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A METHOD FOR AUTOMATIC COMPUTATION OF STELLAR EVOLUTION*

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ABSTRACT

A method is described for obtaining time sequences of stellar configurations automatically by means of high-speed digital computers. The time-dependent differential equations of stellar strucutre are replaced by second-order difference equations. These non-linear difference equations, together with the boundary conditions, are solved by an iterative method which is a generalization of the Newton-Raphson method for obtaining roots of functions. With the UNIVAC computer of the Livermore site of the Lawrence Radiation Laboratory, the method has been used in connection with various previously published studies of stellar evolution.

I. INTRODUCTION

The detailed theoretical study of stellar evolution requires extensive use of modern high-speed electronic computers. The nature of the logical structure of the programs prepared for such machines requires that the methods of computation be designed to be as nearly automatic as possible, since the intervention of a human operator slows up the calculation enormously and creates a danger for the introduction of errors.

An automatic program is greatly expedited by limiting the method of integration to the solution of simple difference approximations to the differential equations. Furthermore, the inclusion of the boundary conditions from the start in the equations of condition describing the problem tends to eliminate the need for human decisions, as is often needed in "eigenvalue" methods of the integration of boundary value problems.

Since the equations are non-linear in an essential way, any solution along these lines must be based on an iterative procedure. The method described in this paper has been used in connection with the calculations described in Henyey, LeLevier, and Levee (1959), hereafter referred to as "HLL," and in Henyey, LeLevier, and Levee (1955*a*, *b*).

The basic ideas were first worked out by two of us (Henyey and Wilets) several years ago. A description of the method of solution is given by Berg and Wilets (1955). Certain additions and modifications were developed subsequently by the other authors of this paper.

II. THE BASIC DIFFERENTIAL EQUATIONS

Since we are interested in the time variations in stellar configurations, it is convenient to use a Lagrangian spatial co-ordinate. A possible choice for this co-ordinate is the mass inclosed within a specified sphere. Actually, its variation throughout the star is such as to make it quite inconvenient, since in both the outer and the central regions the mass varies very slowly. It is preferable to use some parameter whose variation with radius is reasonably uniform, so that the truncation errors involved in the difference equations,

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discussed in Section IV, are thereby made much less serious. Let j be such a suitable chosen parameter. Its characteristics will be precisely defined by specifying the relationship between mass, m, and j by means of a function $m(j)$.

The basic differential equations may now be written as follows:

$$\frac{\partial P}{\partial j} = \frac{-4\pi G m}{r^2 V}\frac{\partial r}{\partial j}, \tag{1}$$

$$\frac{\partial T}{\partial j} = -\frac{3\kappa l}{4acT^3 V r^2}\frac{\partial r}{\partial j}, \tag{2}$$

$$\frac{dm}{dj} = \frac{r^2}{V}\frac{\partial r}{\partial j}, \tag{3}$$

$$\frac{\partial l}{\partial j} = \frac{r^2}{V}\left(\epsilon - \frac{\partial E}{\partial t} - P\frac{\partial V}{\partial t}\right)\frac{\partial r}{\partial j}. \tag{4}$$

These equations follow directly from those given by Chandrasekhar (1939) with certain modifications. Instead of the density, ρ, we use the specific volume, V, the reciprocal of ρ. Also we use $4\pi m$ and $4\pi l$ in place of Chandrasekhar's $M(r)$ and $L(r)$. The rate of production of thermonuclear energy per unit mass is represented by ϵ. The second and third terms in equation (4) are discussed by Thomas (1931). The third term represents the rate of conversion of gravitational potential energy into heat and radiative flux. The pressure, P, the internal energy per unit mass, E, the opacity, κ, and ϵ are known functions of V, T, and the chemical composition. Their precise forms are not relevant to the present discussion.

In the presence of adiabatic convective equilibrium, equation (2) must be replaced by

$$\frac{\partial E}{\partial j} + P\frac{\partial V}{\partial j} = 0. \tag{5}$$

The numerical integration of the differential equations requires that they be replaced by suitable difference equations. Since the difference equations which we consider involve truncation errors of second order, they should involve differences of dependent variables whose variations are as nearly linear as possible. To meet this requirement, we have found it quite satisfactory to use the variable P/T^3 instead of P in equation (1) and l/r^2 instead of l in equation (4). The final form of the differential equations we then write as follows:

$$\frac{\partial}{\partial j}\left(\frac{P}{T^3}\right) + \frac{3}{T}\left(\frac{P}{T^3}\right)\frac{\partial T}{\partial j} + \frac{4\pi G m}{r^4 T^3}\frac{dm}{dj} = 0, \tag{6}$$

$$\frac{\partial T}{\partial j} + \frac{\text{Я} l}{4T^3 r^4}\frac{dm}{dj} = 0, \tag{7}$$

$$r^2\frac{\partial r}{\partial j} - V\frac{dm}{dj} = 0, \tag{8}$$

$$\frac{\partial}{\partial j}\left(\frac{l}{r^2}\right) + \frac{2}{r}\left(\frac{l}{r^2}\right)\frac{\partial r}{\partial j} - \frac{1}{r^2}\left(\epsilon - \frac{\partial E}{\partial t} - P\frac{\partial V}{\partial t}\right)\frac{dm}{dj} = 0. \tag{9}$$

For conciseness, we have made the following substitution in equation (7):

$$\text{Я} = \frac{3\kappa}{ac}. \tag{10}$$

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In addition to these equations of stellar structure, we must consider the equations describing the time variation in the chemical composition. For simplicity, we limit ourselves to a discussion of the conversion of hydrogen into helium.

At a point in radiative equilibrium we have

$$\frac{\partial X}{\partial t} = -R_{\rm pp}X^2 - R_{\rm CN}X X_{\rm CN}\,, \tag{11}$$

where $R_{\rm pp}$ and $R_{\rm CN}$ are the reaction rates for the proton-proton and carbon-nitrogen processes, each defined for unit concentrations by mass of hydrogen, X, and the carbon-nitrogen isotopes, $X_{\rm CN}$. These reaction rates are known functions of T and V whose precise forms are not relevant in this paper.

Throughout a convective region it is necessary to consider the composition as homogeneous. Since the mass, m^*, inclosed by the convective region can vary with time, the time rate of change of the hydrogen within it is given by

$$\frac{dX}{dt} = -\frac{1}{m^*}\int_0^{m^*}(R_{\rm pp}X^2 + R_{\rm CN}X X_{\rm CN})\,dm + (X' - X)\frac{1}{m^*}\frac{dm^*}{dt}\,. \tag{12}$$

In this expression X' is the value of X immediately outside the boundary if dm^*/dt is positive and is equal to X otherwise. Thus the second term, which represents the exchange of hydrogen between the homogeneous core and its immediate environment, is zero when the core interface is not advancing.

In all cases the helium concentration, Y, is computed from

$$X + Y = 1 - Z = \text{Constant}\,. \tag{13}$$

III. THE DIFFERENCE EQUATIONS

In order to accomplish the integration of the differential equations, we must consider the evaluation of the various parameters at a discrete set of points or shells in the stellar configuration. In the scheme we have adopted for the difference equations, which replace the differential equations, the set consists of two subsets: the first of these contains points labeled by integral values of j running from zero at the center to J at the surface; the second contains points labeled by half-odd integral values of j and located intermediately between the points in the first set. The first set consists of the points at which r and l are evaluated, and the second of those at which T, V, and the chemical composition parameters are evaluated.

While the specific choice of the points is somewhat arbitrary, we require that the relationship between m and j be represented by a smooth function. Furthermore, the relationship between r and j should be reasonably uniform. Unfortunately, it is not possible to maintain a linear relationship between r and j because the evolutionary changes produce non-uniform variations in the positions of the points. In practice, a rational expression in j can be designed to define the march of m and specifically dm/dj over the sequence of points in the combined set and to have the desired properties. An example is given graphically in HLL.

Equations (6), (7), and (5) are replaced by the finite difference approximations,

$$\left(\frac{P}{T^3}\right)_{j+1/2} - \left(\frac{P}{T^3}\right)_{j-1/2} + 3\,\frac{(P/T^3)_{j+1/2} + (P/T^3)_{j-1/2}}{T_{j+1/2}+T_{j-1/2}}(T_{j+1/2} - T_{j-1/2}) + \frac{32\pi G m_j m'_j}{r_j^4 (T_{j+1/2}+T_{j-1/2})^3} = 0\,, \tag{14}$$

$$T_{j+1/2} - T_{j-1/2} + \frac{\text{Я}_{j+1/2} + \text{Я}_{j-1/2}}{(T_{j+1/2}+T_{j-1/2})^3}\frac{l_j m'_j}{r_j^4} = 0\,, \tag{15}$$

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$$E_{j+1/2} - E_{j-1/2} + \tfrac{1}{2}(P_{j+1/2} + P_{j-1/2})(V_{j+1/2} - V_{j-1/2}) = 0 , \tag{16}$$

where m' is dm/dj.

These conditions are considered as being applied at integral values of j. The derivatives are accordingly evaluated as the slope of the cords joining values at $j + \frac{1}{2}$ and, $j - \frac{1}{2}$. The quantities needed at j, which are normally evaluated at $j \pm \frac{1}{2}$, must be evaluated by averaging. In this averaging process it is important that one operate on quantities whose variations are expected to be roughly linear.

A similar point of view is used for the finite difference approximations to equations (8) and (9). They are considered as being applied at points labeled $j + \frac{1}{2}$:

$$\tfrac{1}{4}(r_{j+1} + r_j)^2 (r_{j+1} - r_j) - V_{j+1/2} m'_{j+1/2} = 0 , \tag{17}$$

$$\left(\frac{l}{r^2}\right)_{j+1} - \left(\frac{l}{r^2}\right)_j + 2\,\frac{(l/r^2)_{j+1} + (l/r^2)_j}{r_{j+1} + r_j}(r_{j+1} - r_j) - \frac{4\,m'_{j+1/2}}{(r_{j+1} + r_j)^2} \times \left\{\epsilon_{j+1/2} - \frac{1}{\Delta t}\left[E_{j+1/2} - E^n_{j+1/2} + \tfrac{1}{2}(P_{j+1/2} + P^n_{j+1/2})(V_{j+1/2} - V^n_{j+1/2})\right]\right\} = 0 . \tag{18}$$

In the equations (14), (15), (16), (17), and (18) all quantities are to be regarded as being evaluated at the advanced epoch, $t^{n+1} = t^n + \Delta t$, except quantities labeled with superscript n, which are evaluated at the earlier epoch, t^n. The manner in which the time differences are represented is in accordance with requirements of stability. A discussion of this method of differencing in connection with the solution of the simple heat equation is given by Richtmyer (1957). His example 3, from Table I (p. 93) illustrates the method.

The difference approximation to equation (11), which represents the conversion of hydrogen into helium, is

$$\frac{X_{j+1/2} - X^n_{j+1/2}}{\Delta t} = -\tfrac{1}{2}\left[R_{pp} X^2 + R_{CN} X X_{CN} + R^n_{pp} (X^n)^2 + R_{CN} X^n X_{CN}\right]_{j+1/2} . \tag{19}$$

The subscript occurring after the bracket on the right-hand side of equation (19) indicates that all quantities within the brackets are evaluated at $j + \frac{1}{2}$. The form of this expression is based on the trapezoidal rule. If T and V are known at the advanced time, t^{n+1}, the reaction rates are also known, and hence this quadratic equation may be solved for $X_{j+1/2}$.

The corresponding difference approximation to equation (12) can be written in a form similar to equation (19) if we replace the reaction rates by mean values computed as follows:

$$\langle R_{pp} \rangle = \frac{\sum_{\text{core}} (R_{pp})_{j+1/2} \Delta m_{j+1/2}}{m^*} , \tag{20}$$

where $\Delta m_{j+1/2}$ is equal to $m'_{j+1/2}$ for zones completely within the convective core and is the appropriate fraction of $m'_{j+1/2}$ for the transition zone. A similar average is used for $\langle R_{CN} \rangle$. The expression then becomes

$$\frac{X_{j+1/2} - X^n_{j+1/2}}{\Delta t} = - \left[\langle R_{pp} \rangle X^2 + \langle R_{CN} \rangle X X_{CN} + \langle R^n_{pp} \rangle (X^n)^2 + \langle R^n_{CN} \rangle X^n X_{CN}\right]_{j+1/2} , \tag{21}$$

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when the last term of equation (12) can be neglected. In the machine program which we have used in our investigations we have made provision for handling this term by means of a very complex subroutine. However, we omit discussion of it here, since the core was increasing in mass *only* when the star was homogeneous. After inhomogeneities occurred, dm^*/dt was negative, as described in HLL.

IV. THE SOLUTION OF THE DIFFERENCE EQUATIONS

The basis of the present method is the simultaneous solution of the complete system of difference equations and boundary conditions. Since the equations are non-linear, it is necessary to resort to an iterative procedure. We adopt a form of the Newton-Raphson technique.

Equations (14), (15), (17), and (18) may be written in the following concise functional forms:

$$\begin{aligned} Q'_j(T_{j-1/2},\ V_{j-1/2},\ r_j,\ T_{j+1/2},\ V_{j+1/2}) &= 0\ , \\ Q''_j(T_{j-1/2},\ V_{j-1/2}\ ,\ r_j,\ l_j,\ T_{j+1/2},\ V_{j+1/2}) &= 0\ , \\ Q'_{j+1/2}(r_j,\ V_{j+1/2},\ r_{j+1}) &= 0\ , \\ Q''_{j+1/2}(r_j,\ l_j,\ T_{j+1/2},\ V_{j+1/2},\ r_{j+1},\ l_{j+1}) &= 0\ , \end{aligned} \tag{22}$$

where it is assumed that P, E, $\mathcal{H}$, and ϵ are known functions of T and V. The occurrence of T and V in the last equation refers only to the advanced epoch, since their values at the earlier epoch are presumed known.

The Newton-Raphson method as applied to a system

$$f_k(x_1,\ x_2,\ \ldots,\ x_K) = 0\ , \qquad k = 1,\ 2,\ \ldots,\ N \tag{23}$$

proceeds by considering an approximate set of values for $x_1, x_2, \ldots$, which are corrected on the basis of changes $\delta x_1, \delta x, \ldots$, determined by

$$f_k + \sum_{l=1}^{K} \frac{\partial f_k}{\partial x_l}\,\delta x_l = 0\ , \tag{24}$$

where the f_k's are not in general zero because the variables do not satisfy the equations of condition (22). When reasonably good approximations are available, the method converges rapidly.

Fortunately, equations (22) involve values of the unknown variables only at three consecutive points, provided that we consider the integral and half-add integral points as following each other consecutively. Thus the number of terms in the application of equation (24) is very greatly limited. In fact, at most, six derivatives times their corresponding increments occur in any one variational equation.[1]

Let Q_j and $Q_{j+1/2}$ represent the following number pairs:

$$Q_j = \begin{pmatrix} Q'_j \\ Q''_j \end{pmatrix}\ , \qquad Q_{j+1/2}\ \begin{pmatrix} Q'_{j+1/2} \\ Q''_{j+1/2} \end{pmatrix}\ . \tag{25}$$

These may be regarded as vectors in an abstract two-dimensional space. Similar two-vectors are conveniently defined thus:

$$W_j = \begin{pmatrix} \delta r_j \\ \delta l_j \end{pmatrix}, \qquad W_{j+1/2} = \begin{pmatrix} \delta T_{j+1/2} \\ \delta V_{j+1/2} \end{pmatrix}. \tag{26}$$

[1] This technique has been known and used for some time but apparently without any formal published account. A description has only recently been given by Richtmyer (1957).

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It is also convenient to define the following second-order matrices:

$$A_j = \begin{pmatrix} \frac{\partial Q_j'}{\partial T_{j+1/2}} & \frac{\partial Q_j'}{\partial V_{j+1/2}} \\ \frac{\partial Q_j''}{\partial T_{j+1/2}} & \frac{\partial Q_j''}{\partial V_{j+1/2}} \end{pmatrix}, \tag{27}$$

$$B_j = \begin{pmatrix} \frac{\partial Q_j'}{\partial T_{j-1/2}} & \frac{\partial Q_j'}{\partial V_{j-1/2}} \\ \frac{\partial Q_j''}{\partial T_{j-1/2}} & \frac{\partial Q_j''}{\partial V_{j-1/2}} \end{pmatrix}, \tag{28}$$

$$C_j = \begin{pmatrix} \frac{\partial Q_j'}{\partial r_j} & 0 \\ \frac{\partial Q_j''}{\partial r_j} & \frac{\partial Q_j''}{\partial l_j} \end{pmatrix}, \tag{29}$$

$$A_{j+1/2} = \begin{pmatrix} \frac{\partial Q_{j+1/2}'}{\partial r_{j+1}} & 0 \\ \frac{\partial Q_{j+1/2}''}{\partial r_{j+1}} & \frac{\partial Q_{j+1/2}''}{\partial l_{j+1}} \end{pmatrix}, \tag{30}$$

$$B_{j+1/2} = \begin{pmatrix} \frac{\partial Q_{j+1/2}'}{\partial r_j} & 0 \\ \frac{\partial Q_{j+1/2}''}{\partial r_j} & \frac{\partial Q_{j+1/2}''}{\partial l_j} \end{pmatrix}, \tag{31}$$

$$C_{j+1/2} = \begin{pmatrix} 0 & \frac{\partial Q_{j+1/2}'}{\partial V_{j+1/2}} \\ \frac{\partial Q_{j+1/2}''}{\partial T_{j+1/2}} & \frac{\partial Q_{j+1/2}''}{\partial V_{j+1/2}} \end{pmatrix}. \tag{32}$$

The occurrence of zero elements in these matrices arises from the absence of certain variables in equations (22).

Using standard matrix notation, the variational equations derived from equation (22) in accordance with the scheme given in equation (24) may be written in the following concise form:

$$Q_j + A_j W_{j+1/2} + B_j W_{j-1/2} + C_j W_j = 0 , \tag{33}$$

$$Q_{j+1/2} + A_{j+1/2} W_{j+1} + B_{j+1/2} W_j + C_{j+1/2} W_{j+1/2} = 0 . \tag{34}$$

In the case when convective equilibrium must be considered, the second equation in (22) must be equation (16). Formally, however, there is no change in the structure of the derived matrix elements.

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The difference equations must be supplemented by suitable boundary conditions at the center:

$$W_0 = 0 \ . \tag{35}$$

Therefore, we may solve the matrix equation (34) with $j = 0$ for the vector $W_{1/2}$ in the form

$$W_{1/2} = K_{1/2} - L_{1/2}W_1 \ . \tag{36}$$

Clearly,

$$\begin{aligned} K_{1/2} &= -\ (C_{1/2})^{-1}Q_{1/2} \ , \\ L_{1/2} &= (C_{1/2})^{-1}A_{1/2} \ . \end{aligned} \tag{37}$$

Consider next the case $j = 1$ with respect to equation (33). The values for $W_{1/2}$ given in equation (36), may be substituted to give the expression involving $W_{3/2}$ and W_1. This may now be solved for W_1 in terms of known quantities and $W_{3/2}$. Similarly, W_1, thus obtained, may be eliminated from equation (34), for $j = 1$, and the resultant expression solved for $W_{3/2}$ in terms of known quantities and W_2.

Evidently this procedure of elimination of the vector with the lowest subscript in equation (33) or (34) may be accomplished and a solution for the vector of next higher subscript be obtained in one or the other of the following forms:

$$\begin{aligned} W_j &= K_j - L_jW_{j+1/2} \ , \\ W_{j+1/2} &= K_{j+1/2} - L_{j+1/2}W_{j+1} \ . \end{aligned} \tag{38}$$

The appropriate recurrence equations may be obtained by substituting these in the variational equations (33) and (34). It will then be found that

$$\begin{aligned} L_{j+1/2} &= (C_{j+1/2} - B_{j+1/2}L_j)^{-1}A_{j+1/2} \ , \\ K_{j+1/2} &= -\ (C_{j+1/2} - B_{j+1/2}L_j)^{-1}(Q_{j+1/2} + B_{j+1/2}K_j) \ , \\ L_{j+1/2} &= (C_{j+1} - B_{j+1}L_{j+1/2})^{-1}A_{j+1} \ , \\ K_{j+1} &= -\ (C_{j+1} - B_{j+1}L_{j+1/2})^{-1}(Q_{j+1} + B_{j+1}K_{j+1/2}) \ . \end{aligned} \tag{39}$$

The cyclic application of these relations enables the calculation of all exhibited quantities to be carried out systematically after A, B, and C have been evaluated as given in equations (27)–(32). This process is carried out in the sense of increasing j until the surface is reached where A_J, B_J, and C_J are undefined. This deficiency must, of course, be met by the use of certain suitable surface-boundary conditions. We have used the following crude representation of a model atmosphere.

Using equations (1), (2), and (10), we obtain

$$\frac{dP}{dT} = \frac{16\pi GmT^3}{\text{Я}^l} \ . \tag{40}$$

Assuming an opacity law of the form

$$\text{Я} = \frac{\text{Я}_0}{T^\lambda V} \tag{41}$$

and using the perfect gas equation of state, it follows that, at the center of the last zone,

$$(P_{J-1/2})^2 = \frac{32\pi G}{\lambda + 5}\frac{m_J P_{J-1/2}}{\text{Я}_{J-1/2}l_J}(T_{J-1/2})^4 + P_J^2 \ , \tag{42}$$

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where the very small variations of m and l in the last half-zone have been neglected. The photospheric pressure, P_J, is approximately

$$P_J = \frac{4\pi G m_J}{r_J^2 \kappa}. \tag{43}$$

Here κ is the atmospheric opacity evaluated at the photospheric pressure and the photospheric temperature:

$$T_J = \left(\frac{l_J}{4\pi\sigma r_J^2}\right)^{1/4}. \tag{44}$$

It therefore follows that this opacity may be expressed as a function of m_J/r_J^2 and l_J/r_J^2. We have succeeded in representing the calculations given by Vitense (1951), modified for $X = 0.68$ and $Z = 0.01$, by the following interpolation formula:

$$\kappa = \left(\frac{m_J}{r_J^2}\right)^{1/3}\left[1.9\times 10^{-47}\,\frac{(l_J/r_J^2)^4}{9\times 10^{-22}\,(l_J/r_J^2)^2+1} + 1.5\times 10^{-5}\right], \tag{45}$$

where all quantities are expressed in cgs units. The last term approximately represents the effect of the contribution of free electrons by the metals at low temperatures.

If we assume that $\mathcal{H}m'/T^3$ is constant from $J - \frac{1}{2}$ to J, equation (7) can be differenced and reduced to the following form:

$$T_{J-1/2}^4 - \frac{32\mathcal{H}_{J-1/2} l_J m'_{J-1/2}}{(3r_J + r_{J-1})^4} = 0. \tag{46}$$

The surface temperature and the variation in l are considered negligible, and the average of the radius is reckoned at $J - \frac{1}{4}$.

As indicated previously, the iterative procedure described by equations (39) terminates with the calculation of $L_{J-1/2}$ and $K_{J-1/2}$. Were W_J available, equations (38) could be used to evaluate successively $W_{J-1/2}$, W_{J-1}, $W_{J-3/2}$, . . . , W_1, $W_{1/2}$.

Equations (42), (43), (45), and (46) involve the basic variables r_{J-1}, $T_{J-1/2}$, $r_{J-1/2}$, r_J, l_J, as well as several subsidiary quantities therein defined. These subsidiary quantities, P_J and κ, may be eliminated, leaving two independent conditions on the basic variables. Corresponding to these equations, one can derive variational equations by equation (24) involving the increments δr_{J-1}, $\delta T_{J-1/2}$, $\delta V_{J-1/2}$, δr_J, and δl_J linearly. These, in turn, may be solved in conjunction with the four equations (38), evaluated at $j = J - 1$, for these five variations and δl_{J-1}. Thus W_{J-1} is determined, and equation (38) may be used to evaluate all remaining variations.

The increments thus determined are added to the values of the corresponding variables, in order to provide improved values of r, l, T, and V throughout the configuration. With the new values of r and V we may now compute improved reaction rates and therefore improved values of the chemical composition parameters.

In the event that a convective core exists, the parameter m^* is determined after improved values of r, l, T, and V are available but before the chemical composition is corrected. The value of m^* is determined by interpolating between the appropriate two successive values of m_j, where the condition for marginal stability is achieved in passing radially outward through the configuration. This condition is

$$\frac{\mathcal{H}l}{16\pi G T^3 m}\left[\frac{\partial P}{\partial T} - \frac{(\partial E/\partial T)/(\partial P/\partial V)}{(\partial E/\partial V) + P}\right] = 1. \tag{47}$$

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All the new improved values may in turn be used to improve the Q's as given by equation (25) and their partial derivatives as given by equations (27) to (32) inclusive. This process is repeated until the increments are sufficiently small in terms of some preassigned tolerance. In general, satisfactory values are achieved after two or three iterations.

After the iterative procedure has yielded sufficiently precise values of the variables, we proceed to the next epoch. It then becomes necessary to estimate provisional values, in order to start a new iterative cycle. These are determined by extrapolating linearly from the two previous epochs. In practice it was found that, in general, an extrapolation of 80 per cent was better than 100 per cent and in exceptional cases avoided dangers involved in obtaining provisional values outside the region of convergence of the iterative technique.

At the end of calculation for a given epoch the timewise changes in the specific volume and chemical composition are scrutinized by the computer in order to determine whether the time interval should be changed in accordance with preassigned tolerances for the maximum changes in these variables.

The values at the initial epoch are determined as described in HLL.

V. CONCLUDING REMARKS

Although the method described in the preceding section has proved basically to be extremely efficient for automatically following an evolving sequence of stellar configurations, there exist certain details which lead to awkward complications in connection with the boundary of a convective region and the surface boundary conditions. These may be avoided to a certain extent by evaluating all quantities at the same points. The relaxation of the equations of condition then becomes slightly more complex, although the total computing time remains approximately the same. An investigation of this revised method is under way.

The development and testing of the methods herein described was made possible by the availability of the UNIVAC computer of the Livermore site of the Lawrence Radiation Laboratory of the University of California. We wish therefore to acknowledge gratefully our debt to the Atomic Energy Commission and the staff of the Laboratory.

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