

BALMER DECREMENTS: THE DIFFUSE NEBULAE

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ABSTRACT

The possibility that diffuse nebulae may have a total optical depth in the early Balmer lines of the order of unity is considered. The Balmer decrement is found as a function of the total optical depth of a plane atmosphere in $H\alpha$, assuming that the atmosphere has a constant temperature and density (less than 10^6 electrons/cm³) and is optically thick to Lyman-line radiation. It is found that the $H\alpha/H\beta$ ratio increases with increasing optical depth in $H\alpha$. The observations of $H\alpha/H\beta$ in diffuse nebulae are high enough to make any other interpretation beside that of optical thickness in $H\alpha$ appear doubtful. The measurements of $H\gamma$ and $H\delta$ tend to confirm this interpretation.

I. INTRODUCTION

The present state of knowledge concerning line formation does not permit computing the Balmer decrement for an arbitrary atmosphere. We must specify simple conditions under which the Balmer decrement is to be derived and then justify these conditions for a particular astronomical object. All too often, however, this justification involves only a comparison of the observations with the theory, a comparison rendered difficult by unknown corrections for interstellar reddening and uncertain observations.

In particular, the theory of Cillié (1932, 1936) and Baker and Menzel (1938) (and extended somewhat by Chamberlain 1953) has been used to compare observations of the Balmer decrement in gaseous nebulae, shell stars, and other stellar atmospheres. It is the purpose of this paper to investigate the assumptions used by these authors and to show (1) that, because of the neglect of collisional processes originating from the second level, their results are not applicable to atmospheres whose electron densities are greater than 10^7 electrons/cm³ and (2) that, because of the neglect of radiative excitations from the second level of hydrogen, the theory is of doubtful application to diffuse nebulae.

The assumption used throughout this paper is that the part of the atmosphere from which Balmer-line radiation originates is optically very deep in Lyman-line radiation. This will be true if diffuse nebulae are bounded by the ionizing radiation available from within, in which case the total optical thickness in Lyman continuum radiation is about 10, and consequently the optical thickness in Lyman- α is about¹ 10^5 . This same value is obtained by calculating the ground-state population, using parameters typical of a diffuse nebula, thus confirming the assumption.

One of the earliest attempts to describe theoretically the Balmer decrement was made by Cillié (1932, 1936), who solved, first, a seven-level and then a fourteen-level hydrogen atom under the following conditions: (1) detailed balance in the Lyman lines, (2) no collisional processes involved, and (3) radiative transitions upward occur only from the ground state and not from the excited levels. This work was improved by Baker and Menzel (1938), who solved an arbitrary n -level atom under the same conditions. The

¹ Strömgren (1939, 1948) showed that ionization-bounded regions have a total optical thickness in the Lyman continuum of about unity. The more recent work of Jefferies and Pottasch (1959), taking into account the scattered radiation and the frequency dependence of the absorption coefficient beyond the Lyman series limit, increases this value to about 10. The ratio of the optical depth at the Lyman continuum series limit to that at the center of Lyman- α depends only on the ratio of the respective absorption coefficients, which is a slowly varying function of the kinetic temperature (assumed of the order of 10000° K).

Balmer decrement obtained by Baker and Menzel is sometimes called the "capture decrement" and is sometimes called "Baker and Menzel Case B." We shall discuss this approach more fully in the second section. Their work was particularly aimed at discussing the Balmer decrement in planetary nebulae, but references in the current literature apply it to stellar atmospheres, in particular Be shell stars as well.²

Thomas (1949*a, b*, 1950, 1952), also Matsushima (1952), in a series of papers aimed specifically at interpreting the Balmer decrement in the solar chromosphere began by abandoning conditions 2 and 3. They first included collisions upward from the ground state, then collisions in general, and showed that in the solar atmosphere appreciable self-absorption is evidenced by the Balmer lines to quite high-energy levels. In both the sun and planetary nebulae, Thomas showed that condition 3, when applied to an atmosphere thick in Lyman-line radiation, is both wrong and unnecessary.

Miyamoto (1953) limited himself to a five-level atom, ignored collisions and the Paschen continuum radiation field, but considered the simultaneous transfer of H α and H β in an atmosphere of constant electron density and electron temperature. His effort was directed specifically at interpreting Be-star Balmer decrements.

Chamberlain (1953) solved the same problem as Baker and Menzel with one exception: collisional processes upward from the ground level are included. All comments in the remainder of this paper which refer to the conditions used by Cillié and Baker and Menzel also refer to Chamberlain's work.

All the previous work has not taken any account of the orbital angular momentum degeneracy of the hydrogen atom but has assumed that the population of a level n was divided among the various sublevels in proportion to the statistical weight. Recently Burgess (1958) and Searle (1958) have solved the same problem as Cillié and Baker and Menzel but have taken this degeneracy into account. Their results differ only slightly from those of Cillié and Baker and Menzel. Burgess has also pointed out that assumptions 2 and 3 cannot both be correct, since no mechanism for depopulating the 2p level is included.

II. LIMITATIONS OF THE WORK OF CILLIÉ AND BAKER AND MENZEL

We can best see the limitation of the analysis of Cillié and of Baker and Menzel if we try to determine the population of the second level, N_2 . An atom may enter the second level by direct recombination from the ionized state or by recombination to a higher level and subsequent cascade to the second level. Leaving the second level under the conditions specified by Baker and Menzel is impossible.³ Spontaneous de-excitation to the ground state is exactly balanced by radiative excitations, since the atmosphere is assumed deep in Lyman radiation. There is the possibility of two-quantum emission from the 2s level (which was not taken into account by Cillié or by Baker and Menzel), but, in order to leave the 2p level, we must invoke either a collisional transition or a radiative transition upward from the 2p level.⁴ Thus, if we take literally the three conditions given, the population of the 2p level will continue to increase with time until either collisional transitions from the 2p level become important or radiative excitations from the 2p level become important or so much of the population is in the second level that

² Several references (e.g., Burbidge and Burbidge 1953; Pagel 1959) have indicated that self-absorption might be important in determining the Balmer decrement in Be stars. However, the attempts of these authors to reproduce theoretically the observed decrements have met with little success. This, I believe, is primarily because of their neglect of transitions due to electron collisions, which, as will be shown in Section II, cannot be neglected in atmospheres having electron densities greater than 10^7 electrons/cm³.

³ This has been pointed out by Thomas (1949*b*) and by Burgess (1958).

⁴ We have divided the second level into two sublevels, 2s and 2p, according to the azimuthal quantum number. We consider that there is no radiative transition between the sublevels—only a collisional transition. This is justified in Appendix I.

the depopulation of the ground level makes the atmosphere thin in Lyman-line radiation. We assume, however, that the observed atmospheres are optically thick in Lyman-line radiation. Therefore, either collisional transitions or radiative excitations from the 2p level are important, or both. This will, in general, invalidate the results of Cillié-Baker-Menzel *except* when the depopulation of the 2p level takes place either by a collisional de-excitation to the ground level or by a collisional transition to the 2s level followed by two-quantum emission. Only if one of these two processes occurs, will the equations of statistical equilibrium reduce to those used by Cillié-Baker-Menzel.

First, we shall assume an atmosphere in which radiative excitations from the second and higher levels are negligible (but which is still optically deep to Lyman-line radiation). We shall show that in such an atmosphere the results of Cillié-Baker-Menzel are no longer correct, because of collisional excitation from the second level to the third level, when the electron density exceeds $10^7/\text{cm}^3$. Then in the following sections we shall discuss a specific atmosphere (diffuse nebula) with an electron density of about 10^2 and see how radiative excitations in this atmosphere effect the Balmer decrement.

We can calculate the effect of collisional transitions from 2p populating the third level by comparing this value with the number of transitions from the continuum to the third level. To compute this, we must solve the general equations of statistical equilibrium

TABLE 1
COLLISION TRANSITION RATES: SECONDS⁻¹, $T_e = 10000^\circ \text{K}$

$a \rightarrow b$	C_{ab}	$a \rightarrow b$	C_{ab}	$a \rightarrow b$	C_{ab}
2p $\rightarrow$ 1	$8.4 \times 10^{-9} N_e$	2p $\rightarrow$ 3	$1.0 \times 10^{-7} N_e$	2s $\rightarrow$ 2p	$4.8 \times 10^{-4} N_e^*$
2p $\rightarrow$ 1	$8.7 \times 10^{-9} N_e$	2p $\rightarrow$ 2s	$1.6 \times 10^{-4} N_e^*$		

* Assuming $N_i = N_e$, since proton impact is most important in determining the collision rate.

librium and specify the collisional transition rates. The collision rates, C_{ab} , are given in Table 1. The first three are taken from Jefferies and Pottasch (1959) and the last two from Seaton (1955).

A solution of the equations of statistical equilibrium for an atom with four discrete levels plus a continuum (under conditions stated above) is as follows:

$$\frac{N_{2p}}{N_e} \cong \frac{A_{i2p} + A_{i3} \mathfrak{P}_{32s} + A_{i4} (\mathfrak{P}_{42s} + \mathfrak{P}_{43} \mathfrak{P}_{32s})}{C_{2p2s} \mathfrak{P}_{2s1} \mathfrak{P}_{1i}}, \quad (1)$$

where $P_{ab} = A_{ab} + C_{ab}$, and A_{ab} is the rate of radiative transitions $a \rightarrow b$ per second, and C_{ab} is the rate of collisional transitions $a \rightarrow b$ per second. The total number of collisional transitions $a \rightarrow b$ would be $N_a C_{ab}$ per cubic centimeter per second.

$$\mathfrak{P}_{ab} = \frac{P_{ab}}{\sum_s P_{as}}$$

is essentially that fraction of the total number of transitions, occurring from level a , which go from $a \rightarrow b$. If the atmosphere is optically deep in $a \rightarrow b$ radiation, then P_{ab} may be replaced by C_{ab} , because of the consequent detailed balance in the radiative processes.

In Table 1 the collision rates $2s \rightarrow 3$, etc., are not included because, with the assumption of detailed balance in the Lyman lines, at electron densities less than $10^4/\text{cm}^3$ the population of the 2s level is much less than the 2p level (while at electron densities great-

er than $10^4/\text{cm}^3$ the two levels are populated according to the statistical weight). This is rigorously demonstrated in Appendix II.

Collisions from the ground state enter the calculation in the term $\mathfrak{P}_{1i}$, the ratio of the number of transitions directly from the ground state to the continuum, to the total number of transitions from the ground state. Since the Lyman continuum radiation and the electron temperature will vary with the particular situation, we can assume the two extremes, i.e., $\mathfrak{P}_{1i} = 1$, indicating that the number of radiative transitions from the ground state due to Lyman continuum radiation is much greater than all collisional transitions, and $\mathfrak{P}_{1i} = 5.1 \times 10^{-2}$, assuming that the Lyman continuum radiation field is negligible compared to collisions. The first value is probably of importance astrophysically if the electron temperature is $\sim 10000^\circ \text{K}$, while the latter value might be more appropriate at $T_e \sim 40000^\circ \text{K}$. No general rule can be formulated, and each case must be computed individually.

Under the assumptions used by Cillié-Baker-Menzel (and our four-level atom) the number of transitions into the third level is $N_e A_{i3}$, where $A_{i3} = 3.4 \times 10^{-14} N_e$ for $T_e = 15000^\circ \text{K}$. An additional number of transitions $N_2 C_{23}$ enters the third level by electron collisions from the second level. When the latter quantity is as great as, or

TABLE 2
COMPARISON OF ROLES PLAYED BY RECOMBINATION AND
BY COLLISION IN POPULATING N_3

N_e	$N_e A_{i3}$	$\mathfrak{P}_{1i} = 1$ $N_{2p2} C_{2p3}$	$\mathfrak{P}_{1i} = 5.1 \times 10^{-2}$ $N_{2p} C_{2p3}$
10^2	3.4×10^{-10}	8.2×10^{-13}	4.2×10^{-11}
10^3	3.4×10^{-8}	8.7×10^{-11}	4.4×10^{-9}
10^4	3.4×10^{-6}	1.3×10^{-8}	6.6×10^{-7}
10^5	3.4×10^{-4}	5.6×10^{-6}	2.8×10^{-4}
10^6	3.4×10^{-2}	4.9×10^{-3}	2.5×10^{-1}
10^7	3.4	4.9	2.5×10^2
10^8	3.4×10^2	4.9×10^3	2.5×10^5

greater than, the former quantity, assumption 2—that collisional transitions upward from the excited levels are unimportant—clearly breaks down, and the Cillié-Baker-Menzel decrements are no longer valid. We can easily calculate the values of these two quantities as functions of electron density with the help of equation (1). The results are shown in Table 2.

By comparing the second column of Table 2 with the last two columns it can be seen that, for $\mathfrak{P}_{1i} = 1$, recombinations are the most important factor in populating the third level below electron densities of $10^7/\text{cm}^3$, and for $\mathfrak{P}_{1i} = 5 \times 10^{-2}$ they are the most important factor below electron densities of $10^5/\text{cm}^3$. For all higher electron densities the collisional excitations from the second to the third level cannot be ignored.

Thus if radiative transitions upward from excited levels are negligible, theories which ignore collisional transitions from the second level upward are limited to electron densities below 10^5 – 10^7 . They may be further limited, of course, by the fact that radiative transitions upward from excited levels may be important. We shall discuss this effect now for diffuse nebulae ($N_e < 10^3$).

III. THE OPTICAL DEPTH OF DIFFUSE NEBULAE IN BALMER-LINE RADIATION

We have just shown that for diffuse nebulae, since $N_e < 10^5$, we are justified in ignoring collisional transitions from excited levels. We shall also ignore collisional transitions from the ground state, since at $T_e = 10000^\circ \text{K}$ they will be small compared to the tran-

sitions due to ionizing radiation. (Assuming that the ionizing radiation comes from a black body at $T = 40000^\circ \text{K}$, $R = 10$ solar radii, and a reasonable dilution factor. If $T_e > 20000^\circ \text{K}$, collisional transitions may have to be included.)

The population of level 2p can now be calculated, assuming no radiative excitation from this level, using equation (1).

The results are shown in Table 3 for five different diffuse nebulae. The values of N_e for the first four are taken from Pottasch (1956) corrected as suggested by Jefferies and Pottasch (1959). The value of N_e in the Orion Nebula is probably better determined and was taken from Osterbrock and Flather (1959). The total optical depth at the center of H α is given in the last column (assuming a Doppler half-width corresponding to 10000°K). The optical depths obtained are rather rough because of the uncertainties in T_e and N_e but are almost certainly correct within an order of magnitude. Inclusion of collisional transitions from the ground level because of higher T_e would increase the 2p level population and hence increase the optical depth in H α .

TABLE 3
H α OPTICAL DEPTHS IN FIVE DIFFUSE NEBULAE

Nebula	$2R_0$ (parsecs)	N_e	$\tau(\text{H}\alpha)$	Nebula	$2R_0$ (parsecs)	N_e	$\tau(\text{H}\alpha)$
NGC 7000	43	27	2 3	NGC 6523	4 5	780	6 2
NGC 2244	31	69	3 9	NGC 1976	6 0	10^4 variable	17
NGC 281	13	138	3 3				

IV. BALMER DECREMENTS IN LOW-DENSITY GAS CLOUDS WHOSE H α OPTICAL DEPTH IS OF THE ORDER OF UNITY

In Section II we have shown that, in atmospheres with electron densities as low as in diffuse nebulae, collisional processes from the second level to higher or lower levels are unimportant. In Section III we have shown that the optical depth in H α is high enough that radiative excitations in the Balmer lines cannot be ignored. In this section we consider the effect of radiative excitation from the second level on the Balmer decrement.

In order to compute the radiation fields in the Balmer lines, we solve the transfer equation, which in the Eddington approximation for a plane-parallel atmosphere is

$$\frac{1}{3} \frac{d^2 J_\nu}{d\tau_\nu^2} = J_\nu - S_\nu, \quad (2)$$

where, neglecting stimulated emission, the source function, S_ν , for the Balmer line $X \rightarrow 2$ is

$$S_\nu = \frac{2 h \nu_{2x}^3}{c^2} \frac{\varpi_2}{\varpi_x} \frac{N_x}{N_2},$$

and the ratio N_x/N_2 is found from the equations of statistical equilibrium:

$$\frac{N_x}{N_2} = \frac{\varpi_x}{\varpi_2} \frac{c^2}{2 h \nu_{2x}^3} (1 - \lambda_{2x}) \int J_{2x} \Phi_\nu d\nu + \lambda_{2x} b_{2x}, \quad (3)$$

where, for H α and H β (and a five-level atom),

$$\lambda_{23} b_{23} = \frac{C_{2p2s} [\mathfrak{P}_{2s3} + \mathfrak{P}_{2s1} \mathfrak{P}_{1i} (\mathfrak{P}_{i3} + \mathfrak{P}_{14} \mathfrak{P}_{43}) + \mathfrak{P}_{2s1} (\mathfrak{P}_{13} + \mathfrak{P}_{14} \mathfrak{P}_{43})] + A_{2p4} \mathfrak{P}_{43}}{A_{32} (1 - \mathfrak{P}_{1i} \mathfrak{P}_{i1})}, \quad (4a)$$

$$\lambda_{24} b_{24} = \frac{C_{2p2s} [\mathfrak{P}_{2s4} + \mathfrak{P}_{2s1} (\mathfrak{P}_{1i} \mathfrak{P}_{i4} + \mathfrak{P}_{14})] + A_{2p3} \mathfrak{P}_{34}}{(A_{42} + A_{43}) (1 - \mathfrak{P}_{1i} \mathfrak{P}_{i1})}, \quad (4b)$$

and Φ_ν is the profile of the line normalized so that

$$\int \Phi_\nu d\nu = 1.$$

As a first approximation we shall assume that the integral $\int J_{2x} \Phi_\nu d\nu$ can be replaced by a sum consisting of one term, its value at the line center.⁵

The values of the scattering parameter, λ_{2x} , are given as follows:

$$\lambda_{23} = \frac{C_{32} (1 - \mathfrak{P}_{1i} \mathfrak{P}_{i1}) + P_{3i} \mathfrak{P}_{i2} + P_{34} \mathfrak{P}_{42}}{A_{32} (1 - \mathfrak{P}_{1i} \mathfrak{P}_{i1})},$$

$$\lambda_{24} = \frac{A_{43}}{A_{43} + A_{42}}.$$

A word should be said about the physical meaning of the scattering parameters λ_{23} and λ_{24} . They represent the probability that a quantum of radiation will be absorbed rather than scattered. Thus λ_{23} is the ratio of the number of transition out of the third level not including a direct radiative transition $3 \rightarrow 2$ —which would only reproduce (scatter) the originally absorbed quantum—to the total number of transitions from the third level.⁶ We can see in this way why λ_{24} will be much closer to unity than λ_{23} . If a quantum of $H\beta$ is absorbed, there is about equal probability that it will be scattered and that it will decay to produce Paschen- α plus $H\alpha$, thus λ_{24} is about $\frac{1}{2}$. However, if $H\alpha$ is absorbed, it will mostly be scattered, because the only way by which it can leave the third level without reproducing itself is by a collisional transition of some sort or by absorption of Paschen radiation. Under the condition that we are dealing with in diffuse nebulae, the most important of these transitions will be an absorption of Paschen- α radiation for which the transition rate is calculated to be (approximately)⁷ $10^{-4} N_e \tau_{H\alpha}^0$. At an electron density of $10^3/\text{cm}^{-3}$ in an atmosphere whose optical depth in $H\alpha$ is unity, this transition rate of 0.1 is very small compared to the spontaneous radiation rate A_{32} of 4.4×10^7 . Thus, for these conditions, all but one quantum of 4.4×10^8 quanta of $H\alpha$ absorbed is scattered, while half the quanta of $H\beta$ absorbed are scattered. This is the primary physical reason why the absorption of Balmer-line radiation effects the Balmer decrement: the $H\alpha$ radiation is simply scattered while the $H\beta$ radiation is partly absorbed, steepening the decrement. A secondary effect arises because each quantum of $H\beta$ radiation which is removed from the radiation field by absorption and recombination to the third level produces an $H\alpha$ quantum, thus further steepening the decrement.

This state of affairs is very much dependent on the fact that conditions in diffuse nebulae are very far from thermodynamic equilibrium. The scattering parameter λ_{23} for $H\alpha$ radiation is so small because of the low electron density (the collisional transition rates are extremely small) and because of extremely low populations of the upper levels, which reduces the Paschen radiation.

The equation of transfer in $x \rightarrow 2$ radiation then becomes

$$\frac{1}{3} \frac{d^2 J_{2x}}{d\tau_{2x}^2} = \lambda_{2x} J_{2x} - \lambda_{2x} b_{2x} \left(\frac{2 h \nu_{2x}^3}{c^2} \frac{\varpi_2}{\varpi_x} \right), \quad (5)$$

and the solution near the edge of the atmosphere (assuming b_{2x} and λ_{2x} to be constant) is

$$J_{2x} = \frac{2 h \nu_{2x}^3}{c^2} \frac{\varpi_2}{\varpi_x} b_{2x} \left\{ \frac{\sqrt{\lambda_{2x}} (1 + \tau_{2x} \sqrt{3}) (1 - \exp[-\tau_{2x}^0 \sqrt{3\lambda_{2x}}])}{1 + \sqrt{\lambda_{2x}} + (1 - \sqrt{\lambda_{2x}}) \exp[\tau_{2x}^0 \sqrt{3\lambda_{2x}}]} \right\}, \quad (6)$$

⁵ This entire procedure of determining the source function is discussed in greater detail by Jefferies and Pottasch (1959).

⁶ Lyman-line transitions are assumed to be balanced in detail.

⁷ See Appendix III.

where τ_{2x}^0 is the total optical depth of the plane atmosphere. The solution was carried out subject to the following boundary conditions: at the outside edge there is no incident radiation field (Krook 1955)

$$\tau_{2x} = 0, \quad J_{2x} = \frac{1}{\sqrt{3}} \frac{dJ_{2x}}{d\tau_{2x}},$$

and at the inside edge there is likewise no incident radiation field, and the problem is symmetric about the center, so that

$$\tau_{2x} = \tau_{2x}^0, \quad J_{2x}(\tau = 0) = J_{2x}(\tau = \tau_{2x}^0).$$

When $\tau_{2x}^0 \sqrt{3} \lambda_{2x} \ll 1$, the emergent mean intensity, $J_{2x}(\tau = 0)$, is

$$J_{2x} = \frac{\sqrt{3}}{2} \frac{\varpi_2}{\varpi_x} \frac{2 h \nu_{2x}^3}{c^2} \lambda_{2x} b_{2x} \tau_{2x}^0. \quad (7)$$

The ratio of the integrated mean intensity leaving the atmosphere in H α to that in H β when $\tau_{2x}^0 \sqrt{3} \lambda_{2x} \ll 1$, is

$$\begin{aligned} \frac{\text{Intensity}(H\alpha)}{\text{Intensity}(H\beta)} &= \frac{J_{23}(\tau = 0) \Delta\nu_{23}}{J_{24}(\tau = 0) \Delta\nu_{24}} = \frac{\lambda_{23} b_{23} \kappa_{23} \Delta\nu_{23} \nu_{23}^3 \varpi_4}{\lambda_{24} b_{24} \kappa_{24} \Delta\nu_{24} \nu_{24}^3 \varpi_3} \\ &= \frac{\lambda_{23} b_{23} A_{32} \nu_{23}}{\lambda_{24} b_{24} A_{42} \nu_{24}}. \end{aligned}$$

If we assume that the optical depth in the Balmer lines is small and that radiative excitations from the second level are unimportant, we obtain the results previously derived under these conditions by Cillié, Baker and Menzel, and Chamberlain. For if, in equations (4a) and (4b), we let $A_{2s3} = A_{2s4} = A_{2p4} = A_{2p3} = 0$ (negligible radiative excitations from the second level); $\mathfrak{P}_{2s1} = 1$; $\mathfrak{P}_{1i} = 1$, $P_{13} = P_{14} = 0$ (no collisions from the ground level, the only unbalanced radiative transition upward is absorption of Lyman continuum radiation), then

$$\frac{\lambda_{23} b_{23}}{\lambda_{24} b_{24}} = \frac{(A_{42} + A_{43})(P_{i3} + P_{i4} P_{43})}{A_{32} P_{i4}} = \frac{(A_{42} + A_{43}) A_{i3} + A_{i4} A_{43}}{A_{32} A_{i4}},$$

which is exactly the population ratio N_3/N_4 that would have been predicted by Cillié and by Baker and Menzel, had they used only a five-level atom. Had we assumed that the only upward transitions from the ground level were collisions,

$$\mathfrak{P}_{13} = \frac{C_{13}}{\sum_i C_{1i}},$$

we would have obtained Chamberlain's result for the population ratio N_3/N_4 :

$$\frac{\lambda_{23} b_{23}}{\lambda_{24} b_{24}} = \frac{(A_{42} + A_{43}) \{C_{1i}(P_{i4} P_{43} P_{i3}) + C_{13} + C_{14} P_{43}\}}{A_{32} (C_{1i} P_{i4} + C_{14})}.$$

Thus we see that the results of Cillié, Baker and Menzel, and Chamberlain are a special case and are correct at these electron densities as long as radiative excitations from the second level have negligible effect.

We can easily take into account the optical depth in the Balmer lines and radiative excitations from the second level. When the total optical depth of the atmosphere in H α

is of the order of 10, we may still use the approximate solution for the emergent intensity for H α given by equation (7) because $\tau_{23}^0 \sqrt{(3\lambda_{23})} \ll 1$, since $\lambda_{23} \simeq 10^{-6}$. However, we must use equation (6) to describe the emergent intensity in H β . Thus the ratio of the flux leaving the atmosphere in H α to that in H β is

$$\frac{\text{Intensity (H}\alpha\text{)}}{\text{Intensity (H}\beta\text{)}} = \frac{A_{32}\nu_{23}\lambda_{23}b_{23}}{A_{42}\nu_{24}\lambda_{24}b_{24}} \times \frac{1}{2} \frac{1 + \sqrt{\lambda_{24}} + (1 - \sqrt{\lambda_{24}}) \exp[-\tau_{24}^0 \sqrt{(3\lambda_{24})}]}{1 - \exp[-\tau_{24}^0 \sqrt{(3\lambda_{24})}]} \tau_{24}^0 \sqrt{(3\lambda_{24})}, \quad (8)$$

where for convenience the last term will be referred to as $f(\lambda_{24}, \tau_{24}^0)$. The ratio $\lambda_{23}b_{23}/\lambda_{24}b_{24}$ can be computed from equation (4), realizing that

$$A_{24} = 4\pi \int \frac{J_{24} \kappa_{24} d\nu}{h\nu_{24}},$$

where κ_{24} is the absorption coefficient (cm^2) for H β .

The resulting H α :H β ratios are shown in the last column in Table 4, along with the

TABLE 4
H α /H β RATIOS $T_e=10000^\circ\text{K}$

$\tau_{23}^0(\text{H}\alpha)$	$\lambda_{23}b_{23}/\lambda_{24}b_{24}$	$f(\lambda_{24}, \tau_{24}^0)$	Five-Level Atom H α /H β	$\tau_{23}^0(\text{H}\alpha)$	$\lambda_{23}b_{23}/\lambda_{24}b_{24}$	$f(\lambda_{24}, \tau_{24}^0)$	Five-Level Atom H α /H β
0 . . .	0 791	1 0	3 08	15	1 41	2 40	13 2
3×10^{-1}	0 806	1 03	3 20	20 . . .	1 50	3 13	18 2
1	0 841	1 07	3 48	25 . . .	1.56	3 86	23 0
3	0 936	1 20	4 36	30 . . .	1.60	4 61	28 5
7	1 14	1 55	6 87	35 . . .	1 64	5 37	33 8
10 . . .	1 28	1 78	8 86	40 . . .	1 68	6 00	39 0

values of $\lambda_{23}b_{23}/\lambda_{24}b_{24}$ as a function of τ_{23}^0 , the total optical depth of the atmosphere in H α . We have assumed no collisions from the ground state.

This increase in the H α :H β ratio with optical depth will continue until $3\lambda_{23} \tau_{23}^0 \sim 1$. With the approximate value of λ_{23} (and A_{34}) previously given, this should occur at an optical depth $\tau_{23}^0 \sim 300$ for an atmosphere with $N_e = 10^3 \text{ cm}^{-3}$. At greater optical depths the value of λ_{23} should continue to increase and thus the values of H α :H β should begin to decrease. Only a qualitative estimate is given because for these conditions the transfer equations in H α , H β , and Paschen- α must be solved simultaneously.

Calculations have also been carried out for a seven-level atom under exactly the same conditions as discussed for the five-level atom. The results are shown in Table 5.

In Table 5 two columns are given for each ratio. The first gives the results for a seven-level atom, while the second column is the same as the first, multiplied by a factor so that the results at $\tau(\text{H}\alpha) = 0$ are those of Baker and Menzel.

The reason that H γ and H δ increase with respect to H β with increasing optical depth is simply that more H β is absorbed than H γ or H δ . This result may be important in trying to separate the effect of the optical depth from that of interstellar reddening in the observations, since both effects act the same way in steepening H α /H β , but in opposite ways in the ratios H γ /H β and H δ /H β .

In the following sections we compare the H α :H β ratios shown in Table 5 with the observations of diffuse nebulae and planetary nebulae.

V. COMPARISON WITH OBSERVATIONS: DIFFUSE NEBULAE

A completely quantitative comparison of the theoretical values just computed with the observed $H\alpha/H\beta$ ratios cannot be made. We have neglected the line profile in our derivation; we have not considered microscopic motions in the atmosphere; we have solved the plane-parallel atmosphere and not the spherical atmosphere. Further, we do not know precisely the electron temperature of the diffuse nebulae or the amount and quality of the radiation exciting them, to determine how much relative effect radiative and collisional transitions from the ground state will have. Finally, we do not know the electron density very well. Only a semi-quantitative comparison can be made.

TABLE 5
THEORETICAL BALMER DECREMENTS, $T_e = 10000^\circ \text{K}$, $N_e < 10^6 \text{ cm}^{-3}$

$r_{23}(H\alpha)$	$H\alpha/H\beta$		$H\gamma/H\beta$		$H\delta/H\beta$	
0	2 81	2 50	0 447	0 51	0 254	0 31
3×10^{-1}	2 95	2 63	461	525	262	32
1	3 16	2 82	469	535	266	324
3	3 86	3 44	485	544	286	348
7.	5 89	5 25	557	635	344	418
10	7 45	6 64	585	666	371	451
15	10 7	9 51	675	770	454	551
20 .	14 5	12 9	0 774	0 882	0 545	0 663

TABLE 6
 $H\alpha/H\beta$ RATIOS IN DIFFUSE NEBULAE

Nebula	Distance (parsecs)	$H\alpha/H\beta$	Nebula	Distance (parsecs)	$H\alpha/H\beta$
NGC 6523	1300	6 75	λ ORI .	600	7 5
6888	1500	14 9	NGC 1976	500	4 2
7000	900	16 6	2175	2100	15 1
IC1936b	700	16 8	2264	700	8 9
NGC 281	2500	10 6	2237	1400	11 2
1499	300	13 0	IC2177	1100	11 8
IC 405	580	15 4			

To my knowledge, the most extensive attempt to measure the $H\alpha/H\beta$ ratios in diffuse nebulae was carried out by Johnson (1953). He made 30 measurements in 13 different nebulae. Table 6 summarizes the average values of the $H\alpha/H\beta$ ratio for the 13 nebulae.

The median $H\alpha/H\beta$ value is 11.7, and the average value is the same. It would take an excessively large value of interstellar reddening to reduce these ratios to those predicted by Baker and Menzel. Another argument against interstellar reddening being a large factor is the fact that the average value of $H\alpha/H\beta$ for nebulae at distances less than 1000 parsecs is 11.6 (six nebulae) and for nebulae greater than 1000 parsecs the average $H\alpha/H\beta$ is 11.8 (seven nebulae). The nearest nebula (300 parsecs) has an $H\alpha/H\beta$ of 13.0. The farthest nebula (2500 parsecs) has a value $H\alpha/H\beta = 10.6$.

Other observers give similar results with somewhat smaller $H\alpha/H\beta$ ratios. For example, Greenstein and Henyey (1939) find a mean $H\alpha/H\beta$ of 8 for a group of nine nebulae. They also measure an $H\alpha/H\beta$ ratio of 42 for the nebula IC 1318, indicating that high values do exist.

These observations, combined with the calculations of optical depth in $H\alpha$ in the diffuse nebulae shown in Table 3, are consistent with the theoretical values given in Table 5 and lend support to the hypothesis that radiative excitation from the second quantum level are important in determining the Balmer decrement in diffuse nebulae. There is no detailed agreement, however, since Table 3 indicates the $H\alpha$ optical depth to be greater in the Orion Nebula (NGC 1976), while the measurements in Table 6 would indicate that only a small optical depth is expected in this nebula.

Johnson did not measure any other hydrogen lines. Recently Mathis (1957) measured an $H\gamma/H\beta$ of 0.56 and $H\delta/H\beta$ of 0.34 in the Orion Nebula. Thackeray (1950) measured an $H\gamma/H\beta$ ratio of 0.56 in NGC 6514 and a ratio of 0.50 in NGC 6523. Since interstellar reddening only increases $H\gamma/H\beta$, it is impossible to use interstellar reddening to get $H\alpha$, $H\beta$, and $H\delta$ to fit the results of Baker and Menzel.

VI. COMPARISON WITH OBSERVATIONS: PLANETARY NEBULAE

The observed values of $H\alpha/H\beta$ in planetary nebulae are lower than those observed in diffuse nebulae but still higher than what would be predicted (2.55) by Baker and Menzel. The observed values (Minkowski and Aller 1956) are shown in Table 7. In the

TABLE 7
OBSERVED VALUES OF $H\alpha/H\beta$: PLANETARY NEBULAE

Nebula	$H\alpha/H\beta$	Nebula	$H\alpha/H\beta$
NGC 7662	3 97	NGC 2022	3 06
2392	3 80	1535	3 61
	{ 5 81 weak knot	7027	6 50
40*	{ 8 00 center		
	{ 9 04 strong knot		

* $H\alpha + (N II)$ were inseparable

past, most workers in the field have assumed the value given by Baker and Menzel and called the difference interstellar reddening. This leads to values of reddening which might be considered high. (No one has taken issue with the values obtained in this way.)

Computations of $H\alpha$ optical depth carried out for planetary nebulae in the same way as done for diffuse nebulae in Section III yield smaller values, but there is evidence that inhomogeneities may exist in these objects. From the observations it is not possible to say clearly whether radiative excitations from the second level are important or not in planetary nebulae.

VII. SUMMARY

We give the following conclusions from computing the Balmer decrement for an atmosphere which is optically very thick in Lyman-line radiation:

1. If radiative excitations from the second level are unimportant, collisional transitions from the second level cannot be neglected at electron densities greater than $10^7/\text{cm}^{-3}$, while they may be neglected at electron densities below $10^5/\text{cm}^{-3}$.
2. We have computed the effect of radiative excitations from the second level of hydrogen on the Balmer decrement as a function of the optical depth of the atmosphere in $H\alpha$.
3. There is good reason to suppose that the Balmer decrements observed in diffuse nebulae are affected by radiative excitations from the second level.

The author is indebted to Drs. R. N. Thomas and J. T. Jefferies for many discussions of the theoretical aspects of this problem and to Dr. H. M. Johnson for discussion of the observations.

We wish here to justify the inclusion of only collisional transitions and not radiative transitions between the 2s and 2p levels of hydrogen.

The structure of the first and second levels of hydrogen are shown in Figure 1. The splitting of the second level is seen to be almost a factor of 10^6 smaller than the total $2 \rightarrow 1$ transition. The energy differences in the second level are found from the well-known experiments of Lamb and Retherford (1950, 1951).

Knowing the energy differences, we can now compare the lifetimes in the $2^2S_{1/2}$ and $2^2P_{3/2}$ for electric dipole transitions. The results are shown in Table 8.

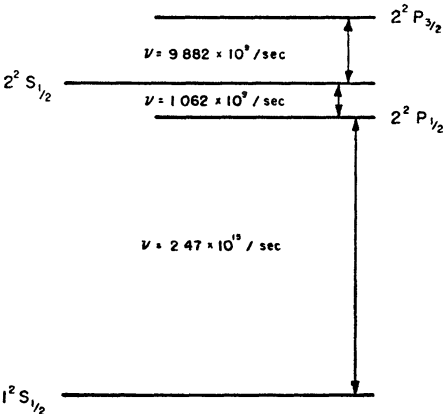


FIG. 1.—The structure of the second level of hydrogen

TABLE 8

Transitions	Lifetime (seconds)	Transitions	Lifetime (seconds)
$2S_{1/2}$ – $2P_{1/2}$ (electric dipole)	6×10^8	Collision $2s$ – $2p$ ($N_e=10^9$)	20
$2P_{3/2}$ – $2S_{1/2}$ (electric dipole)	6×10^6	Collision $2p$ – $2s$ ($N_e=10^9$)	60

From Table 8 we see that collisions are the principal means of intra-level transitions and completely dominate over radiative transitions.

APPENDIX II

We shall here justify the statement that if the atmosphere is optically very deep in Lyman-line radiation, the population of the 2s level is always less than that of the 2p level. This can be seen to be true qualitatively: since the atmosphere is assumed optically very deep in Lyman- α radiation, any spontaneous transition from 2p to 1 is balanced in detail by the reverse process. Thus the 2p level cannot be depopulated by emission of Lyman- α . All collisional processes will tend to keep the two levels populated in the ratio of their statistical weights, as will excitations from the second level. When two-quantum emission is important, it will act to depopulate the 2s level relative to the 2p level. Thus the only process which does not act to keep the two levels in the ratio 3:1 is the two-quantum emission depopulating the 2s level.

Quantitatively, for a four-level atom, the population ratio can be written:

$$\frac{N_{2s}}{N_{2p}} = \frac{C_{2p2s} (1 - \mathfrak{P}_{1i} \mathfrak{P}_{i1}) + A_{2p3} \mathfrak{P}_{32s}}{C_{2s2p} (1 - \mathfrak{P}_{1i} \mathfrak{P}_{i1}) + A_{2s1} \mathfrak{P}_{1i} (\mathfrak{P}_{i2p} + \mathfrak{P}_{i3} \mathfrak{P}_{32p}) + A_{2s3} \mathfrak{P}_{32p}}.$$

Using the collisional rates shown in Table 2 and ignoring the H α radiation field ($A_{2s1} = 8.3$), the population ratio N_{2s}/N_{2p} is shown in Figure 2. At electron densities greater than 10^4 cm^{-3} , the ratio is constant (0.33); it decreases at lower electron densities.

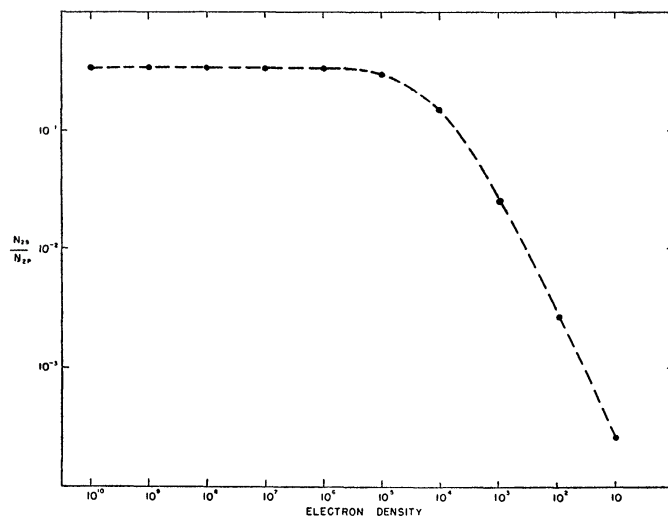


FIG. 2.—Ratio of the 2s population to the 2p population as a function of the electron density. The atmosphere is assumed deep in Lyman-line radiation.

APPENDIX III

To find the transition rate for the absorption of Paschen- α quanta, A_{34} : The solution of the transfer equations in Paschen- α (in the same manner as for H α and H β) is

$$J_{34} = \frac{2h\nu_{34}^3}{C^2} \frac{\varpi_3}{\varpi_4} \frac{C_{34} + A_{32}\mathfrak{P}_{24}}{A_{43} + A_{42}\mathfrak{P}_{23}} Q,$$

where

$$Q = \frac{\{ \sqrt{\lambda_{34}} + 1 - \exp[-\tau_{34}\sqrt{(3\lambda_{34})}] \} \{ 1 - \exp[-\tau_{34}^0\sqrt{(3\lambda_{34})}] \}}{(1 + \sqrt{\lambda_{34}}) + (1 - \sqrt{\lambda_{34}}) \exp[-\tau_{34}^0\sqrt{(3\lambda_{34})}]}$$

and the transition rate

$$A_{34} = 4\pi \int \frac{J_{34} \kappa_{34} d\nu}{h\nu_{34}}.$$

Since the total optical depth of the atmosphere we are considering, τ_{34}^0 , is much less than unity,

$$Q = \frac{\sqrt{3}}{2} \lambda_{34} \tau_{34}^0.$$

Very approximately, then,

$$A_{34} = A_{43} \frac{J_{24}}{J_{23}} \frac{\varpi_4}{\varpi_3} \frac{\nu_{23}^3}{\nu_{24}^3} \frac{\sqrt{3}}{2} \lambda_{34} \tau_{34}^0,$$

since

$$\frac{J_{24}}{J_{23}} \leq \frac{1}{3}, \quad \lambda_{34} < 1, \quad A_{34} \leq 9.4 \times 10^5 \tau_{34}^0 = 9.4 \times 10^5 \frac{\kappa_{34}}{\kappa_{23}} \frac{N_3}{N_2} \tau_{23}^0,$$

and, from equations (3) and (4a),

$$\frac{N_3}{N_2} = 3.1 \times 10^{-11} N_e,$$

and thus

$$A_{34} \leq 17.5 \times 10^{-5} N_e \tau_{23}^0.$$

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