STATISTICAL MECHANICS OF THE SIMPLEST TYPES OF GALAXIES

K. F. Ogorodnikov

In accordance with the principles given by the author in another paper* quasi-stationary stellar systems are considered. These have finite phase volumes and times of relaxation which are not longer than the characteristic time-scales (of the order of magnitude of the rotational period). In these systems the actual phase distribution cannot differ appreciably from the most probable distribution.

In order to find the most probable phase distribution the author applies the method of "phase cells," (known in statistical physics) together with the method of additive parameters. Every stellar system, from the point of view of a dynamical system of material particles, is characterized first of all by three additive parameters expressing properties of motion: the mass – M, the sum of the energies of motion of the stars – E, and the sum of the rotational moments of the stars – K. Every stellar system completely defined by the values of these three parameters is called dynamically definable or briefly a D-system. These are systems of the simplest type, as generally three dynamic parameters are not sufficient for uniquely defining a stellar system (e.g. our galaxy).

In the present paper only D-systems are considered. It is shown that the following characterize the most probable state of D-systems:

1) Uniform rotation (Theorem I); 2) "isothermal" Maxwellian distribution of peculiar stellar velocities (Theorem II). However, Maxwell's law is valid only within a limited range of velocities.

The application of Poisson's equation together with Theorems I and II leads to the conclusion that in the first approximation the stellar density is constant throughout the main body of the D-galaxy and is proportional to the square of the angular velocity of rotation (Theorem III). D-galaxies appear to be dynamically unstable as they do not comply with the requirements of Poincare's criterion for the upper limit of angular velocity (Theorem IV).

Nevertheless we can associate with every D-system a certain classical equilibrium figure of a uniform, gravitating liquid mass. Maclaurin's ellipsoids of the 1st type (planetary) correspond to E-galaxies and to the nuclei of normal S-spirals. Maclaurin's ellipsoids of the 2nd type (disk-shaped) correspond to SO - and normal S-spirals. Finally, the prolate ellipsoids of Jacobi correspond to a new type of galaxies which we shall term needle-shaped (acicular), or simply A-galaxies.

These galaxies are very long and very thin bodies. They are observed in sufficient numbers in the skies but ordinarily they are confused with genuine normal spirals seen edge-on. Their presence among edge-on spirals is revealed by

* This paper will appear in Vol. 34, No. 6, of this Journal.

the apparent excess of the number of such objects as compared to theoretical predictions based on the assumption that all these objects are disk-shaped. Such an excess was pointed out as early as 1920-22 by J. H. Reynolds [5]. The main features of A-galaxies are: a very tufty appearance with an absence of any regular structure (nuclei, spiral arms, annular structure and the like). They represent, probably, one of the earliest stages of galactic evolution. Typical representatives of the A-class of galaxies are: NGC 2188, 2796, 3034, 3556.

Owing to their great instability, the A-galaxies are observed rather seldom in their pure form. In the majority of cases they form the main bodies of SB-spirals displaying a state of disruption through an outflow of streams of stellar gas from their outermost points. (NGC 7741 is a typical example.)

Pear-shaped asymmetry, theoretically predicted for liquid masses by G. Darwin, H. Poincare and A. Liapunov, is also possible for galaxies. It is actually observed in many instances (e. g. NGC 4455, 4631, 7479). In the extreme cases the asymmetry leads to the formation of one-arm spirals (e.g. NGC 4038, 4254).

In the second approximation a differential equation of the Helmholtz type for the additional star density was derived. For its solution a suitable boundary problem based on the uniform equilibrium body of the first approximation must be formulated. In the simplest case of a very thin and flat disk it is possible to apply the Fourier method of separating variables. This leads to a Bessel equation for the distribution of the additional star density along the radius. Different particular solutions correspond either to the central nucleus or to a system of concentric rings. A "barometric" formula was obtained for the density distribution in the z-direction. In the azimuthal direction a harmonic solution was found. Mathematically this solution is identical with the well-known solution for an elastic round membrane.

The above solution permits the interpretation, at least qualitatively, of a number of different structural details of SO- and SB-spirals, which have an annular structure and a strong condensation of mass towards the center. Owing to low density in the end parts of the main body, which is now less prolate, the outflow of stellar gas goes on slower and the streams are thinner than in the case of "classical" SB-spirals, such as NGC 7741 with no central condensation. (Typical examples: NGC 488, 1398, 4725.) The existence of the solution of the Helmholtz equation ensures the stability of the central nucleus and of the annular pattern and prevents their dissipation. At the same time the existence of a flat Maclaurin ellipsoid of equilibrium prevents the majority of stars, ejected from other parts of the old prolate main body, from dissipating and permits them to form a new disk-shaped body.

In all the above cases the spiral arms must be trailing.

1. Introduction

On the basis of the ideas presented in another paper* we will consider that in every star system there exists an inner part, or the main body, and an outer part, or the halo. The character of a star system is determined by the structure of its main body. The halo surrounds the main body in the form of a structureless star cloud.

The dynamical theory must first of all describe the equilibrium shape of the main body of a star system. Then the structure of the halo can be determined by obtaining a suitable function to describe the distribution of the orbit elements of the individual stars moving in a gravitational field which depends almost entirely on the attraction of the main body.

The phase density has been determined only inside the main body [I].

* See the footnote to the abstract. In the following we will refer to it by the symbol [I].

We will assume that the relaxation time T_0 is considerably less than the time necessary for a star, moving with a velocity $\underline{v}$ equal to the mean value of the residual velocities, to cross a given star system along its diameter D*:

$$T_0 < D/v. \tag{1}$$

Doing this, we assume that, at least inside the main body, there exists an efficient system of irregular forces, analogous to viscosity in gases, which enables us to consider the star system as a continuous medium and, consequently, to use the Boltzmann equation and the equations of stellar hydrodynamics. On the other hand, because of the action of irregular forces,the phase distribution must differ little from the most probable one for the given star system. Only by combining two methods, statistical and hydrodynamical, are we able to eliminate the indeterminacy in the solution of the dynamical problem.

Observations show that the distinguishing feature of galaxies is their axial rotation which is either rigid-body rotation, as in E- and, apparently, SB-type galaxies, or quasi-rigid-body rotation as in S-type galaxies, in the inner part up to the distance corresponding to the maximum velocity of rotation (for this see [I]).

We will assume that the galaxies investigated by us, firstly, are isolated dynamical systems; secondly, we will assume that they are in a quasi-stationary state, by which we mean that the star system is in a state of dynamical equilibrium at every instant of time. With dynamical equilibrium the phase density and, consequently, all the internal physical parameters of the galaxies,can be considered to be explicitly independent of time.** We will not consider any interaction forces apart from gravitational forces. Although we have sufficient data to accept the existence of magnetic fields in certain types of galaxies (for example, in ours), as yet it is insufficiently clear what is the role played by the fields in galactic dynamics. It is equally uncertain whether their action is the same in all types of galaxies. It appears that they are only important in gigantic spirals of the type of our galaxy and, even there, their role is probably limited to the influence they have on the distribution of gas and dust. As regards stellar motion, it is not affected by magnetic fields.

2. The Method of Additive Parameters

According to Jeans' Theorem, the phase density f must be a function of only the integrals of motion of a free point. For example, in the case of a star system whose mass distribution is rotationally symmetric*** there will be two such integrals: the energy integral and the angular momentum integral ("integral of areas"). In cylindrical coordinates they will have the form

$$I_1 = \mathrm{P}^2 + \Theta^2 + Z^2 - 2U(\rho, z); \quad J_2 = \rho\Theta; \tag{2}$$

where $U(\rho, z)$ is the gravitational potential.

*For example, for our galaxy we can take D = 16 kps, v = 20 km/sec and then we find $T_0 < 4 \cdot 10^8$ years.

**By investigating equilibrium bodies we also take the necessary first step towards the study of nonstationary galaxies. The nonstationary motion can be regarded as due to the breakdown of initial stationary conditions. For example, we can consider SB-spirals to be formed by the disruption of a primary elongated cylindrical or ellipsoidal body by an escape of a gas of stars from its ends. In the same way, we can consider the formation of dark bands along the equator of E-galaxies to be due to the disruption of an ellipsoidal body. In both cases it is natural to assume that the disruption is produced by the centrifugal force due to rotation.

***In the following we will distinguish between the rotational and axial symmetry of a function. In the former case, the function depends only on ρ, the distance from the axis of symmetry, and the coordinate $\underline{z}$. In the latter case, each point (ρ, θ, z) has a point $(\rho, \theta + \pi, z)$ which is situated symmetrically with respect to the axis. The level surfaces of the function will be surfaces of rotation in the first case, but not in the second. A spheroid is an example of a function with rotational symmetry, an ellipsoid that of a function with axial symmetry.

Therefore,

$$f = F(I_1, I_2), \tag{3}$$

where F is an arbitrary function.

On the other hand, with the condition that in a star system there exists an efficient mechanism of relaxation in the form of irregular forces, the phase density must differ little from the most probable distribution for the star system with the given physical characteristics.

In the method of additive parameters we form functions $\varphi(I_1, I_2)$ of the integrals so that their arithmetic sum for all the stars of the given system represents an additive parameter, i.e., a quantity which describes some characteristic property of this system.

Let

$$P_1 = \sum_{i=1}^{N} m_i \varphi_1(I_{1i}, I_{2i}); \quad P_2 = \sum_{i=1}^{N} m_i \varphi_2(I_{1i}, I_{2i}); \ldots; P_n = \sum_{i=1}^{N} m_i \varphi_n(I_{1i}, I_{2i}) \tag{4}$$

be a system of additive parameters satisfying the conditions that: (1) they are independent and (2) that they determine uniquely the star system. Here I_{1i}, I_{2i}, m_i denote the values of the integrals and the mass of the i-th star, $\varphi_1, \varphi_2, \ldots, \varphi_n$ are the characteristic functions obtained by the method described above.

Any system of additive parameters which satisfies conditions (1) and (2), we will call a defining system. Each star system, generally speaking, has an infinite number of equivalent defining systems of parameters. This is seen, if only from the fact that any linear combination of additive parameters of a given system is itself an additive parameter.

Having found a defining system of additive parameters of the type (4), we then look for the most probable phase distribution satisfying the condition that the parameters $P_1, P_2, \ldots, P_n$ have the specified values. In this case, as we will see below, Jeans' Theorem is automatically satisfied. The method of additive parameters, obviously, can be extended to any number of independent integrals of motion (greater or less than two).

The simplest type of star systems is the dynamically-definable systems or, briefly, D-systems. We can arrive at an understanding of dynamically-definable systems by the following method.

The simplest additive parameters are

$$1)\ P_1 = \sum_{i=1}^{N} m_i; \quad 2)\ P_2 = \sum_{i=1}^{N} m_i I_{1i}; \quad 3)\ P_3 = \sum_{i=1}^{N} m_i I_{2i}. \tag{5}$$

We will show that these parameters have constant values as the result of the general theorems for an isolated system of mass points. In fact, the physical meaning of P_1 is obvious: it is equal to the mass of the star system. Therefore, the invariance of P_1 is an expression of the law of mass conservation. We will call it the mass parameter and will denote it in the following by the symbol M. The second parameter is equal to the sum of the kinetic energies of the individual stars moving in the total gravitational field of the star system. It has the expanded form:

$$P_2 = \sum_{i=1}^{N} m_i (\mathrm{P}_i^2 + \Theta_i^2 + Z_i^2 - 2U_i), \tag{6}$$

where we have written for brevity

$$U_i = U(\rho_i, z_i) = G \sum_{k=1}^{N}{}' \frac{m_k}{r_{ik}}, \tag{7}$$

where G is the gravitational constant, r_{ik} denotes the distance between the k-th and i-th stars, and the prime in the summation sign means that the term for k = i has been omitted in it. From this it is seen that

$$\sum_{i=1}^{N} m_i U_i = -2W, \tag{8}$$

i.e., twice the potential energy of the system (with opposite sign) since in the summation each star is taken into account twice. As regards the sum

$$\sum_{i=1}^{N} m_i (\mathrm{P}_i^2 + \Theta_i^2 + Z_i^2) = 2T, \tag{9}$$

it is equal to twice the kinetic energy of the system. Therefore,

$$P_2 = 2(T + 2W). \tag{10}$$

Even if P_2 is not equal to twice the total energy of the system, it is constant because in the case of a quasi-stationary system of mass points, not only is the law of conservation of energy obeyed, i.e., T + W = constant, but also the virial theorem according to which 2T + W is zero. Therefore, in a quasi-stationary system the kinetic and the potential energies are constant separately, but only their combination P_2 has the additive property with respect to the integrals of motion of the individual stars. We will call P_2 the energy parameter and in the following we will denote it by the symbol E.

Finally, P_3 is the sum of the rotational moments of the individual stars about the axis of symmetry of the star system. In the following we will restrict ourselves to the investigation of only the star systems rotating about the axis of symmetry.* Then P_3 will be constant as the result of the law of conservation of angular momentum. In the following we will denote P_3 by the symbol K. We will refer to M, E and K as dynamical parameters.

In general, the dynamical parameters do not form a defining system, i.e., they are insufficient for a unique description of a given type of star system. The general theorems of mechanics, however, do not give any integrals, except the energy and momentum integrals, for an arbitrary isolated system of mass points.** From this follows that all the other additive parameters must be nondynamical, i.e., they must be statistical in nature. We can thus divide star systems into two classes: dynamically definable and dynamically nondefinable systems. It is clear that every star system among its defining parameters must, in any case, have three dynamical parameters. Therefore, from the point of view of statistical mechanics, dynamically definable star systems are systems of the simplest type and for this reason we will at first restrict ourselves to the investigation of such systems only, leaving the investigation of more complicated dynamically nondefinable systems to a future paper.

3. The Most Probable Phase Distribution for the Simplest Type of Galaxies

We will investigate a D-system of stars (a galaxy) which is defined by three additive dynamical parameters: the mass M, the energy E, and the angular momentum K. We will consider that the system is rotating about its axis of rotational symmetry. For simplicity, we will assume that all the stellar masses are equal. Then, denoting by N the number of stars in the galaxy, we will have three parameters:

$$N,\ E = \sum_{i=1}^{N} (\mathrm{P}_i^2 + \Theta^2 + Z_i^2 - 2U_i);\quad K = \sum_{i=1}^{N} \rho_i \Theta_i. \tag{11}$$

*In other words, we will not investigate the cases in which, in addition to pure rotation, there are precessional and nutational motions, since these are unlikely to arise in a star system not subject to the influence of external forces.

**We do not, of course, include the integral which describes the uniform motion of the center of mass as it does not enter into our calculations.

Taking into account that the main body (which we have in mind here) of the galaxy occupies a finite volume, while the velocities are limited by the escape velocity, we conclude that the phase volume of the galaxy Γ is limited. To find the most probable phase distribution we will use the method of phase cells, which is well-known in physics. For this we divide the phase volume into a very large number m of very small phase cells which, however, must satisfy the condition

$$1 \ll m \ll N, \tag{12}$$

i.e., m must be very large and, at the same time, very small by comparison with N. In other words, the average number of stars in one phase cell N/m must be larger than unity (for example, several tens or several hundred). We denote the volume of the k-th cell by γ_k. γ_k satisfies the relation

$$\sum_{k=1}^{m} \gamma_k = \Gamma. \tag{13}$$

Let us now assume, as is usual in such cases, that the a priori probability for any star to lie in the k-th cell is the same for all stars, is independent of the position of the cell in the phase volume Γ, and is equal to the geometrical probability

$$p_k = \gamma_k / \Gamma. \tag{14}$$

Obviously, this hypothesis is equivalent to the assumption that in the absence of the constraints imposed by the conditions E = constant and K = constant, the phase distribution will be uniform over the whole volume Γ.

The probability that with a random distribution of stars there will be $n_1, n_2, \ldots, n_m$ stars in the cells is given by the binomial distribution as

$$P = \frac{N!}{n_1!\, n_2! \ldots n_m!}\, p_1^{n_1}, p_2^{n_2} \ldots p_m^{n_m}. \tag{15}$$

Within each cell we can take the phase coordinates of the points to be equal. Hence, the quantities given by (11) can be rewritten in the form

$$N = \sum_{k=1}^{m} n_k; \quad E = \sum_{k=1}^{m} n_k \left(\mathrm{P}_k^2 + \Theta_k^2 + Z_k^2 - 2U_k\right) = \sum_{k=1}^{m} n_k E_k; \tag{16}$$
$$K = \sum_{k=1}^{m} n_k \rho_k \Theta_k = \sum_{k=1}^{m} n_k K_k,$$

where the subscript k refers to the value of the given quantity for the k-th cell.

The problem now reduces to the determination of the maximum of the probability P as a function of the variables $n_1, n_2, \ldots, n_m$ which are subject to the conditions (16). Since, after finding the maximum we intend to go over to a continuous and differentiable phase density (which obeys Boltzmann's equation), we can consider the variables n_m to be continuous.

The Lagrangian for the variational problem can be taken as

$$L = \ln P + k_1 \sum_k n_k + k_2 \sum_k n_k E_k + 2k_3 \sum_k n_k K_k, \tag{17}$$

where k_1, k_2, and k_3 are Lagrange multipliers.

Differentiating with respect to n_k, we obtain

$$\frac{\partial L}{\partial n_k} = -\frac{d \ln (n_k!)}{d n_k} + \ln p_k + k_1 + k_2 E_k + 2k_3 K_k.$$

To find the meaning of the derivative of $\ln(n_k!)$ we will assume that the occupation number for the given cell is

$$n_k \gg 1. \tag{18}$$

Then, with sufficient accuracy, we have that

$$\frac{d \ln (n_k!)}{dn_k} = \frac{\ln [(n_k+1)!] - \ln (n_k!)}{(n_k+1) - n_k} = \ln(n_k+1) = \ln n_k. \tag{19}$$

Substituting this expression into the equation of $\partial L/\partial n_k$, we find after rearrangement that

$$n = pe^{k_1+k_2E+2k_3K}, \tag{20}$$

where the subscript k has been discarded, as it is no longer necessary. Instead of the occupation number of a phase cell, we now introduce the phase density f:

$$n = fd\gamma/\Gamma, \tag{21}$$

and obtain the following expression for the most probable phase density:

$$f = e^{k_1+k_2E+2k_3K}. \tag{22}$$

The condition $n_k \gg 1$ has an important basic significance. In the present problem it is determined by the necessity to change over from a discrete distribution to a continuous one and is therefore not associated with the mathematical method used by us, but with the nature of the physical problem. At the same time it has a very important consequence, namely, that all conclusions about the phase distribution are meaningful only for those values of the phase coordinates for which the occupation of the phase cells is sufficiently high. In particular, the Expression (22) for the phase density has no meaning for large values of the coordinates (outside the main body), as well as for high velocities (greater than the escape velocity).

4. Investigation of the Phase Distribution for Dynamically-Definable Galaxies

To obtain the physical meaning of the solution found, we transform the exponent in Eq. (22):

$$\begin{aligned} k_2E + 2k_3K &= k_2\,(\mathrm{P}^2 + \Theta^2 + Z^2 - 2U) + 2k_3\rho\Theta = \\ &= k_2\,[\mathrm{P}^2 + (\Theta - \Theta_0)^2 + Z^2 - 2U_1], \end{aligned} \tag{23}$$

where

$$\Theta_0 = -\frac{k_3}{k_2}\rho \tag{24}$$

and

$$U_1 = U + \frac{1}{2}\Theta_0^2. \tag{25}$$

Here, Θ_0 is the velocity of the centroid. We see that it is at right angles to the galactic meridian through the given point. From the hypotheses that we made at the beginning we have obtained the rotation of the galaxy about its axis of symmetry and Θ_0 gives the velocity of the centroid.

Eq. (24) shows that Θ_0 is proportional to ρ. Therefore, the angular velocity of the centroid,

$$\omega_0 = \Theta_0/\rho = -k_3/k_2 = \mathrm{const} \tag{26}$$

is constant and, consequently, we obtain Theorem I.

Theorem I. The most probable form of rotation for dynamically-definable galaxies is rigid-body rotation.

In our paper [I] we have shown that, judging by all the data, this in fact agrees with observations. On the other hand, from Eqs. (22) and (23) it can be seen that the dispersions of the three velocity components are the same and are equal to

$$\sigma^2 = -\frac{1}{2k_2} = \text{const.} \tag{27}$$

From this we obtain the second theorem:

Theorem II. The most probable distribution of the residual stellar velocities at all points of dynamically-definable galaxies is Maxwell's distribution with the same dispersion for all points of the system ("isothermal" distribution).

Integrating (22) over all the velocity space and taking (23) into account, we obtain the star density ν,

$$\nu = Ce^{\frac{U}{\sigma^2} + \frac{\theta_0^2}{2\sigma^2}}, \tag{28}$$

where

$$C = (2\pi)^{3/2}\sigma^3 e^{k_1} \tag{29}$$

is a constant which is determined by the condition that the integral of ν over all of the galactic volume * must be equal to N, the number of stars in it.

The physical meaning of the Lagrange multipliers has also become clear. If according to (27) k_2 determines the dispersion, k_3 on the basis of (26) determines the angular velocity of rotation, and, finally, according to (29) k_1 determines the mass, for which it is necessary to know the potential (i.e., the mass distribution), the size and shape of the galaxy, i.e., it is necessary, first of all, to solve the dynamical problem. From (28) it follows immediately that the level surfaces of the star density are given by the equation

$$U + \frac{1}{2}\omega_0^2\rho^2 = C. \tag{30}$$

This equation is the same as the analogous equation in the classical problem of the equilibrium figures of rotating liquid bodies and it will be used extensively by us in the following.

To eliminate the potential from Eq. (28), we take its logarithm and apply the Laplace operator. Then, using Poisson's Equation

$$\Delta U = -4\pi G m\nu,$$

where G is the gravitational constant and $\underline{m}$ is the average stellar mass, we obtain a differential equation for the star density:

$$\sigma^2\Delta(\ln\nu) = -4\pi G m\nu + 2\omega_0^2. \tag{31}$$

This is the fundamental equation for the problem of determining the distribution of mass inside the D-galaxies. In expanded form it is written as:

$$\sigma^2\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\ln\nu}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\ln\nu}{\partial\theta^2} + \frac{\partial^2\ln\nu}{\partial z^2}\right\} = -4\pi G m\nu + 2\omega_0^2. \tag{31'}$$

* Here, as everywhere else, we have in mind the volume of the main body of the galaxy.

In the case when the star system possesses rotational symmetry, the second term in the bracket must be omitted.

Eq. (31 ') is too complicated to be worth an attempt to integrate it in the general case. However, a special case of the solution is conspicuous. In fact, in the left-hand side the star density appears only through the derivative of its logarithm. Therefore, if we restrict ourselves to the investigation of sufficiently "smooth" models, then the absolute magnitude of the left-hand side will be small. Hence, neglecting the left-hand side, we obtain the special solution:

$$2\pi G m\nu - \omega_0^2 = 0$$

or

$$\nu_0 = \frac{\omega_0^2}{2\pi G m}. \tag{32}$$

This is a very important result which leads to further conclusions. The following theorem is proved by Eq. (32).

Theorem III. In any dynamically-definable galaxy, if the star density varies smoothly enough, the star density is constant in the first approximation and is proportional to the square of the angular velocity of rotation.

Before we proceed to further deductions, we must make two remarks of a fundamental nature:

1) Firstly, if we note that the product $m\nu$ represents the density of matter d:

$$m\nu = d,$$

then Eq. (32) can be rewritten in the form:

$$\frac{\omega_0^2}{2\pi G d} = 1,$$

which shows that the value of the star density obtained by us as a first approximation does not satisfy Poincare's criterion for the upper limit of the angular velocity [1]. We thus obtain Theorem IV.

Theorem IV. Dynamically-definable galaxies with quasi-constant star density* are dynamically unstable.

2) ν = const cannot be an exact solution of any physical problem. In fact, from Eq. (28) it follows that with ν = const, the potential must be a function of ρ only (Θ_0 is a function of ρ). But this is not physically possible, since in this case the component of the gravitational field parallel to the z axis (axis of symmetry of the galaxy) will be equal to zero, while the component of the force along the radius ρ will be independent of z. Consequently, in this case we would have a uniform circular infinite cylinder, rotating about its axis. We can consider this case to be physically meaningless.

To obtain dynamical stability and a physically real model, we must find the second approximation to the solution of Eq. (31). Taking

$$\nu = \nu_0 + \nu_1, \tag{33}$$

(where ν_0 is the first approximation to ν given by (32), while ν_1 is a small quantity by comparison with ν_0) and substituting ν into (31), we can linearize the equation and obtain, after transformation, the differential equation from which the second approximation can be obtained:

$$\Delta\nu_1 + 2\,\frac{\omega_0^2}{\sigma^2}\,\nu_1 = 0. \tag{34}$$

This equation is well known in mathematical physics and is called the Helmholtz Equation (for example, see [2]). In particular, it was used by H. Poincare and J. Jeans in developing a theory of ellipsoidal figures of equilibrium.

We will study Helmholtz's Equation in greater detail below. We will only make a few brief remarks now.

* That is, with a density constant in the first approximation.

For brevity, in what follows we will refer to the uniform density ν_0 as the primary density and to ν_1 as the additional density. The effect of the additional density is mainly that it determines the formation of central nuclei, which are in fact observed in most galaxies. Because of the additional density, the quantity

$$\Omega = \frac{\omega_0^2}{2\pi G m\nu} \tag{35}$$

decreases and hence the dynamical instability of the galaxy decreases.

In addition, as we will see below, the additional density leads to the formation of certain structural details in the galaxies, in particular, to the formation of an annular structure, as well as of a system of planet-like condensations similar to beads, sometimes observed in galaxies (for example, NGC 4324). The annular structure sometimes exists together with the spiral (for example, in M31).

5. The Equilibrium Figures of Galaxies

As the result of the analysis carried out in the preceding paragraphs, we found that the dynamically-definable galaxies have the following properties: 1) they rotate as a rigid body, 2) they are uniform with respect to the distribution of mass (the density is everywhere the same), and 3) they behave as continuous bodies in the sense that the methods of hydrodynamics are applicable to them. This leads to the possibility of using, in the case of galaxies, the classical theory of the equilibrium figures of rotating uniform liquid bodies.

Referring for details to the textbooks (for example, see [3, 4]), we recall that, depending on the numerical values of the quantity Ω in Eq. (35), there exist the following equilibrium ellipsoids:

Values of Ω	Type of equilibrium ellipsoids
1. From 0 to $\Omega_1 = 0.1871...$	3 ellipsoids of which two are Maclaurin ellipsoids and one is a Jacobi ellipsoid
2. From Ω_1 to $\Omega_0 = 0.2247$	2 Maclaurin ellipsoids
3. With $\Omega = \Omega_0$	1 Maclaurin ellipsoid
4. With $\Omega > \Omega_0$	No ellipsoids

In the cases when there are two Maclaurin ellipsoids ($\Omega < \Omega_0$), to each value of Ω there corresponds one "planetary" and one "disc-shaped" Maclaurin ellipsoid. The planetary ellipsoid is slightly oblate and as Ω tends to zero it tends to a sphere. The disc-shaped ellipsoid is always very oblate, it is always similar to a circular flat disc and as Ω tends to zero, it tends to an infinite thin flat layer. As Ω increases and tends to the value $\Omega_0 = 0.2247...$, the oblateness of the first ellipsoid increases, while that of the second decreases, and when $\Omega = 0.2247$ the ratios of the semi-axes of both ellipsoids become the same and equal to

$$\varepsilon_0 = \frac{a-c}{a} = 0.6323...$$

A Jacobi ellipsoid is an elongated triaxial ellipsoid which rotates about its minor axis. It only exists for small values of $\Omega \leq \Omega_1 \leq 0.1871...$. If we take that $a > b > c$, then as Ω tends to zero the semiaxes b and c tend to equality, and the axis $a \to \infty$, Thus, for small values of Ω Jacobi's ellipsoid is "needle-shaped," i.e., it has the form of a long and thin needle with pointed ends. As Ω increases and tends to $\Omega_1 = 0.1871...$, the axes a and b tend to equality and at $\Omega = \Omega_1$ Jacobi's ellipsoid becomes the biaxial Maclaurin ellipsoid and its oblateness becomes equal to

$$\varepsilon_1 = \frac{a-c}{a} = 0.4172\ ...$$

It is not hard to find examples of galaxies having the form of Maclaurin ellipsoids. E-galaxies and also the nuclei of S-spirals are related to the planetary ellipsoids. The presence in some of them of an equatorial rib and the outflow of matter along it can be easily fitted into the picture provided by the Roche model.

Disc-shaped galaxies also occur in adequate numbers. We can take as belonging to this type all of the varieties of S-spirals, including SO-spirals in which the spiral structure changes into a system of rings (for example, NGC 4826). Their disc-like shape is particularly apparent when these galaxies are observed edgewise (for example, NGC 4565).

Finally, if we accept the hypothesis that SB-spirals represent the result of the outflow of a stream of star gas from the ends of a primary body, then we can consider that SB-spirals are the result of the disruption of needle-shaped ellipsoids (for example, NGC 1300, NGC 7741, and others).

The fact that needle-shaped galaxies are observed most frequently in the form of SB-spirals, characterized by the escape of matter from the ends of the primary body, can be readily understood if we take into account that of all the ellipsoid types, Jacobi ellipsoids on the average correspond to the smallest values of Ω and because of this these galaxies are the least stable. A question arises, however, are there any examples of needle-shaped galaxies which are not SB-spirals? Such spirals must occur among the galaxies usually classified as "spirals seen edge-on." In fact, there is a marked difference in appearance between the various examples of this type of galaxies, as a comparison of such galaxies as, for example, NGC 4565 and NGC 4631 will show. In the former we can clearly see the central nucleus and a dark equatorial band, i.e., the features we can expect to observe in normal spirals; on the other hand, in NGC 4631 we do not see any signs of a nucleus, nor of a dark equatorial band. This gives us a reason for assuming that in the latter case we are observing a needle-shaped galaxy from one side.*

The presence of two different types of galaxies (disc- and needle-shaped) among the S-spirals must affect their observed distribution of oblateness. As can be easily shown, the fraction of galaxies with a high oblateness, usually taken to be the galaxies "seen edge-on," will be larger in the case when needle-shaped galaxies exist than if they are absent. But the excess of the type indicated has been known for a long time, as already in 1920-1922 it was discovered and investigated by Reynolds [5].

So as not to digress, we will restrict ourselves here to the above remarks and postpone a more detailed discussion of this question to another paper. Here, we have still to show that needle-shaped galaxies belong to the class of dynamically-definable galaxies.

6. On the Existence of Needle- and Pear-Shaped Galaxies

In the derivation of the most probable phase distribution in §3, we have assumed that the star system possesses rotational symmetry in its mass distribution. The integral of areas relative to the symmetry axis exists for each star only if this condition holds.

We will now show that the phase distribution obtained by us remains valid for a wider class of bodies of which the triaxial ellipsoid is a special case. That is, we will prove Theorem V.

Theorem V. The class of dynamically-definable star systems includes also any system rotating as a rigid body whose density distribution is symmetrical with respect to a plane passing through the axis of rotation and, at the same time, whose velocity distribution is symmetrical with respect to a plane which is perpendicular to the first plane.

Let $f(x, y, z; \dot{x}, \dot{y}, \dot{z})$ be the phase distribution. If we take the planes of symmetry indicated above to be the coordinate planes (x, z) and (y, z), then the symmetry property expressed in Theorem V can be written down in the form of an equality:

$$f(x, y, z; \dot{x}, \dot{y}, \dot{z}) = f(x, -y, z; \dot{x}, \dot{y}, \dot{z}) = f(x, y, z; -\dot{x}, \dot{y}, \dot{z}) = \\ = f(x, -y, z; -\dot{x}, \dot{y}, \dot{z}). \tag{36}$$

If in dividing the phase volume Γ we take the symmetrically-placed cells to be identical, then the occupation numbers of the phase cells will also have the Symmetry Property (36)

* NGC 2188, 3034, 3556, and 2796 can serve as other examples of needle-shaped galaxies.

$$n_k = n(x_k, y_k, z_k; \dot{x}_k, \dot{y}_k, \dot{z}_k).$$

In a system of coordinates rotating with an angular velocity ω_0 the equations of motion for a free point will be

$$\begin{aligned} &\ddot{x} - 2\omega_0\dot{y} - \omega_0^2 x = \frac{\partial U}{\partial x}; \\ &\ddot{y} + 2\omega_0\dot{x} - \omega_0^2 y = \frac{\partial U}{\partial y}; \quad \ddot{z} = \frac{\partial U}{\partial z}, \end{aligned} \tag{37}$$

where U is the potential.

It is clear that U will have the same symmetry as the mass distribution, i.e.,

$$\frac{\partial U(x, y, z)}{\partial y} = -\frac{\partial U(x, -y, z)}{\partial y}.$$

Multiplying the first of the Eqs. (37) by $-y$, the second by x and adding the two, we obtain, after the simple transformations usual in such cases, the equation

$$\frac{d}{dt}(x\dot{y} - y\dot{x}) + 2\omega_0(x\dot{x} + y\dot{y}) = x\frac{\partial U}{\partial y} - y\frac{\partial U}{\partial x}. \tag{38}$$

Let us now write down the same equation for another star whose phase coordinates are $(x, -y, z; -\dot{x}, \dot{y}, \dot{z})$. We will have

$$\frac{d}{dt}(x\dot{y} - y\dot{x}) - 2\omega_0(x\dot{x} + y\dot{y}) = -\left(x\frac{\partial U}{dy} - y\frac{\partial U}{\partial x}\right).$$

Adding this equation to Eq. (38), we obtain, for a system consisting of two symmetrically-placed stars, the equation:

$$\frac{d}{dt}(x_1\dot{y}_1 - y_1\dot{x}_1) + \frac{d}{dt}(x_2\dot{y}_2 - y_2\dot{x}_2) = 0$$

or

$$(x_1\dot{y}_1 - y_1\dot{x}_1) + (x_2\dot{y}_2 - y_2\dot{x}_2) = C, \tag{39}$$

where the subscripts 1 and 2 denote the coordinates of the first and second stars. Thus, in the case of the symmetry described in Theorem V, the angular-momentum integral exists for each pair of symmetrically-placed stars and as we have agreed to take the occupation numbers of symmetrically-placed phase cells equal, then the expression for the additive parameter

$$K = \sum_{k=1}^{m} n_k \rho_k \Theta_k$$

will remain unchanged. Because of this all the other results connected with the most probable phase distribution will also remain unchanged. The only unimportant difference is that in this case we must consider the star system as formed not from individual stars, but from pairs of symmetrically-situated stars and these will be our material particles.

The applicability of the theorem formally proved by us is considerably wider than is necessary for the physical problem that we are investigating. In fact, with the most probable phase distribution, the distribution of velocities at any point is Maxwellian, i.e., it possesses spherical symmetry. Therefore, it is symmetrical with respect to any plane which passes through the center of the velocity distribution [the point (0, 0, 0)] in the rotating system of coordinates. As regards the mass distribution, the condition of symmetry with respect to a plane passing through the axis of rotation is satisfied not only by triaxial ellipsoids and similar bodies, which are symmetrical with respect to both of their principal planes passing through the minor semiaxis, but also by pear-shaped equilibrium bodies which were investigated classically by A. A. Liapunov, H. Poincare, G. Darwin, L. Lichtenstein, and others.

We have already discussed the question of the occurrence of galaxies with the shape of triaxial ellipsoids. Similarly, asymmetric pear-shaped galaxies are also often observed. We can even state that this type of assymmetry is almost always present to some extent. It is noticeable, for example, in the needle-shaped galaxy NGC 4631 which we have mentioned already. It is also noticeable in the normal-spiral galaxy seen edge-on: NGC 4565. In the most marked cases of this asymmetry it happens that S-galaxies have only one spiral arm (for example, NGC 4038).*

Thus, on the basis of the theory considered by us, the SB-spirals are the result of the rotational instability of ellipsoidal or pear-shaped bodies. Depending on the value of the star density, the angular velocity of rotation, and the presence of a central condensation (the nucleus),we have different forms of SB-spirals. In the absence of a central condensation, with a high average star density and a high angular velocity, we have a large outflow in the form of wide streams (for example, NGC 7741). In the presence of a very marked central condensation we have thin streams, which corresponds to a low value of the star density at the ends of the main body (examples: NGC 1300, 1530, and others).

Using the outflow theory, we can explain fairly simply some of the details of SB-spirals which at first sight seem to be unimportant. For example, the wide streams (as in NGC 7741) curve markedly inwards to the center because of the gradual shortening of the main body as its mass decreases. On the other hand, the thin streams curve inwards very slightly and do not depart significantly from a circular shape, because in this case the shortening of the main body takes place much more slowly.

In many SB-spirals the peculiarity is observed that the beginning of the stream does not coincide with the end of the main body. On the basis of the outflow process described above, this phenomenon can be explained in the following way. The escape of stars from the main body, naturally, mainly takes place at the expense of the stars with the highest absolute (i.e., relative to the system of coordinates with origin at the center of mass of the galaxy and with axes fixed in direction) velocities, because it is necessary for the velocity of a star to exceed the local escape velocity before it can break away. Therefore, the stream of star gas must mainly consist of stars whose absolute velocity is greater than the circular velocity of the local centroid. Because of this, the stars that have broken away will continually overtake the main body as the galaxy rotates. On the other hand, in the remaining part of the main body the circular velocity of the centroid will decrease because of the loss of the stars which overtake the centroid. Therefore, the stream as a whole will start to overtake slightly the main body and this slow advancing will also continue after the outflow has stopped.

We can often observe not one, but two or even more thin streams forming a series of rings or, more accurately, a series of sections of slowly-contracting spirals (for example, NGC 488 and NGC 1398). This can be explained by assuming that at the end of one period of outflow, the main body,because of gravitational contraction,

---

* At this stage it may be useful to point out that the reason why Chandrasekhar and G.Kuzmin, who investigated the problem of finding the general solution for a stationary star system, arrived only at the solution of the special form: to systems with rotational (in our terminology) symmetry (if the solutions with screw symmetry are not considered to be physically completely unreal), is that these authors imposed certain formal mathematical conditions on their solutions.In particular,Chandrasekhar imposed the condition that the solution should give differential motion of the centroids, while Kuzmin the condition that among the integrals of motion in the gravitational field of the system there should be an integral linear in the velocities which, as can be seen from Kuzmin's analysis, is equivalent to the condition for the existence of the angular-momentum integral.

The problem of the statistical mechanics of normal spirals of the type of our galaxy will be investigated by us in another paper, in which we will show how the method of additive parameters can be used to interpret the differential(nonrigid-body)rotation observed in these galaxies. Therefore, here we will restrict ourselves to a few brief remarks on the morphology of the galaxies. If we assume that SB- and SO-spirals are formed as the result of the disruption of dynamically-unstable needle-shaped galaxies, then there can be no objections to the assumption that normal spirals are also produced in the same manner.

takes the shape of another equilibrium figure. As the result of this, its moment of inertia about the axis of rotation is decreased and the angular velocity is correspondingly increased. This may initiate another process of outflow, and so on.

It is clear that these considerations must be verified by quantitative calculations on the basis of the theory of nonstationary processes, which we will not carry out here.

From the above discussion it follows that in SB-spirals the arms must be curled inwards, even if in some cases this is very slight. In this respect the results of the present theory differ from the results of the well-known theory of Lindblad. We can expect that Linblad's theory is applicable to galaxies of the type of NGC 2841, 7217, etc., which have a massive nucleus in the form of an oblate ellipsoid in which from various points along the equator (not necessarily at the opposite ends of a diameter) there is an emission of thin spiral streams as the result of the instability of circular orbits outside the main body of the galaxy.

We will investigate the problem of normal spirals in the next paper.

In conclusion, we will briefly discuss the solution of Helmholtz's Equation and the ring structure of galaxies which follows from it.

7. The Formation of Central Condensations and Annular Structures in Galaxies

Helmholtz's Equation for the additional density has the form

$$\Delta u + k^2 u = 0, \tag{40}$$

where

$$k^2 = 2\frac{\omega_0^2}{\sigma^2}.$$

Inside the main body, the unknown function $\underline{u}$ can take both positive and negative values. However, the negative values cannot be greater in absolute magnitude than the primary density ν_0, otherwise the star density would become negative. Besides, this is also unacceptable for purely formal reasons, since $|u| < \nu_0$. For the same reason, $\underline{u}$ cannot be negative outside the main body.

To solve Eq. (40) it is necessary to formulate the appropriate boundary problem, i.e., to specify the value of either $\underline{u}$, or of its derivative along the normal to some closed surface which represents the external surface of the galaxy.

However, as an example, we will restrict ourselves to the investigation of one simple case, that of a very flat spheroid which we will take as a flat circular disc.

In expanded form in cylindrical coordinates the equation can be written in the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0, \tag{41}$$

whose solutions can be found in textbooks of mathematical physics (for example, see [2], vol. 3, 4th edition, § 154).

To separate the variables, we take

$$u = P\Theta Z, \tag{42}$$

where P, Θ and Z are functions of the variables ρ, θ, and $\underline{z}$, respectively. For brevity we will call them the radial, azimuthal and vertical factors, respectively. Substituting (40) into (39), we obtain for P, Θ, and Z the following equations:

$$\Theta'' + p^2\Theta = 0;$$

$$Z'' - l^2 Z = 0; \tag{43}$$

$$P'' + \frac{1}{\rho} P' + \left[(k^2 + l^2) - \frac{p^2}{\rho^2}\right] P = 0, \tag{43}$$

where p and l are new constants (p is an integer which we can consider to be always positive). Since Θ must be a periodic function with period 2π and it does not matter where we start measuring θ, in the case of the azimuthal factor we will have

$$\Theta = \sin p\theta \text{ (when } p \neq 0\text{) and } \Theta = 1 \text{ (when } p = 0\text{).} \tag{44}$$

For the vertical factor we obtain the formal solution

$$Z = Ae^{lz} + Be^{-lz}.$$

However, Z must obviously satisfy the condition that $Z \to 0$ as $|z| \to \infty$. In addition, we can take $Z(0) = 1$ since the value of the density for $z = 0$ will be determined by the value of the function P.

Thus in the case of the vertical factor we obtain the "barometric" equation*

$$Z = e^{-l|z|}. \tag{45}$$

The third equation, as is well known from the theory of linear equations, has only one solution which remains finite at $z = 0$. This is the Bessel function of order p. If we take

$$k^2 + l^2 = q^2, \tag{46}$$

then the solution will be

$$P = J_p(q\rho). \tag{47}$$

Since the density must be equal to zero on the equator of the galaxy, we obtain for q a number of values, each of which corresponds to a root of the equations

$$J_p(\mu) = 0.$$

If we denote the i-th root of the Bessel function of order p by μ_i^p we will have

$$q_i^p = \frac{\mu_i^p}{\rho_0}, \tag{48}$$

where ρ_0 is the equatorial radius of the galaxy.

In the general case, the solution of Helmholtz's Equation must be the sum of particular solutions of the type (42), i.e.,

$$u = A_0 u_0 + A_1 u_1 + \ldots, \tag{49}$$

where $A_0, A_1, \ldots$ are constant coefficients.

From the theory of Bessel functions it is known that there exists an infinite number of positive roots for each value of p. In our problem the Bessel function $J_0(q\rho)$, of order $p = 0$, is of particular importance. It is the only one of all the Bessel functions which has a value at the coordinate origin different from zero: $J_0(0) = 1$. Therefore, the solution of the form (47) for every galaxy which has a central nucleus must contain a term corresponding to $p = 0$. If the other terms are absent ($A_k = 0$, $k \neq 0$), then the distinguishing feature of the galaxy must be an annular structure with unbroken rings, since when $p = 0$ the aximuthal factor $\Theta = 1 =$ constant.

*It is known that the "barometric" law for the decrease of star density with increasing z coordinate is observed in the case of our galaxy (for example, see [6]).

NGC 4826 is an example of an unbroken annular structure. It has at least three concentric rings surrounding the central nucleus. The elliptical shape of the nucleus and the rings must be the result of an inclination to the line of sight. Two concentric rings are quite clearly visible. The first ring is broken in two places which are diametrically opposite. If this break is real, and is not the result of the absorption of light by dark matter, then this indicates that for this galaxy the Solution (49) contains a term corresponding to $p = 1$.

The galacy NGC 4324 has an even more complicated appearance; in the rings surrounding the nucleus there appear equally-spaced bead-like condensations of approximately equal size and brightness, similar to the annular nebula in the Kant-Laplace nebular hypothesis. If we consider that the beads fill uniformly all of the ring, then there must be 16 of them and the corresponding solution must be of the order $p = 16$.

However, two difficulties arise. The first of these is that the first maximum of the Bessel function $J_{16}(x)$ occurs at $x = 18.1$, while the first few maxima of the function $J_0(x)$ occur at $x = 3.83, 7.02, 10.17, 13.32$ and 16.47. In other words, it appears as if inside the ring corresponding to $J_{16}(x)$ there should be at least five rings corresponding to the solution belonging to $p = 0$ which must undoubtedly exist in view of the presence of the nucleus. This difficulty, however, can be easily eliminated if we remember that the eigenvalue $\underline{q}$, on which the solution depends, is determined from Eq. (46) which contains, in addition to the number $\underline{k}$, depending only on the properties of the galaxy itself, the coefficient l of the "barometric" equation. The bigger l, the faster is the decrease of density with height. For large values of l, and consequently of $\underline{q}$, the first root of the equation $J_{16}(x) = 0$ can be reached for comparatively small values of ρ. Thus, at least in principle, the first difficulty can be easily eliminated.

The situation is more complicated in the case of the second difficulty. Similarly to the ring in NGC 4826, the ring in NGC 4324 is also broken at two points on the opposite ends of a diameter. In the region of the break there are gaps also in the bead-like structure. The fact that, as in NGC 4826, these breaks occur exactly at the ends of the observed larger diameter of the ring, is very striking. This throws doubt upon the correctness of the "natural" assumption that the elliptical shape of the nucleus and the rings is caused only by the inclination of the galaxy to the line of sight. It is probable that in this case the nongravitational (for example, magnetic) forces have an important effect. This is very likely, particularly because an analogous phenomenon, i.e., a break in the ring at the ends of the larger diameter, is also observed as was pointed out by G. A. Gurzadian in some of the planetary nebulae, i.e., in objects of a completely different nature to galaxies; in planetary nebule the magnetic field undoubtedly plays an important part.*

All this shows that the nature of the annular structure of galaxies is quite complicated. We will not discuss it any further since our aim is only to investigate the principles of statistical theory, leaving to a future time its application to the structure of individual objects. We will only remark that the annular structure frequently exists together with the spiral, there being a characteristic superposition of the two structures.

8. Summary

We have seen that to the class of the simplest type of galaxies, or D-galaxies, there belong a number of varieties: 1) SB-spirals (barred), 2) SO-galaxies with an annular structure and, finally, 3) E-galaxies (elliptical). We have discussed the first two forms in greater detail. We have not considered E-galaxies because their structure and evolution can apparently be interpreted sufficiently well in terms of the well-known Roche theory of the outflow from the equatorial rim of a rotating spheroidal body (for example, see [4], vol. 3, § 54). In addition, we concluded that there exists a new type of galaxy, which we have called a needle-shaped galaxy, that has not been previously investigated. To each of the galactic types listed above there corresponds a prototype from the classical theory of equilibrium figures of gravitating rotating uniform liquid bodies.

E-galaxies correspond to Maclaurin ellipsoids of the first kind, or planetary ellipsoids. SO-galaxies correspond to Maclaurin ellipsoids of the second kind, or disc-shaped ellipsoids. Needle-shaped galaxies correspond to Jacobi ellipsoids. The same ellipsoids correspond to the main body in SB-spirals which represent the result of rotational instability. It is probable that the needle-shaped star systems are the most unstable of the galaxies, which is also connected with their flocculent structure. We did not refer to normal spirals which, as is known, are an important variety of galaxies. Our galaxy belongs to this class. The theory of normal spirals must be the

* D. Evans in his article on the bright galaxies of the southern sky in Vistas in Astronomy, vol. 2 (Pergamon Press, 1956) has also pointed out this circumstance.

subject of a separate paper. We will only point out here that in shape, of course, they resemble Maclaurin disc-shaped ellipsoids.

A few words on the meaning of the concept of quasi-stationarity in the investigation of galaxies. As we have seen, this concept allows us to apply the methods used in the study of stationary states to systems which, strictly speaking, are not in a stationary state, provided that the departure from this state takes place sufficiently slowly.

The question arises whether it is valid to apply this method in investigating galaxies, many of which, for example SB-galaxies, are in an obviously nonstationary state.

The main problem of stellar dynamics is the discovery of laws governing the evolution of star systems. Any evolutionary development of a system must be connected with a change of shape, i.e., basically with a redistribution of mass. Therefore, all evolutionary processes are basically forms of nonstationary motion of star systems.

However, it must be recognized that at present we are not able to solve in practice or even formulate the problem of nonstationary motion. At this stage, therefore, we are forced to restrict ourselves to the solution of problems of stationary motion. It can be shown that the solution of stationary problems is an appropriate and necessary stage before we pass to the solution of nonstationary problems. We have to consider the following fact. It is hard to imagine that the evolution of star systems has the same turbulent and catastrophic nature at all stages. It is more reasonable to assume that, at least in some stages, the process of evolution will slow down and during this period we can consider that in the first approximation the star system is in a state of equilibrium. States of this type are just what we have defined as quasi-stationary. In a quasi-stationary state a star system evolves so slowly that at any given instant it can attain the shape of the equilibrium figure corresponding to the dynamical conditions.

Along the line of evolutionary development of star systems, the states of quasi-stationary equilibrium must represent nodal points. Most of the various types of galaxies observed by us must correspond to these nodal points, since the higher the rate of evolution in a given phase, the smaller the number of systems of this type that would be observed. It is very likely that elliptical galaxies, SO- and normal spirals are examples of star systems in a quasi-stationary state. It is obvious, however, that even among quasi-stationary systems there are different rates of evolution. Thus, a star system which has a very flocculent structure will evolve very rapidly because of the loss of stars with velocities exceeding the escape velocity. This agrees with the fact that galaxies with a flocculent structure occur comparatively rarely. On the other hand, there are about a hundred elliptical galaxies, characterized by a very uniform mass distribution and a uniform structure, for every galaxy of another type [7].

Therefore, the investigation of quasi-stationary systems enables us to study the dynamics of the most common type of star systems. This, by itself, is enough to justify the use of the concept of quasi-stationary states as a tool in the study of the dynamics of galaxies. In addition, it enables us also to approach the problem of investigating star systems which are in phases of comparatively rapid evolution, since this phase can be considered as the process of breakdown of the preceding quasi-stationary state. Thus, for example, after the needle-shaped equilibrium figures of the galaxies have been found, the barred galaxies can be considered as the result of the emission of a stream of star gas because of the violation of Poincare's stability criterion.

A. A. Zhdanov
Leningrad State University

Received May 14, 1957

LITERATURE CITED

[1] K. F. Ogorodnikov, Doklady Akad. Nauk SSSR 116, 1 (1957).

[2] V. I. Smirnov, A Course in Advanced Mathematics, vol. 4 [in Russian] (2nd Edition, 1951) §§ 225-231.

[3] M. F. Subbotin, A Course in Celestial Mechanics, vol. 4 [in Russian] (GTTI, 1949).

[4] P. Appel', Equilibrium Figures of a Rotating Uniform Liquid [in Russian] (GTTI, 1936).

[5] J. H. Reynolds, Monthly Notices Roy. Astron. Soc. 81, 129 (1920); 82, 511 (1922).

[6] P. P. Parenago, A Course in Stellar Astronomy [in Russian] (3rd Edition, 1954) p. 264; Astron. J. (USSR) 17, No. 4 (1940).

[7] Iu. I. Efremov, Astron. J. (USSR) 26, 286 (1949).