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THE POLARIZATION OF SYNCHROTRON RADIATION

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ABSTRACT

The spectral distribution and polarization of radiation from an electron gyrating ultra-relativistically in a magnetic field are calculated. The result is used to calculate the principal emissivities of a distribution of high-energy electrons such as occurs in the Crab Nebula. It is shown that the degree of polarization of the emission increases with frequency, from two-thirds at radio frequencies to unity. The depolarizing factors affecting the intensity of the emergent rays are briefly discussed.

I. INTRODUCTION

The synchrotron process by which radiation is generated has assumed considerable importance since the recent studies by Oort and Walraven (1956), Pikelner (1956), and Woltjer (1958). These have virtually confirmed Shklovsky's (1953) hypothesis that the light from the Crab Nebula is due to radiation from high-energy electrons gyrating in a magnetic field. This process can now be invoked to account for the non-thermal emission from other radio stars and from the galactic halo.

The most useful calculations of the radiation from charged particles moving with velocities close to the velocity of light (ultra-relativistic velocities) are those of Schwinger (1949). His procedure was to calculate the radiant power emitted, via the rate at which the particle does work on the field of which it is the source. The field vectors were not required, so that the polarization of the emission was left undetermined.

On general grounds it is to be expected that radiation so intimately connected with a magnetic field would exhibit polarization. Indeed, it was the discovery of strong polarization in the optical emission from the Crab Nebula by Vashakidze (1954) and Dombrovsky (1954) that sealed the adoption of the synchrotron hypothesis. Woltjer (1957) has derived the polarization and intensity distributions in the continuum from an analysis of plates taken by Baade (1956) on the 200-inch telescope through a polaroid filter in the wave-length range $\lambda\lambda$ 5400–6400. He found that both the direction and the degree of (linear) polarization varied systematically over the nebula, the latter from below 50 per cent in the central regions to as high as 80 per cent at the edges.

For the proper interpretation of these measures it is essential that the polarization properties of the emission from a single electron should first be known. Oort and Walraven state that the radiation is totally polarized, with the electric vector parallel to the radius of curvature of the orbit. That this cannot be so, for all directions of emission should become clear from the following considerations. It is well known that the emission from a charged particle moving with an ultra-relativistic velocity βc, where c is the free-space velocity of electromagnetic waves, is effectively restricted to a narrow cone whose angular dimensions are of the order of

$$\xi = \sqrt{(1 - \beta^2)}\ . \tag{1}$$

Within this region the electric vector of the acceleration field takes all directions transverse to the direction of emission. It is well known that for $\beta \ll 1$ the electric-field lines are at any instant meridians through the null points lying at the ends of a diameter through the charge in the direction of the acceleration at the corresponding retarded in-

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stant.[1] It can be shown that, as β approaches unity, the lines are distorted so that the null points lie at an angular distance ξ from the direction of motion, in the plane containing the directions of motion and acceleration. Hence it is to be expected that the total emission from a distribution of high-energy electrons, having various motions in a magnetic field, will not be completely polarized. It is the purpose of this paper to investigate the state of polarization of the resultant emission from such a distribution.

II. THE ELECTRON TRAJECTORY

The relativistic equation of motion of a charged particle, of charge e and mass m, in a uniform magnetic field of induction $\boldsymbol{B}_0$ is

$$\frac{d}{dt}\frac{m\boldsymbol{\beta}}{\sqrt{(1-\beta^2)}} = e\boldsymbol{\beta}\times\boldsymbol{B}_0 , \tag{2}$$

which has the integral $\beta =$ constant. It therefore reduces to the simple form

$$\dot{\boldsymbol{\beta}} = -\boldsymbol{\omega}_B \times \boldsymbol{\beta} \tag{3}$$

where

$$\boldsymbol{\omega}_B = \frac{e}{m}\sqrt{(1-\beta^2)}\,\boldsymbol{B}_0 \tag{4}$$

is a vector whose magnitude is the angular gyrofrequency for the particle in the field. For ultra-relativistic motions its value is considerably reduced from the classical value $|e|B_0/m$. A separation of equation (3) into components parallel and perpendicular to $\boldsymbol{B}_0$ gives

$$\dot{\boldsymbol{\beta}}_{||} = 0 ,$$

$$\dot{\boldsymbol{\beta}}_{\perp} = -\boldsymbol{\omega}_B \times \boldsymbol{\beta}_{\perp} ;$$

whence it appears that the motion consists of a uniform advance with speed $\beta_{||}c$ along a magnetic-field line together with a uniform circular motion of radius $\beta_{\perp}c/\omega_B$ with angular velocity $-\boldsymbol{\omega}_B$ about the direction of $\boldsymbol{B}_0$. For an electron ($e < 0$) the sense of gyration is right-handed with respect to the direction of $\boldsymbol{B}_0$. Thus the trajectory is, in general, a circular helix. If the origin of co-ordinates is taken as the projection from the initial position of the electron on to the axis of the helix and α is the constant angle between $\boldsymbol{\beta}$ and $\boldsymbol{B}_0$, its equation may be written

$$\frac{\boldsymbol{r}}{c} = \frac{\beta}{\omega_B}\left[(\boldsymbol{i}\cos\omega_B t + \boldsymbol{j}\sin\omega_B t)\sin\alpha + \boldsymbol{k}\omega_B t\cos\alpha\right] , \tag{5}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ form a right-handed orthogonal system of unit vectors such that $\boldsymbol{i}$ is in the direction of the initial position and $\boldsymbol{k}$ is in the direction of $\boldsymbol{B}_0$. The electron velocity, then, is

$$\dot{\boldsymbol{r}} = \beta c \boldsymbol{\tau} , \tag{6}$$

where

$$\boldsymbol{\tau} = \sin\alpha\,(-\boldsymbol{i}\sin\omega_B t + \boldsymbol{j}\cos\omega_B t) + \boldsymbol{k}\cos\alpha . \tag{7}$$

By equation (3) the acceleration is always in the direction of $\boldsymbol{k}\times\boldsymbol{\tau}$, which is along the principal normal to the trajectory. The radius of curvature has the constant value $\beta c/(\omega_B \sin\alpha)$.

[1] When the "electrostatic" field is taken into account, the lines representing the total field due to an electron are channeled into the charge, from infinity in the direction opposite to the acceleration and via the meridian lines up from the other null direction.

III. THE SPECTRUM OF THE RADIATION FROM A GYRATING ELECTRON

In calculating the radiation emitted from a gyrating electron it is sufficient to consider only the acceleration field. The electric and magnetic vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ at the point $\boldsymbol{r}$ and instant t are then given in terms of the position $r_1(t')$ of the electron at the retarded time t' by the relation

$$t' = t - \frac{R(t')}{c}, \tag{8}$$

where

$$\boldsymbol{R}(t') = \boldsymbol{r} - \boldsymbol{r}_1(t'). \tag{9}$$

Differentiation of equation (8) yields the time-interval ratio,

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \boldsymbol{\beta}' \cdot \boldsymbol{n}}, \tag{10}$$

where $\boldsymbol{n} = \boldsymbol{R}(t')/R$, and $\boldsymbol{\beta}' = \dot{\boldsymbol{r}}_1(t')/c$.

In terms of a rationalized system of units, electromagnetic or electrostatic, we have the formulae (cf. Landau and Lifshitz 1951, Sec. 9-7)

$$\boldsymbol{E} = \frac{\mu e c}{4\pi R} \frac{\boldsymbol{n} \times [(\boldsymbol{n} - \boldsymbol{\beta}') \times \dot{\boldsymbol{\beta}}']}{(1 - \boldsymbol{\beta}' \cdot n)^3}, \qquad \boldsymbol{B} = \frac{1}{c} \boldsymbol{n} \times \boldsymbol{E}, \tag{11}$$

where μ is the permeability of free space.

We note that the null directions of the acceleration field are where

$$\boldsymbol{n} = \boldsymbol{\beta}' \pm \xi' \frac{\dot{\boldsymbol{\beta}}'}{\dot{\beta}'}.$$

The field vectors are, in general, large when the denominator is small, which occurs when both ξ' is close to unity and ϑ, the angle between $\boldsymbol{\beta}'$ and $\boldsymbol{n}$, is small. More precisely, for emission from an ultra-relativistically moving electron in directions such that[2] $\vartheta = 0(\xi)$,

$$\boldsymbol{\beta}' = \boldsymbol{\tau}\, [1 - \tfrac{1}{2}\xi^2 + 0(\xi^4)],$$

$$1 - \boldsymbol{\beta}' \cdot \boldsymbol{n} = \tfrac{1}{2}(\xi^2 + \vartheta^2) + 0(\xi^4),$$

so that the denominator is of the sixth order; whereas in the numerator, since $|\boldsymbol{n} - \boldsymbol{\tau}| = 0(\vartheta)$,

$$\boldsymbol{n} \times [(\boldsymbol{n} - \boldsymbol{\beta}') \times \dot{\boldsymbol{\beta}}'] = (\boldsymbol{n} - \boldsymbol{\tau})\, \dot{\boldsymbol{\beta}}' \cdot \boldsymbol{\tau}$$

$$+ \tfrac{1}{2}\xi^2 \boldsymbol{\tau} \dot{\boldsymbol{\beta}}' \cdot \boldsymbol{\tau} + (\boldsymbol{n} - \boldsymbol{\tau})\, \dot{\boldsymbol{\beta}}' \cdot (\boldsymbol{n} - \boldsymbol{\tau}) - \tfrac{1}{2}(\xi^2 + \vartheta^2)\, \dot{\boldsymbol{\beta}}' + 0(\xi^3).$$

In the present case of a gyrating electron,

$$\dot{\boldsymbol{\beta}}' = \omega_B \boldsymbol{k} \times \boldsymbol{\tau}\, [1 + 0(\xi^2)], \tag{12}$$

so that $\dot{\boldsymbol{\beta}}' \cdot \boldsymbol{\tau} = 0$ and the field vectors are of magnitude $0(\xi^{-4})$. For $\vartheta \gg \xi$, they are of magnitude $0(1)$.

[2] In the following it will be remarked that we have dropped the primes from the source-point quantities ξ' and $\boldsymbol{\tau}'$.

Since the electron is executing a periodic motion, it emits radiation whose spectrum consists of a superposition of harmonics of the fundamental frequency $\omega_B/2\pi$. The amplitude $\boldsymbol{E}_n$ of the nth harmonic of the electric vector,

$$\boldsymbol{E}(\boldsymbol{r}, t) = \sum_{-\infty}^{\infty} \boldsymbol{E}_n \exp(in\omega_B t), \tag{13}$$

is given by

$$\boldsymbol{E}_n(\boldsymbol{r}) = \frac{\omega_B}{2\pi}\int_0^{2\pi/\omega_B} \boldsymbol{E} \exp(in\omega_B t)\, dt. \tag{14}$$

The evaluation of this integral is facilitated by the two circumstances that the distance $R(t')$ from the charge is large compared with the dimensions of the region in which the motion takes place and that the motion is ultra-relativistic.

In the first place, we may take the reference origin close to the position of the charge during the time of observation. Then we have

$$\boldsymbol{R}(t') = \boldsymbol{r} - \boldsymbol{r}_1(t'), \qquad r_1 \ll r,$$

so that

$$R = r - \boldsymbol{n}\cdot\boldsymbol{r}_1. \tag{15}$$

With these approximations and transforming to the particle time t', we get

$$\begin{aligned} \boldsymbol{E}_n &= \frac{\omega_B}{2\pi} \exp\left(in\omega_B \frac{r}{c}\right)\int_0^{2\pi/\omega_B} \boldsymbol{E} \exp\left[in\omega_B\left(t' - \boldsymbol{n}\cdot\frac{\boldsymbol{r}_1}{c}\right)\right]\frac{\partial t}{\partial t'}\, dt', \\ &= \frac{\mu e c}{8\pi^2 r}\,\omega_B \exp\left(in\omega_B \frac{r}{c}\right)\int_0^{2\pi/\omega_B} \frac{\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta}')\times\dot{\boldsymbol{\beta}}']}{(1-\boldsymbol{\beta}'\cdot\boldsymbol{n})^2} \exp\left[in\omega_B\left(t'-\boldsymbol{n}\cdot\frac{\boldsymbol{r}_1}{c}\right)\right] dt' \end{aligned} \tag{16}$$

on substitution from equations (10) and (11). Into the integrand we may substitute the approximations obtained above.

Since for the purposes of observation we are interested only in the average radiant power received at any point, we may take $\boldsymbol{n}$ in the plane containing the directions of the field $\boldsymbol{B}_0$ and the initial velocity of the charge. Then if ψ is the angle between $\boldsymbol{n}$ and $\boldsymbol{\tau}(0)$, in the direction toward $\boldsymbol{k}$, as in Figure 1, and $\chi = \omega_B t'$,

$$\boldsymbol{n} = \boldsymbol{j}\sin(\alpha-\psi) + \boldsymbol{k}\cos(\alpha-\psi),$$

and

$$\frac{\boldsymbol{r}_1}{c} = \frac{\beta'}{\omega_B}[\sin\alpha(\boldsymbol{i}\cos\chi + \boldsymbol{j}\sin\chi) + \boldsymbol{k}\chi\cos\alpha],$$

$$\boldsymbol{\tau} = \sin\alpha(-\boldsymbol{i}\sin\chi + \boldsymbol{j}\cos\chi) + \boldsymbol{k}\cos\alpha,$$

by equations (5) and (7). We have seen that contributions to the integral from points corresponding to values of ϑ greater than $0(\xi)$ are insignificant. Hence we may replace the integrand by its approximate value for small ψ and χ and replace the terminals by $\pm\infty$. Then, to the first order, we get

$$\boldsymbol{n} - \boldsymbol{\tau} = \boldsymbol{i}\chi\sin\alpha - \psi(\boldsymbol{j}\cos\alpha - \boldsymbol{k}\sin\alpha)$$

$$\boldsymbol{k}\times\boldsymbol{\tau} = -\sin\alpha(\boldsymbol{i}+\boldsymbol{j}\chi),$$

and

$$\vartheta^2 = \psi^2 + \chi^2\sin^2\alpha.$$

The exponent turns out to be a small quantity of the third order,

$$\omega_B \left(t' - \boldsymbol{n} \cdot \frac{\boldsymbol{r}_1}{c}\right) = \tfrac{1}{2} \left(\xi^2 + \psi^2\right) \chi + \tfrac{1}{6} \chi^3 \sin^2 \alpha \,.$$

Substituting from these approximations in equation (16), we get

$$\boldsymbol{E}_n = \frac{\mu\, e\, c \omega_B \sin \alpha}{4 \pi^2 r} \exp\left(i n \omega_B \frac{r}{c}\right) \int_{-\infty}^{\infty} \exp\left[\tfrac{1}{2} i n \chi \left(\xi^2 + \psi^2 + \tfrac{1}{3} \chi^2 \sin^2 \alpha\right)\right]$$
$$\times \frac{\left(\xi^2 + \psi^2 - \chi^2 \sin^2 \alpha\right) \boldsymbol{i} + 2 \psi \chi \sin \alpha \left(\boldsymbol{j} \cos \alpha - \boldsymbol{k} \sin \alpha\right)}{\left(\xi^2 + \psi^2 + \chi^2 \sin^2 \alpha\right)^2} d\chi \,, \tag{17}$$

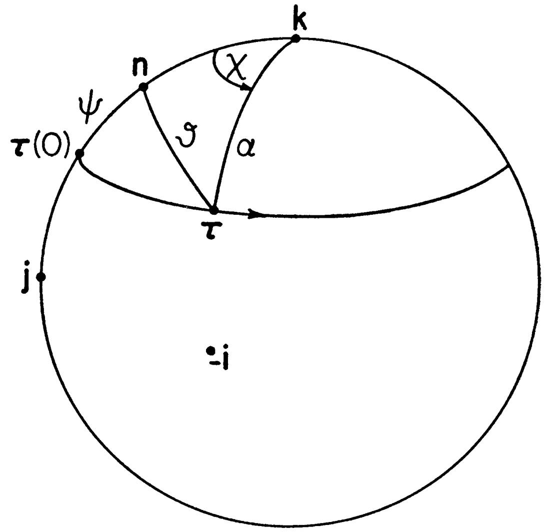

FIG. 1.—The directions of motion and of observation relative to the magnetic field

whose form suggests the possibility of expressing $\boldsymbol{E}_n$ in terms of Airy functions $Ai(x)$ or modified Bessel functions $K_\nu(x)$ of order $\frac{1}{3}$. The Airy function is defined by the integral

$$Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(x u + \tfrac{1}{3} u^3\right) du$$
$$= \frac{1}{\pi \sqrt{3}} x^{1/2} K_{1/3}\left(\tfrac{2}{3} x^{3/2}\right) ,$$

and the integrals involved in equation (17) are of the form

$$\phi_1(\eta) = \int_{-\infty}^{\infty} \exp\left[i \gamma \left(\eta^2 u + \tfrac{1}{3} u^3\right)\right] \frac{2u}{\left(\eta^2 + u^2\right)^2} du \,,$$

$$\phi_2(\eta) = \int_{-\infty}^{\infty} \exp\left[i \gamma \left(\eta^2 u + \tfrac{1}{3} u^3\right)\right] \frac{\eta^2 - u^2}{\left(\eta^2 + u^2\right)^2} du \,.$$

The first integral is easily resolved by observing that the second factor in the integrand is proportional to the derivative of the reciprocal of the derivative of the exponent. Then, on integration by parts, we get

$$\phi_1(\eta) = i \gamma F_\gamma(\eta) \,,$$

where

$$F_\gamma(\eta) = \int_{-\infty}^{\infty} \exp\left[i\gamma\left(\eta^2 u + \tfrac{1}{3}u^3\right)\right] du\,.$$

Again, on applying the same procedure to the integral whose integrand is u times that of $\phi_1(\eta)$, we get

$$\phi_2(\eta) = -\frac{1}{2\eta} F'_\gamma(\eta)\,.$$

Thus equation (17) becomes

$$E_n = \frac{\mu e c}{4\pi^2 r} \exp\left(i n \omega_B \frac{r}{c}\right)\left[-\frac{1}{2\eta} F'_\gamma(\eta)\, \boldsymbol{i} + i\gamma\psi F_\gamma(\eta)\left(\boldsymbol{j}\cos\alpha - \boldsymbol{k}\sin\alpha\right)\right], \tag{18}$$

where

$$\eta = \sqrt{(\xi^2+\psi^2)}\,, \qquad \gamma = \frac{n}{2\sin\alpha}\,.$$

Now

$$\begin{aligned} F_\gamma(\eta) &= 2\pi\gamma^{-1/3} Ai\left(\gamma^{2/3}\eta^2\right) \\ &= \frac{2}{\sqrt{3}}\,\eta K_{1/3}\left(\tfrac{2}{3}\gamma\eta^3\right). \end{aligned}$$

It is more satisfactory to use the representation in terms of Bessel functions, which offers the advantages of various recurrence relations. In particular,

$$\begin{aligned} F'_\gamma(\eta) &= \frac{2}{\sqrt{3}}\left(K_{1/3} + 2\gamma\eta^3 K'_{1/3}\right) \\ &= -\frac{4}{\sqrt{3}}\,\gamma\eta^3 K_{2/3}\left(\tfrac{2}{3}\gamma\eta^3\right), \end{aligned}$$

since

$$K'_{1/3}(x) + \frac{1}{3x} K_{1/3}(x) = -K_{2/3}(x)\,.$$

Hence equation (18) becomes

$$\begin{aligned} E_n = \frac{\mu e c \omega_B}{4\sqrt{3}\pi^2 r} \exp\left(i n \omega_B \frac{r}{c}\right) \frac{n}{\sin\alpha} \Big\{ &(\xi^2+\psi^2)\, K_{2/3}\left[\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{3/2}\right] \boldsymbol{i} \\ &+ i\psi(\xi^2+\psi^2)^{1/2} K_{1/3}\left[\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{3/2}\right](\boldsymbol{j}\cos\alpha - \boldsymbol{k}\sin\alpha) \Big\}, \end{aligned} \tag{19}$$

which is a function of the angle ψ between the direction of observation $\boldsymbol{n}$ and the closest generator of the cone containing the directions of motion $\boldsymbol{\tau}$. Since, to a first approximation,

$$\boldsymbol{j}\cos\alpha - \boldsymbol{k}\sin\alpha = -\boldsymbol{i}\times\boldsymbol{n}\,,$$

the ratio of the two terms within the braces determines the complex polarization of the nth harmonic,

$$Q_n = \frac{\boldsymbol{E}_n\cdot\boldsymbol{i}}{\boldsymbol{E}_n\cdot\boldsymbol{i}\times\boldsymbol{n}}\,,$$

which determines the characteristics of the polarization ellipse. Thus

$$Q_n(\psi) = i\,\frac{(\xi^2+\psi^2)^{1/2}}{\psi}\,\frac{K_{2/3}\left[n/(3\sin\alpha)(\xi^2+\psi^2)^{3/2}\right]}{K_{1/3}\left[n/(3\sin\alpha)(\xi^2+\psi^2)^{3/2}\right]}\,, \tag{20}$$

from which, since both Bessel functions are real-positive, we conclude (cf. Westfold 1959*a*) that the polarization is elliptic, with the axes of the ellipse along directions parallel and perpendicular to the projection of $\boldsymbol{B}_0$ on the plane transverse to $\boldsymbol{n}$. The ratio of the axes is given by $|Q_n|$, with the major axis along the direction of $\boldsymbol{i} \times \boldsymbol{n}$ or $\boldsymbol{i}$ according as $|Q_n| \lesseqgtr 1$; the direction of description of the ellipse is *RH* or *LH* according as $\psi \gtrless 0$. Oort and Walraven's (1956) statement that the polarization is linear with the electric vector parallel to the direction of $\boldsymbol{k} \times \boldsymbol{\tau}(0)$, i.e., parallel to $\boldsymbol{i}$, is seen to be true only when $\psi = 0$.

Since these results depend only on the motion while $\boldsymbol{\tau}$ is within a small neighborhood of $\boldsymbol{n}$, they are also applicable to the radiation from any charge whose acceleration is instantaneously perpendicular to its velocity. The periodicity of the motion in the present case can have no significance. In confirmation of this conclusion, we shall see that most of the radiation is emitted in the range of the spectrum close to the critical frequency f_c. In this region $n = 0(\xi^{-3})$, so that the line spectrum has become practically continuous.

IV. THE SPECTRAL DISTRIBUTION OF THE EMISSION

Since the emitting charge executes periodic motion, the radiated power is distributed among the harmonics of the fundamental frequency. Being interested only in the average power radiated over a period, we may apply Parseval's theorem to the formula

$$\boldsymbol{S} = \boldsymbol{E} \times \frac{\boldsymbol{B}}{\mu} \tag{21}$$

for the Poynting vector. Then, by equation (11), the average

$$\begin{aligned}\langle \boldsymbol{S} \rangle &= \frac{\omega_B}{2\pi} \int_0^{2\pi/\omega_B} \boldsymbol{S} dt \\ &= \frac{\omega_B}{2\pi} \frac{\boldsymbol{n}}{\mu c} \int_0^{2\pi/\omega_B} E^2 dt \\ &= \frac{\boldsymbol{n}}{\mu c} \left(E_0^2 + 2 \sum_1^{\infty} |E_n|^2 \right).\end{aligned}$$

This represents the average flux density of the radiation, such that $\langle \boldsymbol{S} \rangle \cdot \boldsymbol{n} dS$ is the average power flowing normally through the surface element dS distant r from the origin. Thus, if $\langle P(\boldsymbol{n}) \rangle d\Omega(\boldsymbol{n})$ is the average power radiated into the solid angle $d\Omega(\boldsymbol{n})$ subtended by dS and we write

$$\langle P(\boldsymbol{n}) \rangle = \tfrac{1}{2} P_0(\boldsymbol{n}) + \sum_1^{\infty} \langle P_n(\boldsymbol{n}) \rangle ,$$

the average power in the nth harmonic radiated into the solid angle $d\Omega(\boldsymbol{n})$ is $\langle P_n(\boldsymbol{n}) \rangle d\Omega(\boldsymbol{n})$, where

$$\langle P_n(\boldsymbol{n}) \rangle = \frac{2}{\mu c} |E_n|^2 r^2 . \tag{22}$$

It can be seen from equation (19) that $\langle P_n(\boldsymbol{n}) \rangle$ can be resolved into two contributions corresponding to the components of $\boldsymbol{E}_n$ parallel to the directions of $\boldsymbol{i}$ and $\boldsymbol{i} \times \boldsymbol{n}$, for all values of ψ. We shall distinguish such component contributions by the superscripts (2) and (1), respectively, representing intensity components perpendicular and parallel to the projection of $\boldsymbol{B}_0$ in the plane normal to $\boldsymbol{n}$. Thus

$$\begin{aligned}\langle P_n^{(1)}(\boldsymbol{n})\rangle &= \frac{\mu e^2 c\omega_B^2}{24\pi^4}\frac{n^2}{\sin^2\alpha}\psi^2(\xi^2+\psi^2)K_{1/3}^2\left[\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{3/2}\right],\\ \langle P_n^{(2)}(\boldsymbol{n})\rangle &= \frac{\mu e^2 c\omega_B^2}{24\pi^4}\frac{n^2}{\sin^2\alpha}(\xi^2+\psi^2)^2K_{2/3}^2\left[\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{3/2}\right].\end{aligned} \tag{23}$$

The power radiated in the nth harmonic is effectively confined to a small range $0(\xi)$ of ψ. Hence, in calculating the average power radiated in all directions in the nth harmonic, we may take

$$\begin{aligned}\langle P_n\rangle &= \int\langle P_n(\boldsymbol{n})\rangle d\Omega\\ &= 2\pi\sin\alpha\int_{-\infty}^{\infty}\langle P_n(\boldsymbol{n})\rangle d\psi.\end{aligned} \tag{24}$$

In calculating $\langle P_n^{(1)}\rangle$ and $\langle P_n^{(2)}\rangle$ it is simplest to start from the more primitive form (18). Then

$$\langle P_n^{(1)}(\boldsymbol{n})\rangle = \frac{\mu e^2 c\omega_B^2}{8\pi^4}\gamma^2\psi^2\int\int_{-\infty}^{\infty}\exp\{i\gamma[\eta^2(u-v)+(u^3-v^3)]\}du\,dv,$$

$$\langle P_n^{(2)}(\boldsymbol{n})\rangle = \frac{\mu e^2 c\omega_B^2}{8\pi^4}\gamma^2\int\int_{-\infty}^{\infty}\exp\{i\gamma[\eta^2(u-v)+(u^3-v^3)]\}\,uv\,du\,dv.$$

It is convenient to change the variables of integration u and v to x and y, given by

$$2x = u - v, \qquad 2y = u + v.$$

Then

$$\langle P_n^{(1)}(\boldsymbol{n})\rangle = \frac{\mu e^2 c\omega_B^2}{4\pi^4}\gamma^2\psi^2\int_{-\infty}^{\infty}\exp[2i\gamma(\eta^2x+\tfrac{1}{3}x^3)]\,dx\int_{-\infty}^{\infty}\exp(2i\gamma xy^2)\,dy,$$

$$\langle P_n^{(2)}(\boldsymbol{n})\rangle = \frac{\mu e^2 c\omega_B^2}{4\pi^4}\gamma^2\int_{-\infty}^{\infty}\exp[2i\gamma(\eta^2x+\tfrac{1}{3}x^3)]\,dx\int_{-\infty}^{\infty}(y^2-x^2)\exp(2i\gamma xy^2)\,dy.$$

Since

$$\int_{-\infty}^{\infty}\exp(2i\gamma xy^2)\,dy = e^{\pi i/4}\sqrt{\pi}(2\gamma x)^{-1/2},$$

$$\int_{-\infty}^{\infty}\exp(2i\gamma xy^2)\,y^2dy = -\tfrac{1}{2}e^{-\pi i/4}\sqrt{\pi}(2\gamma x)^{-3/2},$$

in which the principal value is denoted by the fractional powers of x, we get the integral formulae

$$\langle P_n^{(1)}(\boldsymbol{n})\rangle = \frac{\mu e^2 c\omega_B^2}{16\pi^3\sqrt{\pi}}(2\gamma)^{3/2}e^{\pi i/4}\psi^2\int_{-\infty}^{\infty}\exp[2i\gamma(\eta^2x+\tfrac{1}{3}x^3)]\,x^{-1/2}dx,$$

$$\langle P_n^{(2)}(\boldsymbol{n})\rangle = -\frac{\mu e^2 c\omega_B^2}{16\pi^3\sqrt{\pi}}(2\gamma)^{3/2}e^{\pi i/4}\int_{-\infty}^{\infty}\exp[2i\gamma(\eta^2x+\tfrac{1}{3}x^3)]\left(x^{3/2}+\frac{x^{-3/2}}{4i\gamma}\right)dx,$$

which are to be interpreted as Cauchy principal values in respect of the singularities of the integrands at $x = 0$. We now substitute from these formulae in equation (24), recalling that $\eta^2 = \xi^2 + \psi^2$. Then the integrations with respect to ψ yield the following results:

$$\langle P_n^{(1)} \rangle = -i \frac{\mu e^2 c \omega_B^2 \sin \alpha}{8\pi^2} \gamma \int_{-\infty}^{\infty} \exp\left[2i\gamma\left(\xi^2 x + \tfrac{1}{3}x^3\right)\right] x^{-2} \frac{dx}{2i\gamma},$$

$$\langle P_n^{(2)} \rangle = -i \frac{\mu e^2 c \omega_B^2 \sin \alpha}{8\pi^2} 2\gamma \int_{-\infty}^{\infty} \exp\left[2i\gamma\left(\xi^2 x + \tfrac{1}{3}x^3\right)\right] x\,dx + \langle P_n^{(1)} \rangle .$$

These integrals can be expressed in terms of Bessel functions in the following manner. Integration by parts gives

$$\int_{-\infty}^{\infty} \exp\left[2i\gamma\left(\xi^2 x + \tfrac{1}{3}x^3\right)\right] x^{-2} \frac{dx}{2i\gamma} = \int_{-\infty}^{\infty} \exp\left[2i\gamma\left(\xi^2 x + \tfrac{1}{3}x^3\right)\right] \frac{\xi^2 + x^2}{x}\, dx .$$

Now

$$\int_{-\infty}^{\infty} \exp\left[2i\gamma\left(\xi^2 x + \tfrac{1}{3}x^3\right)\right] x\,dx = \frac{1}{4i\gamma\xi} F'_{2\gamma}(\xi) = \frac{2i}{\sqrt{3}} \xi^2 K_{2/3}\left(\tfrac{4}{3}\gamma\xi^3\right),$$

and

$$\frac{d}{d\xi} \int_{-\infty}^{\infty} \exp\left[2i\gamma\left(\xi^2 x + \tfrac{1}{3}x^3\right)\right] x^{-1} dx = 4i\gamma\xi F_{2\gamma}(\xi) = \frac{8i}{\sqrt{3}} \gamma \xi^2 K_{1/3}\left(\tfrac{4}{3}\gamma\xi^3\right).$$

Integrating the latter between the limits ξ and ∞, we get

$$\int_{-\infty}^{\infty} \exp\left[2i\gamma\left(\xi^2 x + \tfrac{1}{3}x^3\right)\right] x^{-1} dx = -\frac{2i}{\sqrt{3}} \int_{4\gamma\xi^3/3}^{\infty} K_{1/3}(\eta)\, d\eta .$$

Finally, on substitution from these results and making use of the recurrence relation,

$$-2K'_{2/3}(x) = K_{5/3}(x) + K_{1/3}(x),$$

we get

$$\langle P_n^{(1)} \rangle = \frac{\sqrt{3}\mu e^2 c \omega_B^2 \sin \alpha}{8\pi^2 \xi} F^{(1)}\left(\frac{n}{n_c}\right),$$
$$\langle P_n^{(2)} \rangle = \frac{\sqrt{3}\mu e^2 c \omega_B^2 \sin \alpha}{8\pi^2 \xi} F^{(2)}\left(\frac{n}{n_c}\right), \tag{25}$$

where

$$F^{(1)}(x) = \tfrac{1}{2}x\left[\int_x^{\infty} K_{5/3}(\eta)\, d\eta - K_{2/3}(x)\right],$$
$$F^{(2)}(x) = \tfrac{1}{2}x\left[\int_x^{\infty} K_{5/3}(\eta)\, d\eta + K_{2/3}(x)\right], \tag{26}$$

and

$$n_c = \frac{3 \sin \alpha}{2\xi^3}. \tag{27}$$

Since they believed that all the radiation emitted belonged to $\langle P_n^{(2)} \rangle$, Oort and Walraven gave a graph and table of values of the function

$$F(x) = F^{(1)}(x) + F^{(2)}(x) = x \int_x^\infty K_{5/3}(\eta)\, d\eta\,, \tag{28}$$

only. For large x we have the asymptotic formula,

$$F(x) \sim \sqrt{\left(\frac{\pi}{2}\right)}\, e^{-x} x^{1/2} \left(1 + \frac{55}{72x} - \frac{10151}{10368x^2}\right). \tag{28'}$$

TABLE 1

$$F(x) = x \int_x^\infty K_{5/3}(\eta)\, d\eta$$

$$F_p(x) = x K_{2/3}(x)$$

x	F	F_p	x	F	F_p
0	0	0	1 .0	0.655	0.494
0.001	0 213	0.107	1 2	.566	.439
.005	.358	.184	1 4	.486	.386
.010	.445	.231	1 6	.414	.336
.025	.583	.312	1 8	354	.290
.050	.702	.388	2.0	.301	.250
.075	.772	.438	2 5	.200	.168
.10	.818	.475	3 0	.130	.111
.15	.874	.527	3 5	0845	.0726
.20	.904	560	4 0	0541	0470
25	.917	.582	4 5	0339	0298
.30	.919	.596	5 0	0214	.0192
.40	.901	.607	6 0	0085	0077
.50	.872	603	7 0	0033	0031
.60	.832	590	8 0	0013	0012
.70	.788	.570	9.0	00050	00047
.80	.742	.547	10.0	0.00019	0 00018
0.90	0 694	0 521			

It is represented in Table 1 and Figure 2, together with the complementary function,

$$F_p(x) = F^{(2)}(x) - F^{(1)}(x) = x K_{2/3}(x)\,, \tag{29}$$

whose asymptotic form is given by

$$F_p(x) \sim \sqrt{\left(\frac{\pi}{2}\right)}\, e^{-x} x^{1/2} \left(1 + \frac{7}{72x} - \frac{455}{10368x^2}\right). \tag{29'}$$

The values of $F(x)$ up to $x = 5.0$ are taken from Oort and Walraven's paper (save that the value for $x = 4$ has been altered from 0.0522). Since $F(x)$ has its greatest values where $x = 0(1)$, it follows that most of the radiation is emitted in harmonics $n \simeq n_c$, the critical value. The spectral character of the emission is thus dependent on the energy $\mathcal{E} = mc^2/\xi$ of the electron. In this range of n the harmonics are so closely spaced that the radiation is quasi-continuous. If we write $\langle P_f \rangle df$ for the mean power radiated in the band $(f, f + df)$ and

$$f = \frac{n\omega_B}{2\pi}, \qquad f_c = \frac{n_c\omega_B}{2\pi},$$

we have

$$\langle P_f \rangle df = \frac{\langle P_n \rangle df}{f_B},$$

so that

$$\langle P_f \rangle = \langle P_f^{(1)} \rangle + \langle P_f^{(2)} \rangle , \tag{30}$$

where

$$\langle P_f^{(1)} \rangle = \tfrac{1}{2}\sqrt{3}\,\mu e^2 c f_{B_0} \sin\alpha F^{(1)}\left(\frac{f}{f_c}\right),$$

$$\langle P_f^{(2)} \rangle = \tfrac{1}{2}\sqrt{3}\,\mu e^2 c f_{B_0} \sin\alpha F^{(2)}\left(\frac{f}{f_c}\right), \tag{31}$$

and

$$f_c = 3 f_{B_0} \frac{\sin\alpha}{2\xi^2}, \qquad f_{B_0} = \frac{eB_0}{2\pi m}. \tag{32}$$

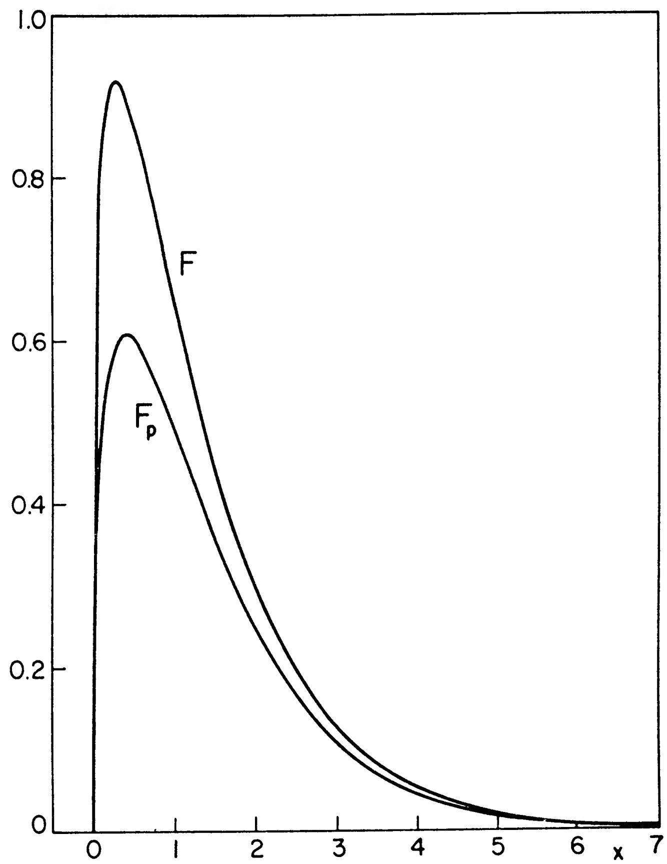

FIG. 2.—The functions $F(x)$ and $F_p(x)$

For B_0 in gauss and a radiating electron,

$$\langle P_f \rangle = 2.34 \times 10^{-25} B_0 \sin \alpha F\left(\frac{f}{f_c}\right) \text{ watt m}^{-2}\,(\text{c/s})^{-1},$$

$$f_c = 4.20 B_0 \frac{\sin \alpha}{\xi^2} \text{ Mc/s}, \qquad f_{B_0} = 2.80 B_0 \text{ Mc/s},$$

with appropriate suffixes where necessary. Thus, for $B_0 = 10^{-4}$ gauss and $f_c = 100$ Mc/s, we must have $\xi \simeq 10^{-3}$.

Finally, as a check on the calculations made in this section, we should show that

$$\int_0^\infty \langle P_f \rangle df = P(t), \tag{33}$$

the total power radiated in all directions. In the present case this is given by the approximate formula (cf. Landau and Lifshitz 1951; and eq. [12]),

$$P(t) = \frac{\mu e^2 c}{6\pi} \frac{\omega_B^2 \sin^2 \alpha}{\xi^4}, \tag{34}$$

so that equation (33) is equivalent to the mathematical result,

$$\int_0^\infty F(x)\, dx = \frac{8\pi}{9\sqrt{3}}.$$

This may be verified by application of the result (see Appendix)

$$\int_0^\infty x^{s-1} \int_x^\infty K_{\nu+1}(y)\, dy dx = \frac{\nu+s}{s} \int_0^\infty x^{s-1} K_\nu(x)\, dx, \qquad \Re s > 0,$$

together with the formula

$$\int_0^\infty x^{s-1} K_\nu(x)\, dx = 2^{s-2} \Gamma\left(\tfrac{1}{2}s - \tfrac{1}{2}\nu\right) \Gamma\left(\tfrac{1}{2}s + \tfrac{1}{2}\nu\right), \qquad \Re s > |\Re \nu|. \tag{35}$$

V. THE EMISSION FROM A DISTRIBUTION OF GYRATING ELECTRONS

At a given field point the synchrotron radiation received from a small volume containing a distribution of ultra-relativistically gyrating electrons effectively originates in the group whose velocities have directions within the conical annulus for which α is within an angular distance $0(\xi)$ of θ, the angle between $\boldsymbol{n}$ and $\boldsymbol{B}_0$. Each member of the group contributes the amounts $\langle P_n^{(1)}(\boldsymbol{n}) \rangle$, $\langle P_n^{(2)}(\boldsymbol{n}) \rangle$, which depend on both the energy,

$$\mathcal{E} = \frac{\mathcal{E}_0}{\xi}, \qquad \mathcal{E}_0 = m c^2, \tag{36}$$

and the small angle $\psi = \alpha - \theta$. The contribution to the emissivity in the direction $\boldsymbol{n}$ from members of the group in the same energy range $(\mathcal{E}, \mathcal{E} + d\mathcal{E})$ is obtained by integrating the incoherent contributions over the range of α. In general, it may be assumed that the distribution of velocities is uniform with respect to α. Then it is clear that the integrations result in two linearly polarized contributions $\langle P_n^{(1)} \rangle / 4\pi$, $\langle P_n^{(2)} \rangle / 4\pi$, in which α is replaced by θ, in the orthogonal directions $\boldsymbol{i} \times \boldsymbol{n}$ and $\boldsymbol{i}$, respectively.

Let $N(\mathcal{E}/\mathcal{E}_0)d\mathcal{E}/\mathcal{E}_0$ be the local number density of electrons having energies within the range $(\mathcal{E}, \mathcal{E}+d\mathcal{E})$. Then the monochromatic emissivity η_f consists of the two oppositely polarized components $\eta_f^{(1)}$ and $\eta_f^{(2)}$ of the form

$$\eta_f^{(1)}(\boldsymbol{n}) = \frac{1}{4\pi}\int_0^\infty N\left(\frac{\mathcal{E}}{\mathcal{E}_0}\right)\left\langle P_f^{(1)}\left(\frac{f}{f_c}\right)\right\rangle \frac{d\mathcal{E}}{\mathcal{E}_0}, \qquad \text{etc.}, \tag{37}$$

where, by equations (36) and (32),

$$f_c = \tfrac{3}{2} f_{B_0} \sin\theta \left(\frac{\mathcal{E}}{\mathcal{E}_0}\right)^2. \tag{38}$$

Substitution from equation (31) with an appropriate change of variable then yields the result

$$\eta_f^{(1)}(\boldsymbol{n}) = \frac{\mu e^2 c}{8\sqrt{2\pi}}(f f_{B_0}\sin\theta)^{1/2}\int_0^\infty N\left(\left[\frac{2f}{3 f_{B_0}\sin\theta}\right]^{1/2} x^{-1/2}\right) x^{-3/2} F^{(1)}(x)\,dx, \tag{39}$$

etc.

The total emissivity is given by

$$\eta_f = \eta_f^{(1)} + \eta_f^{(2)}, \tag{40}$$

of which the part

$$\eta_f^{(p)} = \eta_f^{(2)} - \eta_f^{(1)} \tag{41}$$

is polarized in the direction of $\boldsymbol{i}$, which is perpendicular to the projection of $\boldsymbol{B}_0$ on the plane transverse to $\boldsymbol{n}$. By equations (28) and (29), η_f and $\eta_f^{(p)}$ are given by formulae like (39), with $F(x)$ and $F_p(x)$, respectively, in the integrand. The degree of polarization of the radiation emitted, then, is

$$p_f = \frac{\eta_f^{(p)}}{\eta_f}. \tag{42}$$

These results are simplified if, as is frequently the case, the energy spectrum of the emitting electrons can be represented by a power law with cutoffs at energies $\mathcal{E}_1$ and $\mathcal{E}_2$. Then

$$N(x) = A x^{-\gamma}, \quad \frac{\mathcal{E}_1}{\mathcal{E}_0} \le x \le \frac{\mathcal{E}_2}{\mathcal{E}_0}, \qquad N(x) = 0, \quad x < \frac{\mathcal{E}_1}{\mathcal{E}_0}, \quad x > \frac{\mathcal{E}_2}{\mathcal{E}_0}, \tag{43}$$

where A is proportional to the local number density of all the electrons and the index γ is a constant. We then have, for the emissivity in a direction making an angle θ with the magnetic field $\boldsymbol{B}_0$ in which electrons are gyrating,

$$\eta_f(\boldsymbol{n}) = A\frac{\mu e^2 c}{8\sqrt{2\pi}}\left(\frac{3}{2}\right)^{\gamma/2}(f_{B_0}\sin\theta)^{(\gamma+1)/2} f^{-(\gamma-1)/2}\left[G\left(\frac{f}{f_{c2}}\right) - G\left(\frac{f}{f_{c1}}\right)\right], \tag{44}$$

where

$$G(x) = \int_x^\infty \xi^{(\gamma-3)/2} F(\xi)\,d\xi, \tag{45}$$

and a similar pair of formulae for $\eta_f^{(p)}$ and G_p. Both functions depend on the frequency of the radiation and the component of the magnetic field transverse to the direction of emission. The dependence is that of a simple power law when $\mathcal{E}_1$ and $\mathcal{E}_2$ are such that $f/f_{c2} \ll 1$ and $f/f_{c1} \gg 1$ in the range of frequencies of interest. Then the degree of polarization is a constant independent of the frequency and the magnetic field. For the Crab

Nebula, Woltjer (1958) finds that the radio-frequency spectrum is indeed given by a power law of index about -0.35, which corresponds to the value $\gamma = 1.7$. For radio frequencies, therefore, the expression in brackets in equation (44) becomes equal to the constant $G(0)$. Clearly, any departure from a power-law spectrum at higher frequencies can be attributed to the circumstance that the values of the argument f/f_{c2} have now become significant. In fact, it is found that the magnitude of the spectral index increases for wave lengths below 3000 A.

In order to find expressions for η_f and $\eta_f^{(p)}$, we therefore need to evaluate the functions

$$G(x) = \int_x^\infty \xi^{(\gamma-1)/2} \int_\xi^\infty K_{5/3}(\eta)\, d\eta\, d\xi\,, \tag{46}$$

$$G_p(x) = \int_x^\infty \xi^{(\gamma-1)/2} K_{2/3}(\xi)\, d\xi\,. \tag{47}$$

It can be shown (see Appendix) that

$$\int_x^\infty \xi^{s-1} \int_\xi^\infty K_{\nu+1}(\eta)\, d\eta\, d\xi = \frac{\nu+s}{s} \int_x^\infty \xi^{s-1} K_\nu(\xi)\, d\xi - \frac{x^s}{s}\left[\int_x^\infty K_{\nu+1}(\xi)\, d\xi - K_\nu(x)\right],$$

whence

$$G(x) = \frac{\gamma+7/3}{\gamma+1} G_p(x) - \frac{2x^{(\gamma-1)/2}}{\gamma+1}\left[F(x) - F_p(x)\right]\,. \tag{48}$$

Thus the emissivity and degree of polarization can be evaluated in terms of the tabulated functions $F(x)$, $F_p(x)$, and the function $G_p(x)$ given by equation (47). The value $G(0)$ is given by equation (35), viz.,

$$G_p(0) = 2^{(\gamma-3)/2} \Gamma\left(\frac{3\gamma-1}{12}\right) \Gamma\left(\frac{3\gamma+7}{12}\right), \qquad \gamma > \tfrac{1}{3}\,. \tag{49}$$

For other values of x the function cannot in general be evaluated in terms of tabulated functions.

There is, however, an exception in the particular case where $\gamma = \frac{5}{3}$. Then we may apply the formula

$$\int_x^\infty \xi^{1-\nu} K_\nu(\xi)\, d\xi = x^{1-\nu} K_{\nu-1}(x) \qquad \Re\nu < 1\,,$$

to get

$$G_p(x) = x^{1/3} K_{1/3}(x)\,, \tag{50}$$

whose asymptotic form is given by

$$G_p(x) \sim \sqrt{\left(\frac{\pi}{2}\right)}\, e^{-x} x^{-1/6} \left(1 - \frac{5}{72x} + \frac{385}{10368x^2}\right). \tag{50$'$}$$

It is fortuitous that this value of γ is applicable to observations of the Crab Nebula. Then

$$G(x) = \tfrac{3}{2} G_p(x) - \tfrac{3}{4} x^{1/3}\left[F(x) - F_p(x)\right]\,, \tag{51}$$

whose asymptotic form is given by

$$G(x) \sim \sqrt{\left(\frac{\pi}{2}\right)}\, e^{-x} x^{-1/6} \left(1 + \frac{43}{72x} - \frac{17375}{10368x^2}\right). \tag{51$'$}$$

The functions $G(x)$ and $G_p(x)$ for $\gamma = \frac{5}{3}$ are represented in Table 2 and Figure 3. The same general forms may be expected for neighboring values of γ. Then the emissivity is given by the formula

$$\eta_f(\boldsymbol{n}) = A \frac{\mu e^2 c}{8\sqrt{2}\pi}\left(\frac{3}{2}\right)^{5/6}(f_{B_0}\sin\theta)^{4/3} f^{-1/3} G\left(\frac{f}{f_{c2}}\right) \tag{52}$$

and the corresponding degree of polarization by

$$p_f(\boldsymbol{n}) = \frac{G_p(f/f_{c2})}{G(f/f_{c2})}, \tag{53}$$

TABLE 2

$$G(x) = \tfrac{3}{2} G_p(x) - \tfrac{3}{4} x^{1/3}[F(x) - F_p(x)]$$

$$G_p(x) = x^{1/3} K_{1/3}(x)$$

x	G	G_p	x	G	G_p
0.	2.531	1.688	2.0.	0.172	0.147
0.2.	1.585	1.158	2.5.	.097	.086
0.4.	1.170	0.888	3.0.	.056	.051
0.6.	0.891	0.696	4.0.	.019	.018
0.8.	0 690	0.551	5.0.	.0068	.0064
1 0.	0.537	0.438	6.0.	.0024	.0023
1.2.	0 425	0.351	7 0.	.00087	.00082
1 4.	0.338	0.281	8.0	.00031	.00029
1.6.	0.271	0.226	9.0.	.000112	.000106
1.8.	0.215	0.182	10.0.	0.000040	0.000038

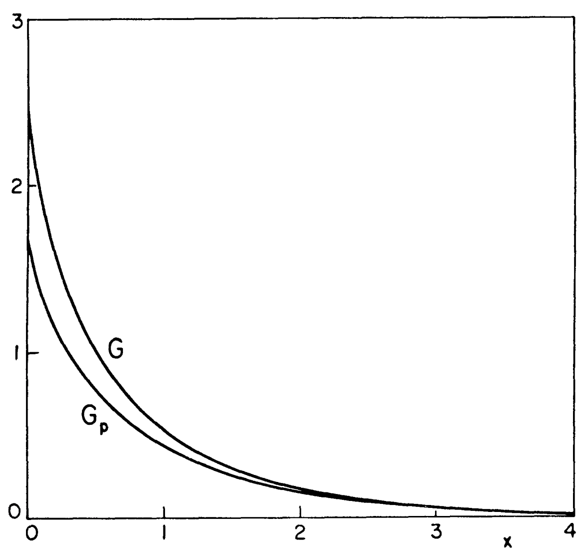

FIG. 3.—The functions $G(x)$ and $G_p(x)$ for $\gamma = \frac{5}{3}$

where, by equation (38),

$$\frac{f}{f_{c2}} = \frac{2f}{3f_{B_0}\sin\theta}\left(\frac{\mathcal{E}_0}{\mathcal{E}_2}\right)^2 . \tag{54}$$

The functional form of equation (53) is represented in Figure 4. For a given direction, the degree of polarization of the radiation emitted increases steadily with frequency, from the value $\frac{2}{3}$ asymptotically to the value 1.

VI. THE INTENSITY

The observed intensity of radiation emitted from a distribution of gyrating electrons is determined by the equation of transfer along a ray trajectory. Such a medium is

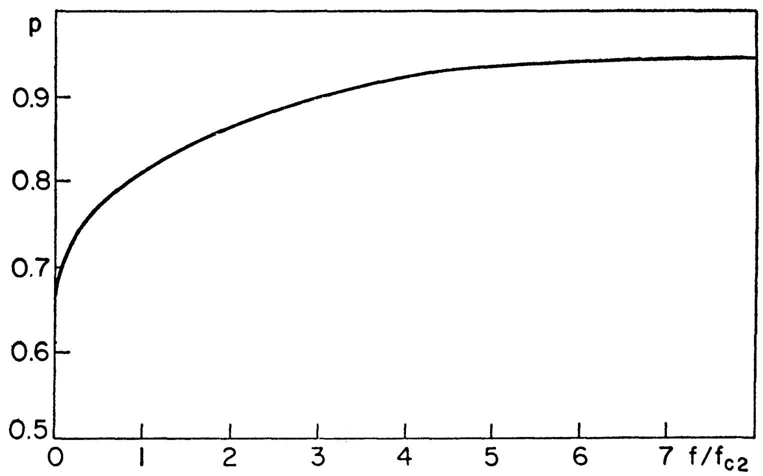

FIG. 4.—The degree of polarization of the emissivity for $\gamma = \frac{5}{3}$

usually so tenuous that the refractive index is unity and absorption[3] is negligible. Thus the trajectories are rectilinear, and the intensity is simply given by the integral along a trajectory,

$$I_f = \int \eta_f d s . \tag{55}$$

Everywhere along the trajectory there are contributions $\eta_f^{(1)}$ and $\eta_f^{(2)}$ polarized in the transverse plane, along the projection of the direction of $\boldsymbol{B}_0$ and in the perpendicular direction, respectively. As long as the orientation of $\boldsymbol{B}_0$ remains unchanged, the intensity will consist of similarly polarized components $I_f^{(1)}$ and $I_f^{(2)}$ given by integrals of the form (55). In particular, if A and the magnitude of $\boldsymbol{B}_0$ also remain constant, the degree of polarization of the ray will be given by equation (53). The direction of polarization measured optically over the Crab Nebula (Woltjer 1957) varies systematically, which indicates a definite structure in the magnetic-field distribution. However, the degree of polarization varies from values below 50 per cent in the denser central regions to as high as 80 per cent at the edges.

In an actual astronomical situation the rays emerging from an object may have traversed regions in which the magnetic field varies considerably in both magnitude and direction. This is more likely in rays from the center than in rays from the edges of the

[3] Given the spontaneous emission from an electron (Sec. IV), the absorption coefficient may be calculated from the Einstein coefficients (cf. Twiss 1954).

object. The over-all effect on an emergent ray is one of depolarization. Its resultant state of polarization is conveniently calculated by defining an emissivity polarization tensor,

$$\mathfrak{n}_f = \eta_f^{(1)} \boldsymbol{i}_1 \boldsymbol{i}_1 + \eta_f^{(2)} \boldsymbol{i}_2 \boldsymbol{i}_2 \ , \tag{56}$$

where

$$\boldsymbol{i}_1 = \boldsymbol{i} \times \boldsymbol{n} = \boldsymbol{k} \operatorname{cosec} \theta - \boldsymbol{n} \cot \theta \ , \qquad \boldsymbol{i}_2 = \boldsymbol{i} = \boldsymbol{n} \times \boldsymbol{k} \operatorname{cosec} \theta \ , \tag{57}$$

are orthogonal unit vectors parallel and perpendicular to the projection of $\boldsymbol{B}_0$ in the plane transverse to the direction of propagation $\boldsymbol{n}$. This tensor, here expressed in diagonal form, i.e., in terms of its eigen-values $\eta_f^{(1)}$ and $\eta_f^{(2)}$ and eigen-vectors $\boldsymbol{i}_1$ and $\boldsymbol{i}_2$, specifies the amount and state of polarization of the radiation emitted at each point of the trajectory. Since $\boldsymbol{k}$ (and θ) in general vary, its integral will yield a non-diagonal intensity polarization tensor $\mathfrak{n}_f$ (Westfold 1959*a*) whose two characteristic values are the intensities of the two oppositely polarized components $I_f^{(\alpha)}$ and $I_f^{(\beta)}$ into which it is possible to resolve an arbitrary superposition of incoherent partially polarized contributions; the characteristic vectors $\boldsymbol{i}_\alpha$ and $\boldsymbol{i}_\beta$ will be in the directions of polarization of the two components. Alternatively, the partially polarized emergent ray may be specified by its total monochromatic intensity,

$$I_f = I_f^{(\alpha)} + I_f^{(\beta)} \ , \tag{58}$$

and the intensity of the excess component,

$$I_f^{(p)} = I_f^{(\alpha)} - I_f^{(\beta)} \ , \tag{59}$$

polarized in the direction of $\boldsymbol{i}_\alpha$. Its degree of polarization is, therefore,

$$p_f = \frac{I_f^{(p)}}{I_f} . \tag{60}$$

The measured degree of polarization will not be given exactly by equation (60), for the measuring instrument is limited by both its resolving power and its frequency bandwidth. The former effect is determined by the diffraction pattern of the instrument, which in the radio case may be different for different polarizations. Within the appropriate pattern, rays of each polarization make their contributions in accordance with the standard relations. The diameter of the Crab Nebula is about 5′. The latter effect will involve a similar averaging over the receiver band width. It will be insignificant unless the rays suffer Faraday rotation in a magneto-ionic medium between the source and the observing instrument. Then each spectral component suffers a different rotation so that the over-all effect is again one of depolarization. This effect, first investigated by Hatanaka (1956) will be discussed in a forthcoming paper (Westfold 1959*b*). The situation is, of course, still more complicated if the emitting region is sufficiently dense for the Faraday effect to be appreciable there.

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APPENDIX

To show that the double integral

$$I(x) = \int_x^\infty \xi^{s-1} \int_\xi^\infty K_{\nu+1}(\eta)\, d\eta\, d\xi$$

can be reduced to the form

$$\frac{\nu+s}{s} \int_x^\infty \xi^{s-1} K_\nu(\xi)\, d\xi - \frac{x^s}{s} \left[\int_x^\infty K_{\nu+1}(\xi)\, d\xi - K_\nu(x) \right].$$

By means of the successive transformations

$$\eta = \xi + u, \qquad \xi = x + t,$$

$I(x)$ can be expressed as a dimetric integral over the first quadrant in the (u, t) plane,

$$I(x) = \iint (x+t)^{s-1} K_{\nu+1}(x+t+u)\, du\, dt.$$

Then, by a change to the variables

$$v = x + t + u, \qquad w = x + t - u,$$

the region of integration is transformed to the quadrant to the right of the lines $u = 0$, $t = 0$ on the (v, w) plane. This is evaluated as a double integral,

$$I(x) = \frac{1}{2} \int_x^\infty K_{\nu+1}(v) \int_{-v+2x}^{v} \left(\frac{v+w}{2} \right)^{s-1} dw\, dv$$

$$= \frac{1}{s} \int_x^\infty v^s K_{\nu+1}(v)\, dv - \frac{x^s}{s} \int_x^\infty K_{\nu+1}(v)\, dv.$$

The desired result follows from an application of the reduction formula,

$$K_{\nu+1}(x) = \frac{\nu}{x} K_\nu(x) - K_\nu'(x),$$

to the first term on the right side, followed by a partial integration.