{3}{2}}}=\frac{1-\sigma^3}{1-(\sigma/\rho)^3} \tag{4}$$

(with $4\sigma^2=1+\rho^2$);

for the hexagonal rosette

$$\mu=\frac{1/(1+\rho)^2+(2\rho-1)/s^3-\rho/\sqrt{3}}{\rho/(1+\rho)^2+(2\rho-\rho^2)/s^3-1/\sqrt{3}\rho^2} \tag{5}$$

(where $s^2=1-\rho+\rho^2$);

and for the octagonal rosette

$$\mu=\frac{(2\rho-\sqrt{2})/p^3+(2\rho+\sqrt{2})/q^3-0.957\rho}{\rho[(2-\sqrt{2}\rho)/p^3+(2+\sqrt{2}\rho)/q^3]-0.957/\rho^2} \tag{6}$$

(where $p^2=1-\sqrt{2}\rho+\rho^2$, $q^2=1+\sqrt{2}\rho+\rho^2$, and $0.957=\frac{1}{4}+1/\sqrt{2}$).

Figure 4 illustrates these relationships under the assumption that m_1 always denotes the larger of the masses which is finite so that μ can be between 1 and 0, these two limits representing the regular polygon ($m_2=m_1$) and the infinitesimal mass ($m_2\rightarrow 0$).

Table II contains some computed values from which the graphs of Fig. 4 were drawn. They also indicate the boundaries which separate domains of possible and impossible equilibrium.

In the case of the rhombus there is only one continuous possible domain which extends from the square with all masses alike to the 60°, 120° rhombus whose short diagonal is equal to the side and for which the far corner masses become infinitesimal as compared to the near corner masses. For any mass ratio $1>\mu>0$ there exists one permissible radius ratio $1<\rho<1.732$

TABLE II. Computed radius ratios and mass ratios (see Fig. 4).

	Rhombus			Hexagon			Octagon	
Radius ratio ρ	Mass ratio μ	Slope $d\mu/d\rho$	Radius ratio ρ	Mass ratio μ	Slope $d\mu/d\rho$	Radius ratio ρ	Mass ratio μ	Slope $d\mu/d\rho$
1.732	0					1.708	1.000	
1.700	0.052					1.700	0.9545	
1.667	0.100		1.620	0		1.667	0.6879	
1.600	0.202		1.600	0.109		1.623	0.2531	
1.500	0.340		1.500	0.520		1.6024	0	
1.400	0.473		1.400	0.736				
1.300	0.599		1.300	0.905				
1.200	0.721		1.200	0.975				
1.100	0.854		1.100	0.998				
1.000	1.000	−1.64	1.000	1.000	−0.028	1.000	1.000	+3.0
						0.955	0.85	
						0.909	0.72	
						0.870	0.61	
						0.833	0.51	
						0.800	0.415	
						0.770	0.321	
						0.741	0.220	
						0.707	0.062	
			0.414	0		0.6934	0	
			0.412	0.0012				
			0.400	0.0092				
			0.380	0.0186		0.586	1.000	
			0.350	0.0259		0.577	0.846	
			0.333	0.0277		0.500	0.368	
			0.320	0.0280		0.400	0.175	
			0.300	0.0270		0.333	0.102	
			0.250	0.0222		0.250	0.0445	
			0.200	0.0145		0.200	0.0233	
			0.100	0.0027		0.100	0.0027	0
			0	0	0	0	0	0

[4]; the smaller masses are always the ones farther out.

For the hexagon a similar domain exists reaching from $1<\rho<1.620$; again with the outer corner masses the smaller ones, the configuration differing the less from a regular hexagon, the nearer the mass ratio is to unity. However, for the hexagon there exists another permissible domain in which the smaller masses are arrayed on a triangle much smaller (ρ considerably <1) and so to speak forming a nucleus inside the larger triangle on whose corners the much larger masses are positioned [5]. The upper limit for this case is $\rho=0.414$; the maximum μ of 0.028 occurs at about $\rho=0.32$.

For the octagon the conditions are reversed: For configurations departing but little from the regular octagon, the smaller masses are closer to the center than the larger ones. The limits are $1>\rho>0.697$. The other permissible domain has a core of heavy masses in a square array interdigitated inside a much larger square of lighter masses, the radius ratio being $\rho<0.624$.

The symmetry requirements would not be violated by the presence of a massive sun in the center of a rosette array, but the law of correlation between the mass ratios of the two kinds of planets and their radius ratios will be influenced by the relative mass of the sun [6]. If the sun's mass is denoted by m_0, then its attraction is readily expressed by an extra term $+m_0/r_1^2$ within the brackets of Eqs. (1), (2), and (3), and by an extra term $+m_0/r_2^2$ in their correlates. This causes an additional term in $\mu_0=m_0/m_1$ to appear in the equations corresponding to Eqs. (4), (5), and (6). For the rhombus, for instance, the solution can be written

$$\mu=\frac{8-(1+\rho^2)^{\frac{3}{2}}[1+4\mu_0(1-1/\rho^3)]}{8-(1+1/\rho^2)^{\frac{3}{2}}}$$

$$=\frac{1-\sigma^3[1+4\mu_0(1-1/\rho^3)]}{1-(\sigma/\rho)^3}$$

$$[\text{with } \sigma=\tfrac{1}{2}(1+\rho^2)^{\frac{3}{2}}]. \quad (4')$$

Or, in terms of $\mu_1=m_1/m_0$ and $\mu_2=m_2/m_0$, which is convenient when m_0 is relatively large,

$$\mu_1\rho^3(1-\sigma^3)-\mu_2(\rho^3-\sigma^3)=4\sigma^3(\rho^3-1). \quad (4'')$$

For the hexagonal and orthogonal configuration the analogous derivations will be obvious.

While the rosette configurations here studied are capable of revolving as a whole in dynamic equilibrium, they are not stable against random perturbations. Should any one of the masses come under the gravitational influence of a foreign massive body approaching it closely, the configuration would be disturbed and might eventually be disrupted.

However, there are two special kinds of periodic oscillations of which these configurations are theoretically capable. To be sure, the mechanism by which they would have to be initially excited may appear

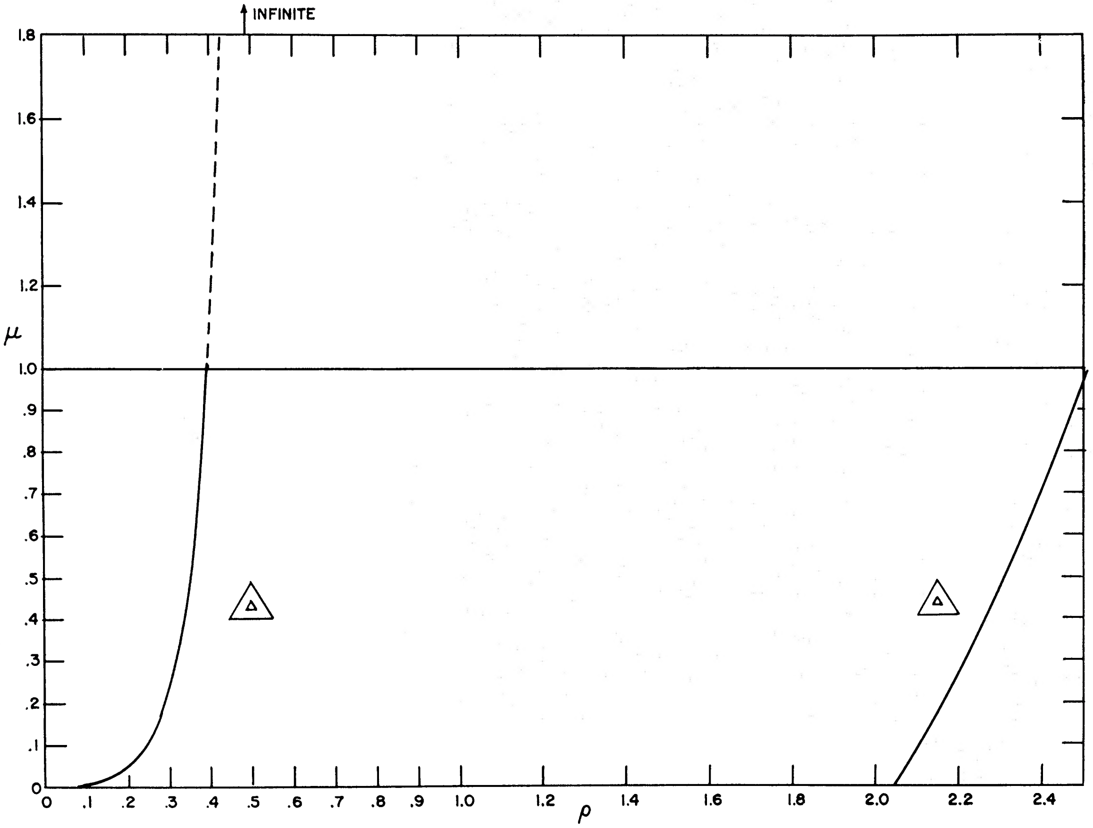

FIG. 5. Relation between mass ratio μ and radius ratio ρ. Case of homothetically oriented triangles.

highly artificial and peculiar, but no more so than the creation of a rosette-type polygon array of celestial bodies would in the first place.

One type of periodic oscillatory motion conceivable is an axial pulsation. If there is no sun in the center, then the system has one degree of freedom, with alternating planets moving to the "north" side of the orbit axis, as it were, and the others toward the south side during one half-cycle of the oscillation and opposite during the other. The planet masses of the two magnitudes m_1 and m_2, can swing through amplitudes of their excursions z_1, z_2, inversely proportional to the masses. This oscillation is harmonic so long as the amplitude is small compared to the distances between the planets. The restoring force equation under the condition of an excursion z_1 of the m_1 system and $z_2=z_1/\mu$ for the $m_2=\mu m_1$ system is then

for the rhombus

$$2G[(m_1+m_2)(z_1+z_2)/(r_1^2+r_2^2)^{\frac{3}{2}}]=-(\ddot{z}_1+\ddot{z}_2)\,; \tag{7}$$

for the hexagonal rosette

$$G(m_1+m_2)\left[\frac{1}{(r_1+r_2)^3}+\frac{2}{(r_1^2-r_1r_2+r_2^2)^{\frac{3}{2}}}\right](z_1+z_2) = -(\ddot{z}_1+\ddot{z}_2)\,; \tag{8}$$

for the octagonal rosette the analogous expressions will be obvious.

Consequently, the respective natural frequencies are, for the rhombus

$$\omega=\left[2G\frac{m_1+m_2}{(r_1^2+r_2^2)^{\frac{3}{2}}}\right]^{\frac{1}{2}}=\Omega\left[\frac{1+\mu}{\sigma^3+\rho\mu}\right]^{\frac{1}{2}}; \tag{7$'$}$$

and for the hexagonal rosette

$$\omega=\left\{G(m_1+m_2)\left[\frac{1}{(r_1+r_2)^3}+\frac{2}{(r_1^2-r_1r_2+r_2^2)^{\frac{3}{2}}}\right]\right\}^{\frac{1}{2}}, \tag{8$'$}$$

and the proportionality with the orbital frequency Ω is furnished by reference to Eqs. (1) and (2).

If the amplitudes of the axial oscillations were to become large and cease to be negligible compared to the distance between adjacent planets, then Eqs. (7) and (8) are no longer a sufficient approximation, the three-dimensional distance enters into the denominator; the motion becomes slower and its character more complex.

If there is a central sun, it does not have to remain at rest: it may also oscillate along the rosette axis against the masses of the planets. In the trivial case of all planet masses being equal and arranged in form of a regular polygon, the situation is very simple: Obviously the excursion amplitude z_0 of the sun of mass m_0 and the amplitude z of the n planets of mass m would be related according to

$$z_0 m_0 = -nzm = -zM,$$

and the natural frequency of the oscillation would be

$$\omega = [G(m_0+M)/r^3]^{\frac{1}{2}} = \Omega[(m_0+M)/(m_0+kM)]^{\frac{1}{2}}. \quad (9)$$

If, however, the planets are of two kinds, arranged in rosette configuration and there is a sun in the middle but also allowed to oscillate along the axis, then the system will, in general, behave like a double pendulum, the motion following the superposition of two oscillations at two separate frequencies with a periodic exchange of kinetic and potential energy between the outer planet ring, the inner planet ring, and the sun.

Another conceivable type of motion pertains to a variation of the entire formation in size. If the angular velocities of all of the bodies of the rosette configuration are not constant, though alike, then their orbits will not be circular. That they will all be Keplerian conics of equal eccentricity can be readily seen when it is considered that, under the conditions of compatible mass ratios and radius ratios between the two sets, as developed for circular orbits, the resultant of all acting gravitational attractions will remain radially directed toward the barycenter and will vary inversely proportionally to the square of the radial distance to any one of the masses, when the configuration merely expands or contracts while retaining geometric (homographic) similarity [7]. Each mass will then behave just as though it were under the influence of only one central force of gravitation emanating from the barycenter. Hence, if the velocities are less than the escape velocities from this central field, then the orbits will be Keplerian ellipses; otherwise they will be parabolas or hyperbolas. Obviously this behavior is not contingent upon the presence or absence of a sun at the barycenter.

The possibility of the existence of multibody configurations of particular symmetry in rotary equilibrium or in periodic motion is interesting from a viewpoint of understanding the working of the forces in the universe. This curious fact need not be imputed to have any immediate practical application. Neither have such constellations—beyond the rhombic Trojans—been observed in the heavens, nor would it be easy to set up artificial systems of rosette symmetry.

On the other hand, it is noteworthy that rosette configurations are capable of rotary equilibrium and periodic motion, not only in the gravitational field governed by an inverse radial distance squared law, but also in unipolar attraction fields characterized by negative exponents different from -2. The analogous derivation of the configuration geometry and equilibrium conditions for them, and an extension of the theory to potential fields involving bipolar reactions, such as electrical attractions and repulsions in addition to gravitation is readily envisaged.

BIBLIOGRAPHIC NOTES

[1] This has already been noted by R. Hoppe, "Erweiterung der bekannten Speciallösung des Dreikörperproblems," *Arch. Math. Phys.* **64**, 218–223 (1879).

[2] Four-body solutions in the form of configurations exhibiting a lesser degree of symmetry have been shown to exist, for instance: the anti-parallelogram (kite shape) by O. Dziobek in *Astron. Nach.* **152** (1900), No. 3627, entitled "Über einen merkwürdigen Fall des Vielkörperproblems," and by J. H. Andoyer, "Sur l'équilibre relatif de n corps," published under *Mémoires et Observations* in *Bull. Astron.* **23**, 50 (1906); also in the latter paper, and in Aurel Wintner's *The Analytical Foundations of Celestial Mechanics* (Princeton University Press, Princeton, New Jersey, 1941), pp. 245–246: the isosceles trapezoid having the barycenter located at the intersection of the diagonals.

[3] This relationship has been previously derived by W. R. Longley in "Some particular solutions in the problem of n bodies," *Bull. Am. Math. Soc.* 13 (1907).

[4] These limits are in precise agreement with those stated more generally by Dziobek, *loc. cit.*, for any quadrangle, namely that each of the diagonals must be longer than any side.

[5] The libration points on the bisecants of the angles of regular polygons were given by M. Lindow in an article entitled "The circular case in the restricted problem of $n+1$ bodies," published in *Astron. Nach.* **228**, No. 5461 (1928), which was brought to our attention by Dr. Gen-ichiro Hori. The existence of two such libration points at $\rho=0.413888$ and 1.619790 outside the domain comprising the regular hexagon cited in the above article prompted us to investigate the separate domain of core-in-ring configurations. These constitute another branch of solutions of Eq. (5).

Incidentally, symmetry about every radius vector is also possessed by systems of similar regular polygons lying one inside another, orientated homothetically rather than interdigitated. For the configuration of two equilateral similar homothetic triangles the limit condition of $\mu=0$ was also given by Lindow, *loc. cit.* and earlier, in an article "The circular case in the problem of 3+1 bodies" in *Astron. Nach.* **220**, No. 5279 (1924), viz., $\rho=2.043817$. It may here be added that such a configuration, which may perhaps be considered a special class of rosette, can also be materialized for any finite mass ratio according to

$$\mu = \frac{1/\sqrt{3} - (1/\rho+2)/(1+\rho+\rho^2)^{\frac{3}{2}} \pm 1/\rho(1-\rho)^2}{1/\sqrt{3}\rho - (\rho+2)/(1+\rho+\rho^2)^{\frac{3}{2}} \pm 1/(1-\rho)^2},$$

which is plotted in Fig. 5, the upper and lower sign pertaining to $\rho \lessgtr 1$, respectively. The points at $\rho=0.398$ (or 2.513) for which all six masses are alike deserve special notice.

In the quoted articles by Lindow not only the libration points but also the possible periodic orbits of infinitesimal bodies remaining in the vicinity of these points are investigated.

The cases of two homothetical noninterdigitated squares as well as interdigitated ones were also sketched by Longley (*loc. cit.*) and some example solutions cited. So was a case with two additional masses on a third inner circle.

[6] The cases of a sun in the center of rhombic and octagonal configurations are also mentioned in the book *Periodic Orbits* by F. R. Moulton (Carnegie Institution, Washington, 1920), Chap. XIII by W. R. Longley.

[7] "Homographic" in the sense defined by Aurel Wintner, *loc. cit.*, in whose book the compatibility conditions of multibody configurations are discussed in Chap. V in great generality (pp. 273–315), and numerous references to related publications are given.