ON THE EFFECTS OF THE SUN AND THE MOON UPON THE MOTION OF A CLOSE EARTH SATELLITE

by

Yoshihide Kozai *
Astrophysical Observatory, Smithsonian Institution

In the present paper I will treat the lunar and solar perturbations of a close earth satellite whose radius is very small compared with that of the moon.

Since the disturbing functions of the sun and the moon both have similar forms, only the method of deriving the perturbation for the moon will be described here.

If we denote the geocentric radius vector of the satellite and of the moon by $\vec{r}$ and $\vec{r}'$, respectively, and expand the disturbing function R into a power series of r/r', a small quantity, we obtain the expression,

$$R = \frac{Gm'}{r'}\left(\frac{1}{r'} + \frac{r^2}{r'^2} S_2 + \frac{r^3}{r'^3} S_3 + \ldots\right), \qquad (1)$$

where G is the constant of gravity, m' is the mass ratio of the moon with respect to the earth, and S_i is the Legendre polynomial of the i-th order; that is,

$$\begin{cases} S_2 = \frac{3}{2} s^2 - \frac{1}{2}, \quad S_3 = \frac{5}{2} s^3 - \frac{3}{2} s, \\ s = \vec{r}\,\vec{r}'/r\,r' . \end{cases} \qquad (2)$$

We can omit the first term, which does not depend on the orbital elements of the satellite. We cannot expect any secular contributions from the odd-order terms, so I will derive the perturbation produced from the second-order term.

Adopting geocentric coordinates, with x-axis directed towards the equinox and z-axis towards the north pole, we have the following three components of $\vec{r}$ (by using the conventional orbital elements):

$$\begin{cases} \frac{x}{r} = \cos(L+\Omega) + 2\sin^2\frac{i}{2}\sin L \sin\Omega, \\ \frac{y}{r} = \sin(L+\Omega) - 2\sin^2\frac{i}{2}\sin L\cos\Omega, \\ \frac{z}{r} = \sin i \sin L, \end{cases} \qquad (3)$$

where L is the argument of latitude. We derive similar expressions for the moon, using primed letters to represent elements referred to the equator. Then, s has the following expression:

---

* Astronomer, Satellite Tracking Program

$$
\begin{aligned}
s &= \frac{xx' + yy' + zz'}{rr'} \\
&= \cos(L + \Omega - L' - \Omega') - 2\sin^2\frac{i}{2}\sin L \sin(L' + \Omega' - \Omega) \\
&\quad - 2\sin^2\frac{i'}{2}\sin L' \sin(L + \Omega - \Omega') \qquad (4) \\
&\quad + 4\sin^2\frac{i}{2}\sin^2\frac{i'}{2}\sin L \sin L' \cos(\Omega - \Omega') \\
&\quad + \sin i \sin i' \sin L \sin L' .
\end{aligned}
$$

It is convenient to express the disturbing function by mean longitudes, λ and λ', and other orbital elements. Dropping all terms depending on the mean longitude of the satellite, which have little effect on the satellite's motion, we have as the principal terms of the disturbing function:

$$
\begin{aligned}
R = n'^2 m' a^2 \Big[& \left\{1 + 3e' \cos(\lambda' - \omega' - \Omega')\right\} \left\{\left(1 + \frac{3}{2}e^2\right) A + \frac{15}{8} e^2 B\right\} \\
& - 4e' \sin(\lambda' - \omega' - \Omega') \left\{\left(1 + \frac{3}{2}e^2\right) A' + \frac{15}{8} e^2 B'\right\}\Big] , \qquad (5)
\end{aligned}
$$

where

$$
\begin{aligned}
A = {} & \frac{1}{4}\left(1 - \frac{3}{2}\sin^2 i\right)\left(1 - \frac{3}{2}\sin^2 i'\right) \\
& + \frac{3}{16}\sin 2i \sin 2i' \cos(\Omega - \Omega') \\
& + \frac{3}{16}\sin^2 i \sin^2 i' \cos 2(\Omega - \Omega') \\
& + \frac{3}{8}\sin^2 i'\left(1 - \frac{3}{2}\sin^2 i\right)\cos 2(\lambda' - \Omega') \\
& + \frac{3}{8}\sin^2 i \cos^4\frac{i'}{2}\cos 2(\lambda' - \Omega) \\
& - \frac{3}{8}\sin 2i \sin i' \cos^2\frac{i'}{2}\cos(2\lambda' - \Omega - \Omega') \\
& + \frac{3}{8}\sin^2 i \sin^4\frac{i'}{2}\cos 2(\lambda' - 2\Omega' + \Omega) \\
& + \frac{3}{8}\sin 2i \sin i' \sin^2\frac{i'}{2}\cos(2\lambda' + \Omega - 3\Omega'),
\end{aligned}
$$

$$
\begin{aligned}
B = {} & \cos^4 \frac{i}{2} \cos^4 \frac{i'}{2} \cos 2(\lambda' - \omega - \Omega) \\
& + \frac{1}{2} \sin^2 i \left(1 - \frac{3}{2} \sin^2 i'\right) \cos 2\omega \\
& + \frac{1}{2} \cos^4 \frac{i}{2} \sin^2 i' \cos 2(\omega + \Omega - \Omega') \\
& + \sin^4 \frac{i}{2} \cos^4 \frac{i'}{2} \cos 2(\omega - \Omega + \lambda') \\
& + \cos^4 \frac{i}{2} \sin^4 \frac{i'}{2} \cos 2(\omega + \Omega + \lambda' - 2\Omega') \\
& + \sin^4 \frac{i}{2} \sin^4 \frac{i'}{2} \cos 2(\lambda' - 2\Omega' - \omega + \Omega) \\
& + \frac{3}{8} \sin^2 i \sin^2 i' \cos 2(\lambda' - \Omega' - \omega) \\
& + \frac{3}{8} \sin^2 i \sin^2 i' \cos 2(\lambda' + \omega - \Omega') \\
& + \sin i \cos^2 \frac{i}{2} \sin i' \cos^2 \frac{i'}{2} \cos (2\lambda' - \Omega' - 2\omega - \Omega) \\
& - \frac{1}{2} \sin i \cos^2 \frac{i}{2} \sin 2i' \cos (2\omega + \Omega - \Omega') \\
& + \frac{1}{2} \sin^4 \frac{i}{2} \sin^2 i' \cos 2(\omega - \Omega + \Omega') \\
& + \frac{1}{2} \sin i \sin^2 \frac{i}{2} \sin 2i' \cos (2\omega - \Omega + \Omega') \\
& + \sin i \sin^2 \frac{i}{2} \sin i' \cos^2 \frac{i'}{2} \cos (2\lambda' + 2\omega - \Omega - \Omega') \\
& - \sin i \cos^2 \frac{i}{2} \sin i' \sin^2 \frac{i'}{2} \cos (2\lambda' + 2\omega + \Omega - 3\Omega') \\
& + \sin i \sin^2 \frac{i}{2} \sin i' \sin^2 \frac{i'}{2} \cos (2\lambda' - 3\Omega' - 2\omega + \Omega) \,.
\end{aligned}
$$

By selecting all terms depending on λ' and by replacing cos by sin, we can also derive expressions for A' and B', from A and B, respectively.

The variations of e and i are obtained from the equations

$$
\left\{
\begin{aligned}
\frac{de}{dt} &= - \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial \omega}, \\
\frac{di}{dt} &= \frac{\cos i}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial \omega} - \frac{1}{na^2 \sqrt{1 - e^2} \sin i} \frac{\partial R}{\partial \Omega}
\end{aligned}
\right. \tag{6}
$$

By using the variations δe and δi, $\delta\omega$ and $\delta\Omega$ can be derived from the formulae

$$\left\{\begin{aligned} \frac{d\omega}{dt} &= -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} \\ &\quad + \frac{d\dot{\omega}}{de}\delta e + \frac{d\dot{\omega}}{di}\delta i, \\ \frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} + \frac{d\dot{\Omega}}{de}\delta e + \frac{d\dot{\Omega}}{di}\delta i, \end{aligned}\right. \tag{7}$$

where

$$\dot{\omega} = \frac{A_2}{p^2} n \frac{4-5\sin^2 i}{2},$$

$$\dot{\Omega} = -\frac{A_2}{p^2} n \cos i .$$

It is remarkable that these disturbing functions do not affect the semimajor axis. In the right-hand sides of the equations, i', the inclination of the lunar path to the equator, is changing gradually, but we may regard it as constant during one year or so.

As for the secular terms of Ω and ω, we have

$$\left\{\begin{aligned} \frac{d\Omega}{dt} &= -\frac{3}{4}\frac{n'^2}{n}m'\frac{\cos i}{\sqrt{1-e^2}}\left(1+\frac{3}{2}e^2\right)\left(1-\frac{3}{2}\sin^2 i'\right), \\ \frac{d\omega}{dt} &= \frac{3}{4}\frac{n'^2}{n}m'\frac{1}{\sqrt{1-e^2}}\left(2-\frac{5}{2}\sin^2 i+\frac{1}{2}e^2\right)\left(1-\frac{3}{2}\sin^2 i'\right), \end{aligned}\right. \tag{8}$$

where

$$\sin^2 i' = \frac{1}{2}\sin^2 J(1+\cos^2 \epsilon) + \sin^2\epsilon\cos^2 J$$
$$+ \frac{1}{2}\sin 2\epsilon \sin 2J \cos N - \frac{1}{2}\sin^2 J \sin^2\epsilon \cos 2N .$$

Here J and N are the lunar inclination and the longitude of the ascending node referred to the ecliptic, and ϵ is the obliquity. We can find the values of J, N and λ' in the American Ephemeris.

If we set $m' = 1$ and $J = 0$, we can derive the solar perturbations from the same equations.