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HEATING OF STELLAR CHROMOSPHERES BY SHOCK WAVES

RAY WEYMANN*
Princeton University Observatory
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ABSTRACT

On the basis of a very simple model for the dissipation of energy by shock waves, the structure of the chromospheres of stars with very different values of the surface gravity is discussed. Sharp jumps in the temperature at certain critical points would be expected. Reasonable chromospheric structures follow from this model for dwarfs, but not for stars of very low surface gravity unless very long-period shock waves are considered. The question of the thermal stability of the solutions is discussed.

I. INTRODUCTION

The fact that nearly all late-type stars show a central emission feature in the H and K absorption lines has generally been accepted as an indication that most of these stars possess a chromosphere-corona. In addition, it has been suggested that stellar coronas may play a role in causing mass ejection in late-type giants. It seems of interest, therefore, to consider the problem of heating a chromosphere of a star with a surface gravity very much lower than that of the sun and to compare it with that of heating a chromosphere of a star with a surface gravity equal to that of the sun. The question of the nature of the chromosphere-corona of giant stars has already been considered in a qualitative way by de Jager (1958).

Although several mechanisms have been proposed for the heating of the solar chromosphere (Biermann 1948; Schwarzschild 1948; Thomas 1948; Piddington 1956), it is not yet clear which of them is the dominant mechanism or whether a combination of them is required. In the following it will be assumed that the heating is due to a stream of shock fronts. The physical process envisaged is one in which the shocks are weak or moderate in strength, with the time between successive passages of shock fronts being fairly small. An idealized system is considered as described by the following assumptions: (*a*) the shock fronts are assumed to be plane and to be propagated normally through a plane atmosphere; (*b*) the pressure, temperature, and velocity of a given mass element are supposed to be strictly periodic functions of the time, with period W, and with one shock front per period passing through every mass element.

A slightly simplified version of the Brinkley-Kirkwood theory (1947) on the dissipation of shocks was used by Schatzman (1949) in his study of the solar chromosphere, and the full theory was applied by Odgers and Kushwaha (1959) in a study of the decay of a single shock running through the atmosphere of a star of early spectral type. The Brinkley-Kirkwood procedure enables one to deal with ordinary differential equations rather than the exact time-dependent set. In the present case of a strictly periodic motion, where the state of the gas immediately ahead of the shock front is not specified, the full Brinkley-Kirkwood theory is not applicable without the introduction of additional assumptions. Instead, therefore, a slightly different approach is used in replacing the exact partial differential equations by approximate time-independent equations. The procedure adopted was to average the exact equations over a period and to make assumptions about the time dependence of the various quantities where necessary.

* Present address: California Institute of Technology, Pasadena, California.

II. DERIVATION OF EQUATIONS

The basic equations outside the shock fronts are, in Lagrangian co-ordinates,

$$\tau_0 \frac{\partial u}{\partial \xi} = \frac{\partial \tau}{\partial t}, \tag{1}$$

$$\frac{\partial u}{\partial t} + \tau_0 \frac{\partial p}{\partial \xi} + g = 0, \tag{2}$$

$$\gamma p \frac{\partial \tau}{\partial t} + \tau \frac{\partial p}{\partial t} + (\gamma - 1) F = 0, \tag{3}$$

where t is the time, ξ is the Lagrangian co-ordinate, τ is the specific volume, p is the pressure, u is the velocity, g is the (constant) surface gravity, F is the net rate of energy loss by radiation per unit mass, and γ is the effective ratio of specific heats; τ_0 is some arbitrary reference specific volume. If this is identified with the time average of the specific volume, then the Lagrangian co-ordinate may be identified with the time average of the Eulerian co-ordinate. Across the shock fronts the Rankine-Huginot relations are

$$u_2 - u_1 = m(\tau_1 - \tau_2), \tag{4}$$

$$m^2(\tau_2 - \tau_1) = p_1 - p_2, \tag{5}$$

$$E_2 - E_1 = \frac{(\tau_1 - \tau_2)(p_1 + p_2)}{2}, \tag{6}$$

where m is the rate of increase of the mass of the gas behind a shock front and E is the specific internal energy of the gas. Taking time averages of equations (1), (2), and (3) and denoting time averages of the various quantities by zero subscripts, one obtains, from equation (1),

$$\frac{du_0}{d\xi} = 0. \tag{7}$$

In order for a strictly periodic solution in these co-ordinates to have meaning, there must be no significant net flow of matter, so that u_0 is set equal to zero. From equation (2) one then obtains

$$\tau_0 \frac{dp_0}{d\xi} + g = 0. \tag{8}$$

Integrating equation (3) over a complete period and rearranging terms, one finds

$$(\gamma - 1) \int p \frac{\partial \tau}{\partial t} dt + \int \frac{\partial}{\partial t}(\tau p) dt + (\gamma - 1) \int F dt = 0,$$

or, since the variables assume the same values at both limits of integration,

$$\int p \frac{\partial \tau}{\partial t} dt = \int (p - p_0) \frac{\partial \tau}{\partial t} dt = -WF_0,$$

where F_0 is the average radiation loss. Hence, by virtue of equations (1), (2), and (8),

$$\tau_0 \int \frac{\partial}{\partial \xi}[(p - p_0) u] dt + \int u \frac{\partial u}{\partial t} dt = -WF_0,$$

which, because of the periodicity, reduces to

$$\frac{d}{d\xi}\left[\frac{1}{W}\int_{t}^{t+W}(p-p_0)\,u\,dt\right]+\frac{F_0}{\tau_0}=0\,. \tag{9}$$

Equation (9) expresses the fact that the net radiation loss is just balanced by the rate of decay of mechanical flux.

Define a shock-strength parameter, σ, by

$$\sigma=\frac{p_2-p_1}{p_0}\,. \tag{10}$$

The specific volume, pressure, and velocity are now assumed to have linear ("sawtooth") time profiles, but this assumption, in conjunction with the Rankine-Huginot relations, will be used only for the purposes of evaluating the integral in equation (9) and for fixing the shock transition solely in terms of σ, p_0, and τ_0. The jump in the velocity and the specific volume are then given in terms of these three quantities by

$$u_2-u_1=\sigma\left(\frac{p_0\tau_0}{\gamma}\right)^{1/2}, \tag{11}$$

$$\tau_1-\tau_2=\frac{\tau_0\sigma}{\gamma}, \tag{12}$$

and equation (9) now becomes

$$\frac{d}{d\xi}\left[\frac{\sigma^2}{12}\left(\frac{p_0^3\tau_0}{\gamma}\right)^{1/2}\right]+\frac{F_0}{\tau_0}=0\,. \tag{13}$$

The energy increment received by an element of matter after the shock front has passed over it may now be expressed in terms of σ, p_0, and τ_0.

Not all this energy, however, will be available to the gas for radiation, since the gas will expand to its original volume and in the process may do work on the surrounding elements. In order to estimate the fraction of the energy increment available for radiation, an assumption must be made regarding the path of a gas element in the p-τ plane between successive shock fronts. The linear time-profile assumption used in the preceding paragraph is not satisfactory for this purpose, since it would imply that all the energy increment received by the gas element is used up in subsequent work as the element expands. Therefore, an assumption on the path in the p-τ plane entirely separate from that of the previous paragraph is made, in order to make this estimate. Since the density and temperature are likely to be highest immediately behind the shock front, most of the radiation will take place just after passage of the shock front. This suggests, then, that the fraction of the energy increment which goes into radiation be calculated as if the gas element radiated energy at constant volume until it reached its original entropy and thereupon expanded adiabatically to the pressure and specific volume it had before the arrival of the shock front. This assumption, together with the relationships between the quantities at the shock front deduced in the previous paragraph, leads to the following expression for the energy available for radiation per gram of matter per shock passage, Q,

$$Q=\left[\frac{p_0\tau_0}{(\gamma-1)}\right]\left\{\left(1-\frac{\sigma}{2\gamma}\right)\left[\left(1+\frac{\sigma}{2}\right)-\left(1-\frac{\sigma}{2}\right)\left(1+\frac{\sigma}{2\gamma}\right)^{\gamma}\left(1-\frac{\sigma}{2\gamma}\right)^{-\gamma}\right]\right\}. \tag{14}$$

The full expression used by Schatzman corresponding to equation (14) amounts to a somewhat different assumption about the path of the element in the p-τ plane, namely, that the gas expands along its new adiabat to its original pressure and then radiates at constant pressure as it contracts to its original volume. In fact, however, these two assumptions lead to equivalent results if the expressions are expanded in powers of σ and only the lowest-order term is kept. Equation (14) then reduces to

$$Q = p_0 \tau_0 (\gamma + 1) \frac{\sigma^3}{12\gamma^2}, \tag{15}$$

and this expression will be used subsequently instead of equation (14). It should be pointed out that the power of σ appearing in equation (15), as well as the numerical coefficient, is a property of the specific assumption made about the path in the p-τ plane. For example, if one assumed that the p-τ cycle could be represented by cooling at constant volume, followed by an isothermal expansion, then in equation (15) Q would be proportional to σ^2, and the subsequent equations would take a slightly different form; for instance, in place of the characteristic number $\frac{5}{4}$ which occurs in the adopted equations, one obtains, instead, $\frac{3}{2}$. Equation (15) will break down unless the shocks are of weak or moderate strength. (In fact, if the linear-profile assumption were taken literally, no physical meaning could be attached to values of σ greater than 2.) Having in mind, then, application of this model only to moderate or weak shock strengths and to situations where the energy radiated between shock passages is not large compared to the mean thermal energy of the gas, it will be supposed legitimate to replace the mean value of the radiation loss by the radiation loss at the mean values of the pressures and specific volume and also to replace $1/\tau_0$ by ρ_0, where ρ is the density. (In the remainder of this paper the zero subscripts will be dropped from the time-averaged quantities.)

Since the deviations from local thermodynamic equilibrium in a chromosphere are presumably severe, the evaluation of F, in principle, demands the simultaneous solution of the radiation-transfer equations, the equations of statistical equilibrium, and the shock equations. It will simply be assumed here, however, that F may be represented by

$$F = \rho \Phi (T), \tag{16}$$

where T is the kinetic temperature and Φ is, for the moment, an unspecified function. The last equation needed in order to make the system determinate is obtained from the condition of thermal equilibrium, $Q = WF$, or, using equations (15) and (16),

$$\frac{p \sigma^3 (\gamma + 1)}{12\gamma^2 W \rho} = \rho \Phi (T). \tag{17}$$

Equations (8), (13), (16), and (17) are the basic equations for this simple shock model. Using the equation of state to eliminite the density and equations (16) and (17) to eliminate F and σ and dividing equation (13) by equation (8), one obtains, after some rearranging, the following equation between the mean temperature and pressure:

$$\frac{d \log T}{d \log p} = \frac{K (\Phi^2 p^2 T^{-1})^{1/6} - 2}{(4 d \log \Phi / 5 d \log T) - 1}, \tag{18}$$

where

$$K = \left[\frac{6}{5} (12)^{1/3} \left(\frac{\mu m_H}{k}\right)^{1/6} (\delta^{\gamma} + 1)^{2/3} \gamma^{-5/6}\right] W^{-2/3} g^{-1}. \tag{19}$$

III. DISCUSSION OF SOLUTIONS

Given the period and initial temperature and pressure, equation (18) may be integrated numerically if the form of Φ is specified; and analytic solutions may be obtained over regions where it is adequate to assume that $\Phi = A T^{a}$, A and a being constant. However, two properties of this model may be discussed simply from the form of equations (18) and (19). First, no run of T versus p without a discontinuity in the temperature is possible over temperature regions where $d \log \Phi / d \log T$ varies from values greater than $\frac{5}{4}$ to values less than $\frac{5}{4}$ with increasing temperature, and the same is true, except for a single special solution, in the case where $d \log \Phi / d \log T$ varies from values less than $\frac{5}{4}$ to values greater than $\frac{5}{4}$ with increasing temperature. (In actuality, of course, the "discontinuity" would be smoothed by thermal conduction.) Since any chromosphere-corona of any consequence (i.e., one hot enough to cause substantial ionization of the hydrogen) will almost certainly contain a temperature region in which $d \log \Phi / d \log T = \frac{5}{4}$, one would conclude from this model that it would be characterized by one or more very steep rises in the temperature.

Second, if one seeks runs of temperature and pressure which are similar in stars with widely different values of g, then, to the extent that the shock model is applicable to both, equation (19) implies that the characteristic period of the waves must be proportional to $g^{-3/2}$.

The form of the function $\Phi(T)$ in the case of the solar chromosphere has been discussed by Athay and Thomas (1956) and exhibits the following qualitative features. In the cooler and denser lower layers, where the hydrogen is neutral, the emission arises chiefly from H^-. With increasing temperature and hydrogen ionization, the emission from hydrogen itself takes over and rises sharply by several orders of magnitude. When the hydrogen is fully ionized, the increase ceases, and the emission drops slightly until a further increase is brought about by the onset of helium ionization. Subsequent local increases in the emission at higher temperatures might be expected from collisional excitation of low-lying permitted levels of the multiple ionized metals, especially carbon, nitrogen, oxygen, neon, and magnesium, as well as from the forbidden coronal lines. These features would be expected to be present in all late-type stars, although the details would depend on the details of the chromospheric structure itself, as well as on the photospheric radiation field. For example, one might expect that, other things being equal, hydrogen ionization in a star with an effective temperature of 3000° would not occur until a somewhat higher temperature than the 6500° indicated for the solar case. Furthermore, in a giant, due to the greater distances involved, the local opacity would be higher than in a dwarf with the same run of p versus T, so that the mean intensity would tend to build up to larger values, with a decrease in the net radiation loss over that expected in a dwarf.

For purposes of numerical illustration, consider two stars, one with $g = g_{\odot}$ and the other with $g = 10^{-3} g_{\odot}$, and consider the run of temperature against pressure over the lower portion of the chromosphere where the kinetic temperature rises from photospheric values up to values where the hydrogen becomes ionized. Suppose that, at the point where shock dissipation commences, one has $p = 2.6 \times 10^3$ dynes/cm² and $T = 4000°$ in both stars. If Φ is represented over this range by $\Phi = AT^{\alpha}$ with $\alpha = 14.5$ and $A = 10^{-34}$, which reproduces the sharp rise in emissivity as H^- emission gives way to hydrogen emission, then, with these numbers, the value of Φ at 10000° is slightly less than that corresponding to the sum of all free-bound emission, while the value at 4000° is slightly greater than that computed for H^- on the basis of local thermodynamic equilibrium.

With this choice of Φ, equation (18) may be integrated to give

$$T = X^{-1/14} \left[C X^{29/53} + \tfrac{70}{87} \right]^{-3/14}, \tag{20}$$

where $X = K^3 A p$ and C is the constant of integration fixed by initial conditions. In these computations γ and μ have been set equal to the constant values 1 and $\frac{2}{3}$, respectively, appropriate values for regions where the hydrogen is becoming ionized. Figure 1 shows typical solutions for three choices of the characteristic period for the case $g = g_{\odot}$: 30, 100, and 300 seconds A, B, C, respectively, and the intermediate value of these periods for the case $g = 10^{-3} g_\odot$ (D). For curves A, B, and C. the ratio of energy radiated per shock passage to the mean energy of the gas is of order 1 or smaller, except for the very coolest portions of solutions B and C, while the shock strengths are nearly constant, varying as $T^{-1/2}$ for B, and at 10000° they have strengths of about 0.6, 1.2, and 2.6, respectively. Thus for B and especially for C, the validity of the weak to moderate shock-strength assumption is dubious. The acoustic flux for curve B varies as $T^{-14.5}$, being equal to about 5×10^8 ergs/cm^2 at 4000° and 6×10^2 ergs/cm^2 at 10000°. While A, B, and C represent reasonable chromospheric structures, the same is not true of curve D. The density at which 10000° is reached in the "chromosphere" represented by D is more typical of that of the interstellar medium, and, although D should be considered only a formal solution, since the physical basis underlying this solution breaks down long before 10000° is reached, it seems clear that a reasonable solution cannot be obtained from this simple shock model with such short periods.

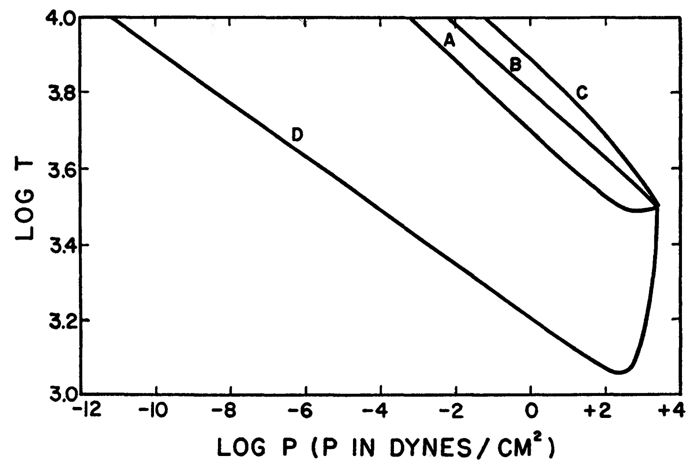

FIG. 1.—Run of the log of the temperature against the log of the pressure for lower portions of chromospheres. Curves A, B, and C represent solutions for $g = g_\odot$ and periods of 30, 100, and 300 seconds, respectively. Curve D represents the (formal) solution for $g = 10^{-3}\, g_\odot$ and a period of 100 seconds.

Physically, the very low value of g would seem to affect the solution in two ways. First, the much gentler pressure gradient in the giant star will inhibit the rate at which the shock strength grows, which, in turn, affects the manner in which the energy is deposited. Second, for identical runs of temperature against pressure in the giant and dwarf, the amount of matter contained between two fixed temperatures will be far greater in the giant than in the dwarf, thus requiring substantially more total energy to balance the radiation losses. One could reproduce the runs of T against p as given in the $g_\odot$ case if, in the giant star, very much longer characteristic periods were considered. To reproduce curve C, for example, would require a characteristic period of a little under 4 months. In view of the fact that the critical period for small-amplitude waves traveling in an isothermal atmosphere of 4000° in the presence of a gravitational field of $10^{-3} g_\odot$ is only about 2 days, it seems doubtful whether such long characteristic periods can be present. Even if one were to consider the possibility of periods as long as this, the resulting shock strengths would be so large that one could not hope to describe them with this model.

The fate of solutions A, B, and C at higher temperatures cannot be discussed without further specification of Φ. Even the conditions governing the temperature jump when $d \log \Phi / d \log T$ crosses $\frac{5}{4}$ are not clear, since it may be that some of the acoustic energy is reflected at the discontinuity. Denoting the acoustic flux by S and letting the subscript 1 stand for the low-temperature side and 2 for the high-temperature side, without further analysis it can only be said that

$$S_2 \leq S_1 \tag{21}$$

with

$$p_2 = p_1 \, . \tag{22}$$

With S given by the bracketed term in equation (13) and making use of equations (21), (22), and (17), plus the equation of state, it can be concluded that the temperature jump should satisfy the inequality

$$\log T_2 - \log T_1 \geq \tfrac{4}{5} (\log \Phi_2 - \log \Phi_1) \, . \tag{23}$$

IV. THERMAL STABILITY OF SOLUTIONS

Athay and Thomas have called attention to the step-like behavior of the temperature in their model of the solar chromosphere, with a region of slowly increasing temperature followed by an abrupt rise to a higher-temperature "planteau." Following the work of Parker (1953), they have interpreted this structure as a manifestation of the thermal instability of certain temperature regions. The model used here implies that temperature discontinuities would arise aside from the question of the stability of these equilibrium solutions. It seems worthwhile, therefore, to ask whether considerations of thermal stability would place further restrictions on solutions obtained from this model.

Strictly speaking, one ought perhaps to consider the stability criterion of Athay and Thomas and of Parker, as applying to the case of incompressible fluids, since arbitrary disturbances will affect the density as well as the temperature of the medium, which will also alter the ability of the gas to radiate. One might expect a tendency toward pressure equilibrium to cause a temperature increase to be accompanied by a density decrease, and, since the rate of radiation per gram is proportional to the density, this effect might cause an otherwise stable situation to become unstable.

As an illustration of this effect, consider exactly the same problem treated by Parker but do not restrict the perturbations to the temperature. Let a uniform, inviscous, but thermally conducting gas be at rest and in thermal equilibrium at some temperature T_0 and some density ρ_0. Suppose, with Parker, that the rate of energy input per gram is fixed (as in, for example, a radioactive source) and again suppose that the gas loses energy by radiation at a rate given by $\rho^2\Phi(T)$ ergs/cm^3/sec. The equations governing small disturbances from this equilibrium state may be written

$$\rho_0 \frac{\partial u}{\partial t} = -\nabla(\delta p) \, , \tag{24}$$

$$\frac{\partial}{\partial t}(\delta \rho) = -\rho_0 \nabla u \, , \tag{25}$$

$$\frac{3}{2}\frac{\partial(\delta p)}{\partial t} = \frac{5}{2}\frac{p_0}{\rho_0}\frac{\partial(\delta p)}{\partial t} - \rho_0^2\left(\frac{d\Phi}{dT}\right)_0 (\delta T) - \rho_0\Phi_0(\delta \rho) + \nu_0(T)\nabla^2(\delta T) \, , \tag{26}$$

where δ denotes the departure from the equilibrium state, $\nu(T)$ is the coefficient of thermal conduction, and the other symbols have the same meanings as before. Letting $\delta(\ )$ denote any one of the perturbed dependent variables, one may consider an arbitrary disturbance to be a superposition of disturbances of the form

$$\delta(\ ) = C \exp(i\kappa x + \omega t) \, . \tag{27}$$

Introducing the dimensionless quantities $\beta = \rho_0\Phi_0/c_0^3|\kappa|$, $\theta = \Phi_0\nu_0 T_0/c_0^6$, $\alpha = (d \log \Phi/d \log T)_0$, and $n = \omega/|\kappa|c_0$, with c_0 the sound velocity, and carrying out the reduction of equations (24) through (26) in the usual way, one obtains the following characteristic equation for the dimensionless frequency, n:

$$n^3 + \tfrac{10}{9}\beta\left(\alpha + \frac{\theta}{\beta^2}\right)n^2 + n - \tfrac{2}{3}\beta\left[1 - \left(\alpha + \frac{\theta}{\beta^2}\right)\right] = 0. \quad (28)$$

From this equation it may be deduced that, for all values of α greater than 1, all disturbances are stable, while, if α is less than 1, there always exists some unstable mode. To see this, it may be observed that, if $\alpha > 1$, then all the coefficients in equation (28) are positive, so that all real roots must be negative. Furthermore, all complex roots must have negative real parts; for, letting

$$n = \eta + i\psi, \quad (29)$$

where η and ψ are real, equation (28) implies that η must satisfy

$$\eta^3 + \tfrac{10}{9}\beta(\alpha + \theta/\beta^2)\eta^2 + \tfrac{1}{4}\{[\tfrac{10}{9}\beta(\alpha + \theta/\beta^2)]^2 + 1\}\eta + \tfrac{1}{8}[\tfrac{10}{9}\beta(\alpha + \theta/\beta^2) - \tfrac{2}{3}\beta(\alpha + \theta/\beta^2 - 1)] = 0, \quad (30)$$

and again, since the coefficients are positive, the real roots of this equation must be negative. On the other hand, if α is less than 1, a wave length such that $\beta = [\theta/(1-\alpha)]^{1/2}$ results in neutral equilibrium, while a disturbance of slightly greater wave length than this will result in instability. The stability criterion $\alpha > 1$ should be compared to that found by Parker, $\alpha > 0$, in the case where only temperature perturbations are considered.

Unfortunately, it is not possible to use such a straightforward procedure in discussing the stability of a more complicated case such as the shock model used here. A plausible criterion may be obtained by considering a small element of matter subject to perturbations which leave the pressure unchanged and which also leave the incident mechanical flux, S, unchanged. Again using equations (17) and (13) and the equation of state, one finds that a small temperature increase leads to a heating of the element, and hence to instability if $d \log \Phi/d \log T$ is less than $\frac{5}{4}$, so that only those regions of temperature where $d \log \Phi/d \log T$ is steeper than $\frac{5}{4}$ would be thermally stable.

V. SUMMARY

The chief conclusions reached in this paper are as follows:

a) Abrupt rises in the temperature of the chromosphere are indicated whenever the logarithmic derivative of the temperature-dependent portion of the radiation-loss expression equals a number of order 1, which in the present model is $\frac{5}{4}$.

b) Thermal instability of the solutions in temperature regions where this derivative is less than $\frac{5}{4}$ is indicated.

c) Reasonable chromospheric structures may be obtained from this model for stars with surface gravity equal to that of the sun, but not for giants with surface gravity a thousand times lower than the solar value, unless very long-period disturbances are considered, in which case the shock strengths turn out to be so large that this model no longer applies.

In evaluating the relevance of these conclusions to the real physical situations, the following assumptions and idealizations which underlie them should be kept in mind:

1. The mechanism responsible for the chromospheric heating has been taken to be a stream of shock waves, instead of other alternatives which have been proposed, such as

jetlike mass motions or magnetohydrodynamic waves. Furthermore, no account was taken of heat flows inward via radiation or thermal conduction from an outer corona, and a much simplified treatment of the net radiation loss was used.

2. Only waves of a single frequency and a single amplitude were considered, whereas one ought to consider the heating effects due to a distribution in both frequency and amplitude of the shock fronts, as well as their interactions.

3. The shock fronts were considered to be plane and to be propagating normally in a plane atmosphere; thus the effects of refraction are neglected, as are inhomogeneities in the structure (which seem firmly indicated in the case of the sun).

4. Assumptions have been made about the time variation of the various quantities between shock passages, in order to convert the system of partial differential equations into a single ordinary differential equation, and the supposition was made that the shocks were not very strong

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