

THE RELATIVISTIC MODEL OF THE STEADY-STATE UNIVERSE

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Summary

It is shown that in the relativistic and Newtonian versions of the steady-state universe the motion of the cosmic matter is indeterminate. The reason is that for matter with equation of state $p + \rho = 0$ both the inertial and passive gravitational mass densities vanish. The conclusion is that these versions are of doubtful value in the prediction of cosmological observations.

1. *Introduction.*—In the steady-state model of the universe space-time is described by a four-dimensional Riemann space with metric

$$ds^2 = -e^{2kt}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + dt^2, \quad (1.1)$$

where k is a constant, r, θ, ϕ are polar coordinates and t is the time. The specification of this metric does not by itself make the model complete. Field equations are needed to connect the expansion rate with the average cosmic density, and to work out the dynamics of individual galaxies.

Attempts to provide a field theory have been made by Hoyle (1948, 1949, 1960). Some advocates of the model do not accept Hoyle's theory (Bondi 1952), and in detailed investigations of the model it has not been customary to use Hoyle's field equations (for example, Sciama 1955). In such work the tendency has been to use general relativity, or even Newtonian mechanics, and to suppose that these can be made to apply to the steady-state model.

The most promising attempt to connect general relativity and the steady-state theory was that of McCrea (1951), and it is with McCrea's work that this paper is chiefly concerned. McCrea's theory has recently been the subject of an interesting investigation by Davidson (1959) and several of the results of this paper are similar to his. The conclusions drawn from the work described here are, however, quite different from those of Davidson.

McCrea (1951) begins by noting that the field equations of general relativity, without cosmological term but with energy tensor T_j^i , viz.,

$$R_j^i - \frac{1}{2}\delta_j^i R = -8\pi T_j^i, \quad (1.2)$$

do not restrict the Riemann space to which they are applied, so it must be possible to use them to describe the space (1.1). To adopt this procedure amounts to using (1.2) to define an energy tensor of matter which would, according to relativity, produce the space-time (1.1). McCrea then shows that the matter concerned may be taken as having uniform positive density and negative pressure, and he concludes that the steady-state model may be described by the field equations of general relativity if the matter in it is assumed to have the necessary physical properties.

This procedure, at once simple and ingenious, seems unfortunately to fail on a crucial point. As will be shown, it so happens that for matter with the equation of state required by McCrea, viz.,

$$\dot{p} = -\rho$$

where p and ρ are the proper pressure and proper density in relativistic units, *the motion is indeterminate*. This weakens very considerably the predictions which can be made from the model, and makes it of doubtful value as an instrument of cosmological theory.

This result is established in Section 2 and Section 3, and is followed in Section 4 by a discussion of McCrea's model in another coordinate system. In Section 5 a Newtonian explanation of the indeterminacy is attempted, and the paper ends with Section 6 in which the effect of the results on the steady-state theory is discussed.

2. *Motion of matter in McCrea's model.*—We assume that cosmic space-time is described by (1.1) and use (1.2) to find the energy tensor of the matter which it contains. The result (McCrea, 1951) is

$$T_1^1 = T_2^2 = T_3^3 = T_4^4 = 3k^2/8\pi ; T_j^i = 0 (i \neq j). \quad (2.1)$$

Let us form the eigenvalue equation at any event P corresponding to this energy tensor, viz.

$$(T_j^i - X \delta_j^i) u^j = 0, (m = 1, 2, 3, 4), \quad (2.2)$$

where X are the eigenvalues of T_j^i and u^j are the eigenvectors; in this equation there is no summation over m . It is easily seen that the eigenvalues are all equal:

$$X^{(1)} = X^{(2)} = X^{(3)} = X^{(4)} = 3k^2/8\pi.$$

Moreover, every vector is an eigenvector for each of the eigenvalues. Now the time-like unit eigenvector of an energy tensor at P gives the unit four-velocity vector of the matter present at P (Lichnerowicz 1955; Synge 1956). Since this eigenvector is arbitrary for T_j^i given by (2.1), it follows that *the velocity of matter in McCrea's model is completely undetermined*.

The usual procedure in the analysis of an energy tensor at a point is next to identify $X^{(4)}$, the eigenvalue corresponding to the time-like eigenvector, with the proper density, and $-X^{(1)}$, $-X^{(2)}$, $-X^{(3)}$ with the principal pressures. In our case, this gives

$$\dot{p} = \dot{p} = \dot{p} = -3k^2/8\pi = \dot{p} ; \rho = 3k^2/8\pi.$$

The material of the model is then a perfect fluid with negative isotropic pressure \dot{p} and density ρ such that

$$\dot{p} + \rho = 0. \quad (2.3)$$

If we write the energy tensor for a perfect fluid in the usual form

$$T_j^i = g_{ja} (\dot{p} + \rho) \frac{dx^a}{ds} \frac{dx^i}{ds} - \delta_j^i \dot{p}, \quad (2.4)$$

we see that (2.3) allows the velocity vector dx^i/ds to be arbitrary, as already found.

One could of course assert arbitrarily that the matter in the model follows a geodesic, in which case one finds that its velocity is

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0, \quad \frac{dt}{ds} = 1, \quad (2.5)$$

and it is at rest in the coordinate system of (1.1). Reasons will be given in Sections 4 and 5 why this assumption would be unsatisfactory, but it may be noted here that one of the most powerful features of relativistic dynamics is that the field equations determine, in general, the motion of the matter present, and it would seem to be quite contrary to the spirit of the theory to assume here a motion which is not demanded by the field equations.

3. *Significance of the indeterminacy of the motion.*—To understand the significance of the statement that McCrea's model does not determine the velocity of the matter present, we compare it with those of ordinary relativistic cosmology. The metrics for these are

$$ds^2 = -e^{g(t)}(dr^2 + r^2 d\theta + r^2 \sin^2\theta d\phi^2)(1 + \frac{1}{4}kr^2)^{-2} + dt^2, \quad (3.1)$$

and to be definite we suppose that one particular model is chosen so that $g(t)$ and k are given. Then, provided $p + \rho \neq 0$, the field equations (1.2) together with the energy tensor (2.4) determine the velocity of the matter present to be

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0, \quad \frac{dt}{ds} = 1,$$

and p and ρ are given in terms of g and its derivatives. This shows that (3.1) refers to a coordinate system in which the particles in the model remain at rest. The model may now be used to make predictions about the red-shift, and other observable data.

If this procedure is followed through for McCrea's model, then, as shown in Section 2, the velocity is not determined, and it is not known how particles are moving relative to the coordinate system being used (1.1).

This relative motion is to a certain extent determinable by direct observation. For example, from the observed spherical symmetry one can assert that $d\theta/ds$ and $d\phi/ds$ are zero. However, in the red-shift formula (and other similar formulae) there will now appear the unknown function dr/ds , depending on r and t . The effect of this is drastically to weaken the model as a means of prediction, since by appropriate choice of this function, it can accommodate widely different observational possibilities. In fact, only a detailed investigation could show whether the model is capable of falsification at all.

4. *The static form of the model.*—The metric (1.1) is that of the de Sitter universe. It is known (Lemaître 1925) that if new variables \bar{r} , \bar{t} are introduced by

$$r = \frac{\bar{r}}{(1 - k^2\bar{r}^2)^{1/2}} e^{-k\bar{t}}, \quad t = \bar{t} + \frac{1}{2}k^{-1} \log(1 - k^2\bar{r}^2), \quad (4.1)$$

(1.1) is transformed into

$$ds^2 = -(1 - k^2\bar{r}^2)^{-1} d\bar{r}^2 - \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2) + (1 - k^2\bar{r}^2) d\bar{t}^2. \quad (4.2)$$

In this form, the space-time of the steady-state theory is static.

If the field equations (1.2) are applied to the metric (4.2) we find that T_j^i has the same values as in (2.1), and, as before, that the motion of the fluid is indeterminate.

We may, if we wish, assume that the matter is at rest in the coordinate system of (4.2), and take

$$\frac{d\bar{r}}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0, \quad \frac{d\bar{t}}{ds} = (1 - k^2\bar{r}^2)^{-1/2}, \quad (4.3)$$

and this might seem to be reasonable because the model (4.2) is static. However, it is easy to see that if we adopt (2.5) for the model (1.1), we may not take (4.3) for the model (4.2). The reason is that (2.5) are the components of a four-vector, and when this is transformed according to (4.1) it does not lead to (4.3). *Thus if we adopt McCrea's assumption (2.5), the matter in the static model (4.2) must be moving with respect to the (\bar{r}, \bar{t}) coordinate system.* In fact, of course, since (2.5) represents a geodesic, its transform must be a geodesic in (4.2). This conclusion, that the matter in the static model (4.2) must be moving, is another argument against the gratuitous assumption that the matter follows a geodesic.

Let us now turn to another curious circumstance, already noted by Davidson (1959). Consider the metric

$$ds^2 = - \left(1 - \frac{2m}{r} - k^2 r^2 \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \left(1 - \frac{2m}{r} - k^2 r^2 \right) dt^2; \quad (4.4)$$

this represents a universe similar to that described by (1.1) or (4.2), but with a mass m at the origin. Now, using (4.4) to evaluate (1.2) one finds again the same values for T_j^i as in (2.1), and the motion is arbitrary even in the presence of the mass. *It seems that the matter with equation of state (2.3) is indifferent to a gravitational field.* We shall return to this point in the next section.

It should be emphasized that these strange conclusions apply to the model which contains this type of matter, and not to the de Sitter model. In the latter the field equations (1.2) are altered by the addition of the cosmological constant, and the energy tensor, the pressure and the density are all zero. A test particle, if introduced, follows a geodesic and its motion is in no way indeterminate.

5. *The physical interpretation.*—To interpret physically the behaviour of matter in the model we start from the clue given towards the end of the last section, that matter satisfying (2.3) seems not to respond to gravitation. Let us adopt a Newtonian point of view, and consider the three ways in which mass density may appear in Newtonian mechanics. We define

- σ_1 = inertial mass density ;
- σ_2 = active gravitational mass density ;
- σ_3 = passive gravitational mass density ;

here active and passive gravitational mass refer to the power of a body to exert gravitation and to respond to a gravitational field in which it is placed.

Of the three densities, only σ_2 has an analogue in general relativity which is known with any assurance. In a static field it appears from the work of Whittaker (1935) that

$$\sigma_2 = T_4^4 - T_1^1 - T_2^2 - T_3^3,$$

and if matter is present in the form of a perfect fluid at rest in the coordinate system, we may write, with McCrea,

$$\sigma_2 = \rho + 3p \quad (5.1)$$

(relativistic units).

To obtain a corresponding expression for σ_1 , we note that for a perfect fluid without pressure

$$T_j^i = \rho g_{ja} \frac{dx^i}{ds} \frac{dx^a}{ds}, \quad (5.2)$$

and here ρ is clearly to be interpreted as the inertial mass density. Comparison of (2.4) and (5.2) suggests for a perfect fluid the tentative identification

$$\sigma_1 = p + \rho. \quad (5.3)$$

Concerning σ_3 very little is known. For a structureless point-particle it is a basic assumption of general relativity that the passive gravitational mass is equal to the inertial mass, but for extended material in the presence of stresses the equality by no means clearly follows. Nonetheless, I shall assume equality and write

$$\sigma_3 = p + \rho. \quad (5.4)$$

Evidence for (5.4) in the case of a perfect fluid at rest exists in the equation of hydrostatic equilibrium (Tolman 1934), which for the interior of a spherically symmetric body is

$$\frac{dp}{dr} + \frac{1}{2}(\rho + p) \frac{d}{dr} \log g_{44} = 0,$$

r being the radial coordinate. Taking $\frac{1}{2} \log g_{44}$ as the analogue of the Newtonian gravitational potential, we see that $p + \rho$ is playing the part of the passive gravitational mass density.

In this connection we may note the following interesting property of matter obeying the equation of state $p + \rho = 0$. Inserting this equation into the energy tensor (2.4) and forming the identities

$$T^a_{i;a} = 0$$

we find, with no assumption whatever about the g_{ij} except that they exist,

$$\frac{\partial p}{\partial x^i} = 0$$

so that the pressure (and density) must be constant throughout space-time. The impossibility of a pressure gradient is readily understood in the light of the preceding argument, since if the inertial mass density σ_1 vanishes (as it will from (2.3) and (5.3)) a pressure gradient would produce an infinite acceleration.

Let us now tentatively adopt (5.3) and (5.4). Then it follows from (2.3) that in *McCrea's model of the steady-state universe both the inertial and the passive gravitational mass densities are zero*. For the Newtonian equation of motion of a small volume dv of matter in a gravitational field \mathbf{g} , we have

$$\sigma_1 dv \mathbf{f} = \sigma_3 dv \mathbf{g} \quad (5.5)$$

\mathbf{f} being the acceleration. Hence, if the densities σ_1 and σ_3 both vanish, \mathbf{f} is indeterminate. We see that the indeterminacy of the motion has a quite natural interpretation on this extended Newtonian view.

If this reasoning is correct, it follows that the Newtonian treatment of the steady-state model (McCrea 1951) is also doubtful. In forming the Newtonian equation of motion for the material substratum of the model, McCrea quite

naturally cancelled out what has here been called σ_1 and σ_3 . But if these are zero this is not permissible. In fact *it seems that the motion of matter is also indeterminate in the Newtonian model.*

In general relativity the postulate that a test particle moves on a geodesic is equivalent to the Newtonian equation of motion in a gravitational field. To remove the indeterminacy in the relativistic model by postulating that the cosmic matter moves on a geodesic in (1.1) or (4.2) would amount, in the Newtonian analogue, to proscribing a particular \mathbf{f} in (5.5). Since (5.5) evidently determines nothing if σ_1 and σ_3 are zero, I conclude once more that there is no justification for the postulate that matter moves on a geodesic.

6. *Conclusion.*—I have argued in this paper that neither the Newtonian nor the relativistic versions of the steady-state theory, at least in their current forms, determine the motion of matter in the universe. Moreover, this lack of determination weakens these versions to such an extent that they seem to be of little value as instruments of prediction in cosmology.

It is just possible that the situation might be retrieved along the following lines. The indeterminacy in the motion arises only if

$$p + \rho = 0 \quad (6.1)$$

is *exactly* satisfied. Could one therefore overcome the difficulty by assuming that (6.1) is only approximately realized in the actual universe? The answer is that the exact fulfilment of the equation is necessary if the universe is to be in a steady state, for the slightest departure from it will mean a secular change in the nature of the universe, which the authors of the theory are at pains to avoid. However, it might be possible to devise a model in which, although (6.1) is not satisfied exactly in any large region of space-time, it *is* satisfied exactly over the model as a whole. Such a model would have different rates of expansion in different regions, and would be essentially inhomogeneous. It would probably be very complicated mathematically, and it might not accord with observation; but the possibility of a model of this sort cannot be ruled out.

The criticisms of the relativistic and Newtonian models given in the preceding sections do not, of course, apply to the steady-state theory itself. Since the latter does not accept general relativity or Newtonian mechanics on a cosmological scale, it is unaffected by defects in models constructed from these theories. What now seems clear, however, if my arguments are correct, is that those who wish to work with the steady-state theory must use a dynamics specifically designed for it, since they are not free to use the existing models of general relativity or Newtonian theory with any degree of confidence.

To illustrate this point I may mention the theory of galaxy formation in the steady-state model. Interesting work with powerful conclusions has been done on this by Sciama (1955). However, Sciama uses Newtonian theory in a way which seems unjustified in the light of this paper. In particular the use of Newtonian mechanics to explain the condensation of newly-created cosmic matter in the presence of existing galaxies is not permissible if the inertial and passive gravitational masses of the created matter are zero.

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References

- Bondi, H., 1952, *Cosmology*, Cambridge University Press.
Davidson, W., 1959, *M.N.*, **119**, 309.
Hoyle, F., 1948, *M.N.*, **108**, 372; 1949, *M.N.*, **109**, 365; 1960, *M.N.*, **120**, 256.
Lemaître, G., 1925, *J. Math. and Phys. (M.I.T.)*, **4**, 188.
Lichnerowicz, A., 1955, *Théories Relativistes de la Gravitation et de L'Électromagnétisme*, Masson et Cie, Paris, p. 13.
McCrea, W. H., 1951, *Proc. Roy. Soc. A*, **206**, 562.
Sciama, D. W., 1955, *M.N.*, **115**, 3.
Synge, J. L., 1956, *Relativity: The Special Theory*, North Holland Publishing Company, Amsterdam, p. 295.
Tolman, R. C., 1934, *Relativity, Thermodynamics and Cosmology*, Oxford, p. 243.
Whittaker, E. T., 1935, *Proc. Roy. Soc. A.*, **149**, 384.