DISRUPTION OF GALACTIC CLUSTERS

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ABSTRACT

The tidal force of passing interstellar clouds accelerates the stars in a galactic cluster, increases the total cluster energy, and leads to the expansion and ultimate disruption of the cluster. The magnitude of this effect is computed on the impulsive approximation, i.e., on the assumption that the stars do not move appreciably as a cloud passes by. Solution of the more general problem, on the simplifying assumption that the gravitational potential of the cluster varies as the square of the distance from the center, shows that the impulsive approximation should usually be adequate. The computed rate of increase of internal cluster energy, resulting from encounters with clouds, is about thirty times the corresponding rate for encounters with field stars, if a cluster radius of 5 pc is assumed. The "disruption time" required for a cluster to dissociate into separate stars varies directly with the cluster density and is about 2×10^8 years for a mean density of $1 M_{\odot}/\text{pc}^3$. The stars of relatively low mass in a cluster will be lost more rapidly, partly by evaporation and partly by enhanced disruption of the extended aura which these stars presumably form about the cluster. This mechanism may account for the scarcity of galactic clusters older than 10^9 years.

I. INTRODUCTION

The evolution of a galactic cluster is dominated by two major effects. The first of these, pointed out several decades ago by Bok (1934), is the increase in mean energy as a result of encounters with passing field stars. Since this effect leads to a gradual expansion of the cluster, we shall refer to it here as "disruption." This process has been analyzed in detail by Mineur (1939) and Chandrasekhar (1942). The second effect, pointed out independently by Ambarzumian (1938) and Spitzer (1940), is the "evaporation" of cluster stars as a result of gravitational encounters between stars in the cluster. In this process, an individual star acquires positive energy and departs from the cluster, leaving the remaining stars with a more negative mean energy than before. As a result, the cluster contracts. This effect has been analyzed by Chandrasekhar (1942, 1943*a*, *b*).

Previous analyses of disruption have referred to the gravitational encounters with field stars only. It has been pointed out by Spitzer and Schwarzschild (1950) that the velocity distribution of type I stars is much more affected by gravitational encounters with cloud complexes than by encounters with the field stars. The purpose of the present paper is to point out that in a similar manner the disruption of clusters by successive encounters with extended interstellar clouds is a full order of magnitude greater than the disruption resulting from passing field stars. The analysis in the subsequent sections indicates that this effect may be of great importance in the evolution of all but the densest galactic clusters.

This analysis was undertaken in an attempt to explain the apparent rarity of old clusters, a fact recently emphasized by Oort (1957). According to Oort's survey of the data, the distribution of ages of galactic clusters, for clusters younger than 5×10^8 years, is about what we would expect on the hypothesis of a uniform rate of formation. The age of a galactic cluster is indicated by the spectral type of the earliest stars, with the age ranging from 10^7 years for stars of type O to 5×10^8 years for types B9 to A0. However, clusters older than 5×10^8 years, in which the earliest stars are of type A1 or later, are almost completely lacking, although, on the assumption of a uniform rate of formation, such clusters should outnumber all others. Oort has suggested that some process leads to the disintegration of most galactic clusters within 10^9 years after their formation. While conclusive results are not now possible, the present analysis strongly suggests that disruption by tidal encounters with interstellar clouds is just such a process.

In Section II the rate of increase of cluster energy, produced by passing clouds, is computed on the impulsive approximation, i.e., with the assumption that the position of the star in the cluster does not change appreciably during the passage of the cloud. The probable range over which this assumption is valid is considered in the next section, where the full time-dependent problem is solved by successive approximations in an idealized case. In Section IV these results are applied to a discussion of cluster evolution. Computations of the velocity-distribution function in an isolated cluster, to yield the rates of evaporation for stars of different masses together with the associated density distributions, are given in a separate paper (Spitzer and Härm 1958).

II. RATE OF ENERGY INCREASE ON THE IMPULSIVE APPROXIMATION

As a cloud passes by a cluster, each star in the cluster experiences a tidal force relative to the center of gravity of the cluster. If the cloud passes by sufficiently rapidly, the velocity change, $\Delta \boldsymbol{v}$, of each star may be computed as though the star were not moving. If an average is taken over all stars, the mean value of $(\Delta v)^2$ is finite and corresponds to an increase in the total kinetic energy, T, of the cluster. From the virial theorem it follows that this increase in the total energy, U, leads to a new equilibrium, in which the total gravitational energy is less negative than before, T is reduced below its former value, and the cluster has expanded. We proceed now to compute the rate at which U increases.

We define a co-ordinate system centered at the cluster center and with the x-axis pointing toward the cloud when it is at its distance, p, of closest approach to the cluster center. The y-axis is taken parallel to the velocity, $\boldsymbol{V}$, which the cloud has, relative to the cluster. The z-axis, of course, is perpendicular to the x- and y-axes.

The tidal acceleration, $\boldsymbol{F}$, is most conveniently expressed in a rotating reference frame, x', y', z', in which x' points to the instantaneous position of the cloud, y' is in the xy-plane, and z' is parallel to z. If we let R be the instantaneous distance of the cloud from the cluster center, then we have the familiar results,

$$F'_x = \frac{2G m_n x'}{R^3}, \qquad F'_y = -\frac{G m_n y'}{R^3}, \qquad F_z = -\frac{G m_n z'}{R^3}, \tag{1}$$

where G is the gravitational constant, m_n is the mass of the cloud or nebula (we use a subscript n for the clouds throughout, reserving the subscript c for properties of the cluster). Evidently, by a proper choice of time origin, we have

$$R^2 = p^2 + V^2 t^2 . \tag{2}$$

The rotating co-ordinate frame is related to the fixed frame by the equations

$$x' = x \cos\theta + y \sin\theta ,$$

$$y' = -x \sin\theta + y \cos\theta , \tag{3}$$

$$z' = z , \tag{4}$$

where $\tan\theta = Vt/p$. After some algebra, we obtain

$$\frac{d v_x}{dt} = \frac{G m_n}{R^3} [x(2 - 3\sin^2\theta) + 3y \sin\theta\cos\theta] , \tag{5}$$

$$\frac{d v_y}{dt} = \frac{G m_n}{R^3} [y(2 - 3\cos^2\theta) + 3x \sin\theta\cos\theta] , \tag{6}$$

$$\frac{d v_z}{dt} = -\frac{G m_n z}{R^3} . \tag{7}$$

Integration of equations (5), (6), and (7), with the use of equation (2), yields

$$\Delta v_x = \frac{2Gm_n x}{p^2 V}, \qquad \Delta v_y = 0, \qquad \Delta v_z = \frac{2Gm_n z}{p^2 V}. \tag{8}$$

It follows that ΔU, the increase in total energy of the cluster, which is equal, of course, to ΔT, is given by

$$\Delta U = \tfrac{1}{2} m_c \left(\frac{2Gm_n}{p^2 V}\right)^2 \tfrac{2}{3} r_c^2, \tag{9}$$

where m^2 is the cluster mass and r_c^2 is the mean-square cluster radius.

The rate of change of U is given by equation (9) multiplied by the number of collisions per unit time, per interval of p, and per interval of m_n, and integrated over p and m_n. In this way we obtain

$$\frac{dU}{dt} = \frac{8\pi G^2 m_c r_c^2}{3V} \int_0^\infty n(m_n)\, m_n^2 d m_n \int_{R_n}^\infty \frac{dp}{p^3}, \tag{10}$$

where R_n is the radius of a cloud whose mass is m_n, and $n(m_n)dm_n$ is the number of clouds, or nebulae, per unit volume with a mass between m_n and $m_n + dm_n$. Encounters for which p is less than R_n will increase U somewhat, but we neglect them.

To evaluate equation (10) approximately, we see that

$$\frac{m_n}{R_n^2} = \tfrac{4}{3}\pi R_n \rho_{in}, \tag{11}$$

where ρ_{in} is defined as the internal density of the cloud, assumed to be a uniform sphere of radius R_n. We assume that this product $(R_n \rho_{in})$ is independent of the size of the cloud, corresponding to an optical extinction per cloud independent of cloud size. As pointed out by Bok (1946), observed clouds seem to have a tendency in this direction. The integral over m_n then gives simply ρ_{an}, the total density of clouds, averaged over that volume of the Galaxy in which the clusters move. The final formula for dU/dt, then, is

$$\frac{dU}{dt} = \left(\frac{4\pi G}{3}\right)^2 \frac{m_c r_c^2 \rho_{an} (R_n \rho_{in})}{V}. \tag{12}$$

As U increases, the cluster expands, increasing the rate of expansion. To evaluate this effect, we must know the dependence of U on r. From the virial theorem we know that U is $\Phi/2$, where Φ, the total gravitational energy of the cluster, may be written

$$\Phi = -\frac{\gamma G m_c^2}{r_c}, \tag{13}$$

where, as before, r_c^2 is the mean-square value of r for all the cluster stars and γ is a numerical constant. For a homogeneous sphere, γ is $(\frac{3}{5})^{3/2}$, or 0.465; and, for a polytropic sphere of index n, γ varies from 0.469 to 0.564 as n increases from 1 to 4; these values are based on radii of gyration computed for polytropes by Motz (1952). We shall let γ be equal to 0.5 in the numerical application of these results.

If we now eliminate r_c from equation (12) by means of equation (13) and integrate, we find that U reaches zero at a "disruption time," t_d, given by

$$t_d = \frac{\gamma}{8\pi G} \frac{V}{\rho_{an} (R_n \rho_n)} \rho_c, \tag{14}$$

where ρ_c is defined as the cluster mass divided by $4\pi r_c^3/3$. This equation will be used in the last section in a discussion of cluster evolution.

III. APPROXIMATE SOLUTION FOR SLOW ENCOUNTERS

The results in the preceding section, while they have the great advantage of simplicity, have the disadvantage that they are based on the assumption that the clusters stars move a negligible amount while the cloud passes by. Since this assumption is not realistic, it is necessary to investigate the more general case, in which the stellar motions are taken into account. Such an investigation will be carried through here with two other simplifying assumptions, as follows: (*a*) the tidal force of the passing cloud is small compared to the gravitational attraction of the cluster; and (*b*) the gravitational potential in the cluster varies as the square of the distance from the center.

Assumption *a* is certainly valid, except for the closest encounters with the densest clouds. This assumption makes it possible to solve the equations by a perturbation method. Assumption *b* provides an idealization of the problem, but it should not falsify the results for actual clusters. According to this assumption, each star moves in simple harmonic oscillation in each co-ordinate, simplifying the analysis very greatly. Actually, we shall consider here only those stars whose initial velocity lies in the *xz*- or the *yz*-plane. The analysis could be extended in a straightforward manner to stars with initial velocities in all three directions, but with considerably greater algebra.

We consider, first, the motion of a star in the *z*-direction; we let the acceleration in this direction equal $-\omega^2 z$. Let the perturbing tidal force be $\lambda z f(t)$, where $f(0)$ equals unity and λ is assumed small compared to ω^2. The equation of motion, then, is

$$\frac{d^2 z}{dt^2}+\omega^2 z=\lambda z f(t)\,. \tag{15}$$

This equation may be solved as a power series in λ,

$$z=z_0+\lambda z_1+\lambda^2 z_2 \ldots . \tag{16}$$

As we shall see, the change in kinetic energy is of order λ^2, and hence we must compute z_2 as well as z_1.

The zero-order solution is obviously

$$z_0=A_0 \cos \omega t+B_0 \sin \omega t\,. \tag{17}$$

The first-order function, z_1, is obtained by inserting z_0 on the right-hand side of equation (15), which is then a known function of t, and solving for z_1 by use of the variation of parameters. In this way we obtain

$$z_1=A_1(t) \cos \omega t+B_1(t) \sin \omega t\,, \tag{18}$$

where

$$A_1(t)=-\frac{1}{\omega}\int_{-\infty}^{t} f(\tau) \sin \omega\tau\,(A_0 \cos \omega\tau+B_0 \sin \omega\tau)\,d\tau\,, \tag{19}$$

$$B_1(t)=\frac{1}{\omega}\int_{-\infty}^{t} f(\tau) \cos \omega\tau\,(A_0 \cos \omega\tau+B_0 \sin \omega\tau)\,d\tau\,. \tag{20}$$

The lower limit of integration has been determined by the condition that A_1 and B_1 both vanish when t is $-\infty$.

Similarly, if z_1 from these equations is inserted in the right-hand side of equation (15), this equation may then be solved for z_2. The result is an equation similar to (18), with A_2 and B_2 replacing A_1 and B_1, where

$$A_2(t) = \frac{1}{\omega^2}\int_{-\infty}^{t} f(\tau)\sin\omega\tau\cos\omega\tau d\tau \int_{-\infty}^{\tau} z_0(u)\, f(u)\sin\omega u du - \frac{1}{\omega^2}\int_{-\infty}^{t} f(\tau)\sin^2\omega\tau d\tau\int_{-\infty}^{\tau} z_0(u)\, f(u)\cos\omega u du\,, \tag{21}$$

$$B_2(t) = -\frac{1}{\omega^2}\int_{-\infty}^{t} f(\tau)\cos^2\omega\tau d\tau\int_{-\infty}^{\tau} z_0(u)\, f(u)\sin\omega u du + \frac{1}{\omega^2}\int_{-\infty}^{t} f(\tau)\cos\omega\tau\sin\omega\tau d\tau\int_{-\infty}^{\tau} z_0(u)\, f(u)\cos\omega u du\,. \tag{22}$$

In the previous section, ΔU was simply the change in kinetic energy produced by the encounter. When the encounter is gradual, this simplification cannot be made, and the total energy, including both potential and kinetic energies, must be considered. The energy in the z co-ordinate may now be taken as the kinetic energy of motion in the z-direction when z vanishes, and we have

$$\Delta\left[\tfrac{1}{2}m\left(\frac{dz}{dt}\right)^2_{z=0}\right] = \frac{m\omega^2}{2}\left[(A_0+\lambda A_1+\lambda^2 A_2\ldots)^2 - A_0^2 + (B_0+\lambda B_1+\lambda^2 B_2\ldots)^2 - B_0^2\right], \tag{23}$$

evaluated at t equal to $+\infty$. To obtain ΔU_z, the total change in the energy of the cluster in the z-direction, we must multiply equation (23) by the number of stars in the cluster and average over all values of A_0 and B_0. Since the phases of the stars are at random, we may write, denoting by broken brackets an average over all stars,

$$\langle A_0 B_0\rangle = 0\,, \tag{24}$$

$$\langle A_0^2\rangle = \langle B_0^2\rangle = z_c^2\,, \tag{25}$$

where z_c^2 is evidently the mean-square value of z for the cluster stars. Because of equations (24) and (25), the terms in λ in equation (23) cancel out. After some straightforward algebra, all the terms in λ^2 may be combined.

The double integrals may be evaluated on integration by parts. For a typical such integral we have, representing trigonometric functions by F and G,

$$\int_{-\infty}^{+\infty} f(\tau)F(\omega\tau)\,d\tau\int_{-\infty}^{\tau} f(u)G(\omega u)\,du = \left|\int_{-\infty}^{t} f(\tau)F(\omega\tau)\,d\tau\times\int_{-\infty}^{t} f(u)G(\omega u)\,du\right|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(t)G(\omega t)\,dt\int_{-\infty}^{t} f(\tau)F(\omega\tau)\,d\tau\,, \tag{26}$$

and hence

$$\int_{-\infty}^{+\infty} f(\tau) F(\omega\tau)\, d\tau \int_{-\infty}^{\tau} f(u) G(\omega u)\, du$$

$$+ \int_{-\infty}^{+\infty} f(\tau) G(\omega\tau)\, d\tau \int_{\infty}^{\tau} f(u) G(\omega u)\, du \tag{27}$$

$$= \left[\int_{-\infty}^{+\infty} f(t) F(\omega t)\, dt\right]\left[\int_{-\infty}^{+\infty} f(t) G(\omega t)\, dt\right].$$

We obtain, finally, for ΔU_z, the change in U_z in a single encounter,

$$\Delta U_z = \frac{m_c}{2} \lambda^2 z_c^2 (I_c^2 + I_s^2), \tag{28}$$

where

$$I_c = \int_{-\infty}^{+\infty} f(\tau) \cos 2\omega\tau\, d\tau, \tag{29}$$

$$I_s = \int_{-\infty}^{+\infty} f(\tau) \sin 2\omega\tau\, d\tau. \tag{30}$$

We now apply equation (28) to the case where $f(t)$ corresponds to the tidal force produced by a passing cluster. The co-ordinate system used is the same as in the first section, and we have, from equation (7),

$$\lambda = -\frac{G m_n}{p^3}, \tag{31}$$

$$f(t) = \left(\frac{p}{R}\right)^3 = \left(1 + \frac{V^2 t^2}{p^2}\right)^{-3/2}. \tag{32}$$

With this expression substituted in equations (29) and (30), we find that I_s vanishes, since $f(\tau)$ is an even function of τ, and equation (29) gives, for I_c,

$$I_c = \frac{2p}{V} \beta K_1(\beta), \tag{33}$$

where β, the effective duration of the encounter, is defined by

$$\beta \equiv \frac{2\omega p}{V}, \tag{34}$$

and $K_1(\beta)$ is the usual Bessel function of imaginary argument. Evidently, β is essentially the time required for a star to pass by the cluster, divided by the oscillation period of a cluster star. These results, substituted in equation (28), give ΔU_z. Motions of the star in the x- and y-directions do not affect this result, as the tidal force in the z-direction is independent of x and y.

To compute ΔU_x and ΔU_y, we must take into account that the x- and y-motions are coupled through equations (5) and (6). To simplify the mathematics, we compute ΔU_x on the assumption that y_0, the initial function $y(t)$ before the encounter, vanishes; however, y_1 contributes directly to ΔU_x and, by affecting x_2, contributes indirectly as well. A similar contribution to ΔU_y is computed on the assumption that x_0 vanishes. The final result, including acceleration in the z-direction also, may be written thus:

$$\Delta U_i = \frac{m_c}{2}\left(\frac{2Gm_n}{Vp^2}\right)^2 \frac{r_c^2}{3} L_i(\beta)\,, \tag{35}$$

where i stands for x, y, or z, and

$$L_x(\beta) = [\beta^2 K_1(\beta) + \beta K_0(\beta)]^2 + [\beta^2 K_1(\beta)]^2\,, \tag{36}$$

$$L_y(\beta) = [\beta^2 K_0(\beta)]^2 + [\beta^2 K_1(\beta)]^2\,, \tag{37}$$

$$L_z(\beta) = [\beta K_1(\beta)]^2\,. \tag{38}$$

The values of these three functions are given in Table 1, together with their sum in the last row.

TABLE 1

VALUES OF $L_i(\beta)$

β	$L_x(\beta)$	$L_y(\beta)$	$L_z(\beta)$	$L_x+L_y+L_z$
0 .	1 000	0 000	1 000	2 000
0 1	1 029	010	0 971	2 010
0 2	1 088	041	0.912	2 042
0 3	1 158	091	0.841	2.089
0 4	1 229	.154	0.763	2.146
0 5	1 294	225	0.686	2.205
0 6	1 347	.298	0.611	2 256
0 7	1 386	370	0 541	2 296
0 8	1 409	.435	0 475	2 319
0 9	1 417	492	0 416	2 325
1 0	1 409	.540	0.362	2 311
1 2	1 352	602	0 272	2 226
1 4	1 254	624	0 202	2.079
1 6	1 130	.611	0.148	1 889
1 8	0 993	.574	0 108	1.674
2 0	0 854	.521	0.078	1 453
2 5	0 543	.365	0 034	0 942
3 0	0 318	.228	0 015	0 561
3 5	0 175	.132	0 006	0.313
4 0	0 092	072	0 002	0 166
4 5	0 047	037	0.001	0 085
5 0	0 023	0 019	0 000	0 042

Examination of Table 1 indicates the extent to which the impulsive approximation is valid in this idealized situation. When the effective duration time, β, vanishes, the stars may be taken as essentially motionless during an encounter, and we recover equation (9) of Section II. With increasing β, $L_x + L_y + L_z$ actually increases at first, finally decreasing again. This initial increase in ΔU with increasing β is due to the possibility of resonance. For a fixed star the tidal force in the xy-plane changes its direction as the cloud passes. A moving star can be passing through the origin when the tidal force has one sign and be far from the cluster center when the force has the opposite sign. Evidently, the impulsive approximation somewhat underestimates ΔU for β about equal to 1.0. Even for β equal to 1.5, $L_x + L_y + L_z$ has almost the same value as for zero β. For a further increase in β, however, ΔU decreases sharply. When the encounter duration time exceeds four periods of oscillation and β exceeds 4, the value of ΔU is reduced an order of magnitude or more below the impulsive approximation. In an actual cluster, where the period is not a constant for all stars but decreases with increasing distance from the cluster center, this decline in the L-functions with decreasing β would not be so abrupt.

In conclusion, we may safely use the results of the impulsive approximation up to about β equal 2 with an accuracy probably better than 20 per cent, on the average. For greater values of β, the value of ΔU decreases rapidly and can probably be neglected.

IV. EVOLUTION OF GALACTIC CLUSTERS

The rate of disruption of a cluster depends inversely on the cluster density. The observational evidence on the density, ρ_c, of galactic clusters will first be summarized briefly. Reliable cluster masses can be obtained only from measured velocity dispersions and use of the virial theorem. Such measures are available only for the Pleiades and Praesepe clusters. Analyses of these data by Mineur (1939) and van Wijk (1949) give total masses of about 500 and 700 $M_\odot$, respectively. The densities depend on the distribution of mass in each cluster.

According to an analysis by Oort (informal communication), the mean value of r^2 for those stars in the Pleiades whose apparent magnitudes lie between 6.0 and 9.9 is 2.9(deg)^2. With the value of 0.011″ for the parallax, given by Mineur (1939), the corresponding value of r_c is 2.7 pc, giving 6.0 $M_\odot$/pc^3 for ρ_c in the Pleiades. The eleven stars brighter than the sixth magnitude are more concentrated toward the center but will not appreciably affect r_c for the cluster as a whole. However, the true value of r_c may be increased somewhat above 2.7 pc by the stars fainter than the tenth magnitude, for which the available data are not sufficiently complete to indicate the spatial distribution. Thus 6.0 $M_\odot$/pc^3 may be regarded as an upper limit for ρ_c. If for the parallax of the Pleiades we take the recent spectroscopic value of 0″.0079 found by Mitchell and Johnson (1957) instead of the value 0″.011 adopted by Mineur, this upper limit on the density is reduced to 3.1 $M_\odot$/pc^3, the cluster mass increasing to 700 $M_\odot$. The star counts for Praesepe have not been analyzed for the spatial distribution.

For other clusters we must use star counts as a measure of the total mass. For the Pleiades and Praesepe the total mass estimated from star counts is about half the dynamical value, and we shall assume this same ratio in other clusters. Much denser clusters than the Pleiades certainly exist. For example, the value of ρ_c in M67, if we assume that r_c equals the radius containing half the mass, is about 70 $M_\odot$/pc^3, according to van den Bergh (1957). Allowance for the difference between star counts and the true value for the total mass increases this density to about 140 $M_\odot$/pc^3. On the other hand, less dense clusters are also known. The structure of the Hyades cluster has been extensively studied by van Bueren (1952). The root-mean-square radius, determined from his data, is 8 pc, giving a value of 0.15 $M_\odot$/pc^3 for ρ_c. If this value is doubled, to allow for undetected masses, the resultant density is still only about 0.3 $M_\odot$/pc^3, which is comparable with the density at which a cluster becomes unstable because of the shearing effect of galactic rotation. While Bok (1934) found instability for a density of 0.09 $M_\odot$/pc, Mineur (1939), in a more elaborate analysis, taking internal motions into account, found that a homogeneous cluster would be unstable at a density of 0.6 $M_\odot$/pc^3. Extension of the analysis to nonuniform clusters has been carried through by van Wijk (1949). Evidently, the Hyades cluster is near the minimum density possible for a cluster that can hold itself together for any appreciable length of time. We shall therefore consider clusters in which the mean density may lie anywhere from 0.3 to 100 $M_\odot$/pc^3.

We pass on to the numerical application, for actual clusters, of the results in the preceding two sections. First, we consider the applicability of the impulsive approximation. We have seen in Section III that this approximation could certainly be used for β equal to or less than 2. Equation (34) then yields the condition that $1/\omega$ be at least p/V. The greatest relevant value of p is roughly the radius R_n of the largest interstellar cloud, which we may here take to be about 50 pc. The value of the relative velocity, V, may be set equal to 10 km/sec, a conservative value for most clusters. The impulsive approximation is therefore valid if

$$\frac{1}{\omega} \geq 5 \times 10^6 \text{ years} . \tag{39}$$

In a cluster of uniform density ρ_c, ω is given by

$$\omega^2 = \frac{4\pi G \rho_c}{3} . \tag{40}$$

Substitution of equation (40) in (39) yields

$$\rho_c \leq 2.2 \frac{M_\odot}{\text{pc}^3} , \tag{41}$$

corresponding to a value in c.g.s. units of 1.5×10^{-22} gm/cm^3. Evidently, the impulsive approximation is entirely valid for the Hyades but is not always correct for M67.

Next we insert numerical values into equation (14) for the disruption time, t_d. The mean density of interstellar matter will be taken as 1.68×10^{-24} gm/cm^3, corresponding to 1 H atom/cm^3. The value of $(R_n \rho_{in})$ may be taken from a recent survey of interstellar cloud types by Spitzer (1957); the data yield a value of 2.0×10^{-3} gm/cm^2, corresponding to a cloud of radius 20 pc, with a mean density of 20 H atoms/cm^3. Inserting these values in equation (14) and letting γ equal 0.5, with V equal to 10^6 cm/sec, we find

$$t_d = 1.9 \times 10^8 \rho_c \left(\frac{M_\odot}{\text{pc}^3}\right) \text{years} . \tag{42}$$

We have already seen in equation (41) that the impulsive approximation, on which this result is based, is valid for ρ_c up to 2.2 $M_\odot$/pc^3. If ρ_c is ten times this upper limit, the process is not of great interest, as the disruption time, t_d, is about 10^{10} years. We discuss briefly the value of t_d to be expected for ρ_c in the range from 2.2 to 22 $M_\odot$/pc^3.

For clusters with ρ_c in this range, the encounters with small clouds will be correctly given by the impulsive approximation, while encounters with large clouds, characterized by a value of β substantially greater than 2, will produce only a small effect. In this case the integral over m_n in equation (10) must extend only over those clouds which are sufficiently small that β is less than 2. It is readily shown that this upper limit on R_n is proportional to $1/\rho_c$. Moreover, if $\rho_{in} R_n$ is assumed constant, as before, the integral over m_n is proportional to the total density of material in these smaller clouds. While it is not possible to state definitely how the integral will vary as the upper limit, or cutoff, on R_n is decreased, it seems likely that the integral will not decrease very rapidly. For example, if all the interstellar material were in small clouds 5 pc in diameter and these were gathered together in vast cloud complexes 50 pc in diameter, each containing some 100 small clouds, then cutting off the integral in equation (10) to exclude the cloud complexes, including only the small clouds, would reduce dU/dt by a factor of only 2. It seems likely that, even for a density of 22 $M_\odot$/pc^3, equation (42) is not in error by as much as an order of magnitude. For densities near the middle of the range, about 6 $M_\odot$/pc^3, equation (42) is probably correct to within a factor of 2. We shall therefore use equation (42) for all densities, with the understanding that, when t_d computed from this equation exceeds 10^9 years, the true value is probably somewhat larger than the computed one.

It is evident from equation (42) that the disruption time is very short for the more extended clusters. The Hyades cluster, for example, will apparently be completely disrupted in about 6×10^7 years. For the Pleiades, on the other hand, the disruption time does not much exceed 10^9 years and may be considerably less. While uncertainty concerning ρ_c prevents very precise statements, it would appear that, for most galactic clusters, the disruption time is between 10^8 and 10^9 years. Only a relatively dense cluster, such as M67, can apparently survive disruption for a period as great as 5×10^9 years. It may be noted that for a cluster of 5 pc r.m.s. radius, the rate of disruption by these tidal encounters is about thirty times more rapid than the value computed by Bok (1934) for encounters with field stars.

These disruption times may be compared with the evaporation times computed by Spitzer and Härm (1958). From their equation (4) we find that the reference time, T_R, is given by

$$T_R = 8.3 \times 10^5 \frac{N^{1/2} [r_c (\text{pc})]^{3/2}}{(m_0/M_\odot)^{1/2} (\log_{10} N - 0.3)} \text{ years}, \tag{43}$$

where N is the number of cluster stars, assumed to be virtually all of the same mass, m_0. The close numerical agreement between equation (43) and the equation for $\bar{T}_E$ given by Chandrasekhar (1942) is fortuitous, since the cutoff factor, $ln\alpha$, used here is 1.5 times that used by Chandrasekhar. The evaporation time, t_{ev}, is defined as the time in which N would decrease by $1/e$ through evaporation, if the relative rate of evaporation remained constant. Values of t_{ev}/T_R are given in Table 1 of Spitzer and Härm.

It may be noted that the values of t_{ev} are probably less reliable than those of t_d. While there are uncertainties in the size distribution of interstellar clouds which affect equation (42), the evaporation time is computed for a hypothetical uniform density, and unknown corrections are required for an actual cluster, where stars of relatively high energy per unit mass may spend most of their time far from the cluster center.

For a star of mass m_0 the evaporation time is $88T_R$. Hence for a typical cluster containing 300 stars of solar mass and with r_c equal to 3 pc, t_{ev} is 3.0×10^9 years, considerably larger than t_d. For a more concentrated cluster, such as M67, r_c may be taken as 1.2 pc, giving 7.6×10^8 years for t_{ev}, considerably less than t_d. Whether a cluster expands by disruption or contracts by evaporation depends primarily on whether r_c is greater or less than about 2 pc; the value at which t_d equals t_{ev} varies only as $N^{1/9}$.

We are now in a position to discuss the evolution of a cluster, formed at some epoch $t = 0$. From a dynamical standpoint, the first process to occur is the establishment of equipartition among stars of the average mass, m_0. This process takes a time T_R, about 10^7 years. Next the stars of lighter mass approach equipartition with the heavier ones, their random velocities increasing correspondingly. The time required for this process has been investigated by Spitzer and Schwarzschild (1951), on the assumption that the stars of lighter mass interact not with one another but with larger masses, which possess a Maxwellian velocity distribution. For stars of mass much less than the average mass, m_0, numerical integration yields the approximate result for the mean-square velocity,

$$\frac{\langle v^2 \rangle}{v_0^2} = \left(1 + 3.2 \frac{t}{T_R} \right)^{2/5}, \tag{44}$$

where v_0^2 is the mean-square velocity of stars of mass m_0; at the time t equal to zero, the lighter and heavier stars are all assumed to have a Maxwellian distribution with the same mean-square velocity. According to equation (44), the mean energy will double in only 1.5 T_R and will triple in a total time equal to 5 T_R. Evidently, equipartition of energy for the less massive stars will be set up in substantially less than 10^8 years in most galactic clusters.

The increase in velocity of the less massive stars will have two important effects. In the first place, these stars will evaporate more rapidly than those of average mass. As shown by Spitzer and Härm (1958), the evaporation rate increases by a factor of about 2 for each numerical decrease of m/m_0 by 0.2. In the second place, the mean-square distance of the less massive stars will presumably increase as the velocity increases, the stars of low mass essentially forming an extended aura around the cluster. While neither of these effects can be regarded as quantitatively established, the general character of the phenomena to be expected seems moderately clear.

If a substantial increase in r^2 for the less massive stars is accepted, an important result follows at once. The rate of energy input into the less massive stars, as a result of tidal forces produced by passing interstellar clouds, will be much greater for less massive stars than for those of average mass. If, for example, the mean value of r^2 for such stars is an order of magnitude greater than for the cluster as a whole, the energy of these stars may increase appreciably in 10^7 years, the estimated time of relaxation. Under such conditions these lighter stars may gain energy more rapidly than they can lose it by encounters in the dense central region and will be lost by tidal disruption much more rapidly than the average stars.

As the distance of a star from the cluster center increases, the star can finally be pulled completely away by one of two effects. On the one hand, the tidal force of the galactic center can overcome the gravitational pull of the cluster at a distance r, if the mean density of the cluster in the sphere of radius r is less than about 0.1 $M_\odot/\text{pc}^3$. In the Pleiades, for example, this occurs for a star more than about 11 pc away from the cluster center. On the other hand, the change in velocity resulting from a single passing cloud may exceed the velocity of escape from the cluster. From equation (8) it will be seen that escape in this manner is possible when the mean cloud density, within a sphere of radius p, exceeds the mean cluster density, within a sphere of radius r, by about the factor $pV^2/2Gm_n$, the square of the ratio of the relative velocity to the escape velocity from the cloud. This factor is probably between 10^2 and 10^3. Since the densities of interstellar clouds are typically within the range from 0.3 to 30 $M_\odot/\text{pc}^3$ (from 10 to 10^3 H atoms/cm^3), it is not clear which of these two effects is usually more important in finally detaching a star from a cluster.

Evidently, many quantitative details of the disruption process remain to be explored. In particular, whether the less massive stars are lost by evaporation or by rapid disruption of an extended aura cannot be indicated definitely at present. The general conclusion may be drawn, however, that all galactic clusters with mean densities between 0.5 and 5 $M_\odot/\text{pc}^3$ will be completely disrupted by successive tidal disturbances in 10^8–10^9 years.

This theoretical result may account in part for the apparent rarity of old clusters (Oort 1957). A more detailed survey would be required, however, to indicate whether or not the present theory is consistent with the observations.

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