THE SECULAR ACCELERATION OF THE MOON, AND THE LUNAR TIDAL COUPLE

C. A. Murray

(Communicated by the Astronomer Royal)

(Received 1957 July 24)

Summary

The discussion of the modern observations of the Sun, Mercury and Venus by Spencer Jones, when taken in conjunction with Jeffreys's theory of tidal friction, is shown to lead to a total dissipation of energy in the oceanic tides which is three times that calculated by Jeffreys in his discussion of tidal data for shallow seas.

The true (negative) acceleration of the Moon corresponding to this value of the dissipation has been included in the expression for the Moon's tabular mean longitude in the *Improved Lunar Ephemeris*. An investigation of the value of this acceleration in ancient times using Hipparchus' eclipse and equinox observations leads to a value twice as large as that given by the modern observations. This result is very satisfactorily confirmed by the observed magnitudes of partial lunar eclipses recorded in the *Almagest*.

It is pointed out that if the acceleration does change, then ephemeris time derived from lunar observations will not be uniform in the Newtonian sense.

---

1. Let the observed correction to the gravitational mean longitude of the Moon be expressed in the form

$$\delta L = a + bT + (q+s)T^2 + B(T) \qquad (1)$$

where q is the real (negative) acceleration due to the reaction of the lunar oceanic tidal couple, s is the apparent acceleration due to any secular retardation of the Earth's rotation and $B(T)$ is the fluctuation due to irregularities in the Earth's rotation; a and b are corrections to observationally determined constants of integration.

If N is the lunar oceanic retarding couple, M, m the masses of the Earth and Moon, and c the mean distance between them (all in c.g.s. units), and if we assume for simplicity that the Moon moves in a circular orbit in the plane of the Earth's equator, then Kepler's third law and the principle of conservation of angular momentum give the rate of change of the Moon's orbital angular velocity as

$$-3N\frac{M+m}{Mmc^2}\ \text{radian/sec}^2.$$

Thus, if q is expressed in seconds of arc and T in centuries,

$$q = -\frac{3}{2}N\frac{M+m}{Mmc^2} \times 2{\cdot}05 \times 10^{24}.$$

Putting $M = 5{\cdot}98 \times 10^{27}$, $M/m = 81{\cdot}5$, $c = 3{\cdot}84 \times 10^{10}$, we have

$$q = -2{\cdot}88 \times 10^{-23}N. \qquad (2)$$

2. Jeffreys (1) has pointed out that, in the case of the solar tidal couple N', the corresponding real acceleration of the Sun is observationally quite negligible.

Accordingly the observed correction to the gravitational mean longitude of the Sun may be expressed in the form

$$\delta L' = a' + b'T + (n'/n)\{sT^2 + B(T)\} \tag{3}$$

where n, n' are the mean motions of the Moon and Sun. In this case the apparent departure from a purely gravitational orbit arises solely from non-uniformity in the Earth's rotation.

3. In his paper on the rotation of the Earth, Spencer Jones (2) adopted definite numerical values for a, b and $(q+s)$ in (1) and thus, from observations of the Moon's longitude δL, defined $B(T)$ as an empirical function of the time. By eliminating $B(T)$ from (3), and from similar equations for Mercury and Venus, he determined $(n'/n)s$ from a discussion of observations extending from, roughly, 1680 to 1930.

He actually adopted $q+s = +5''{\cdot}22$ and derived $(n'/n)s = +1''{\cdot}23 \pm 0''{\cdot}04$. Since $n/n' = 13{\cdot}37$, his discussion gives

$$q = -11''{\cdot}22 \pm 0''{\cdot}53. \tag{4}$$

Thus, from (2),

$$N = (3{\cdot}9 \pm 0{\cdot}2) \times 10^{23} \text{ dyn cm}.$$

Inserting this in Jeffreys's equation for the dissipation (*loc. cit.* p.227), and taking $N/N' = 3{\cdot}4$ we have

$$dE/dt = (3{\cdot}5 \pm 0{\cdot}2) \times 10^{19} \text{ erg/sec}. \tag{5}$$

This is nearly three times the rate of working calculated by Jeffreys from his discussion of tidal data; he remarks, however, that there is probably some dissipation along open shores which he did not include in his calculations.

4. The value of q derived in (4) is of course the correction which has been applied to the theoretical secular acceleration in constructing the *Improved Lunar Ephemeris* (3), as originally proposed by Clemence (4). It is an empirical correction derived from observations extending over the last three centuries. There is no *a priori* reason why it should remain constant over long periods of time, and it is desirable to derive the value which best represents the available ancient observations.

In (1), $sT^2 + B(T)$ is the apparent displacement of the Moon in its orbit due to non-uniformity in the Earth's rotation. It can only be determined in practice from observations of the Sun or planets. (It clearly cannot be determined from observations of the Moon unless q is known *a priori.*) In order to derive a value of q it is thus necessary to have observations of the Moon and Sun (or planets) at as nearly the same epoch as possible.

If δL, $\delta L'$ are corrections to the tabular longitudes of the Moon and Sun at distant epochs T_1, T_2 respectively, the average sidereal secular accelerations ν, ν' over the whole periods T_1, T_2 are defined by

$$\nu = T_1^{-2}\delta L, \quad \nu' = T_2^{-2}\delta L'$$

Thus from (1) and (3), neglecting a, b, a', b'

$$q = \nu - (n/n')\nu' \tag{6}$$

provided that $T_1^{-2}B(T_1) - T_2^{-2}B(T_2)$ is negligible. We assume that this will be the case if T_1 and T_2 do not differ by more than a few years.

5. The observations of Hipparchus in the second century B.C. afford perhaps the most reliable determination of q in ancient times. His series of equinox

observations has been discussed by Fotheringham (5) who derived

$$\nu' = +1''\!\cdot\!95 \pm 0''\!\cdot\!27 \tag{7}$$

for the correction to Newcomb's sidereal acceleration of the Sun for mean epoch −137. His observation of the solar eclipse of −128 Nov. 20 has also been thoroughly discussed by Fotheringham (6) who derived the following equation of condition for corrections to the sidereal accelerations of Brown's and Newcomb's tables:

$$\nu - \frac{7}{4}\nu' = +2''\!\cdot\!10 \pm 0''\!\cdot\!55. \tag{8}$$

Combining (7) and (8) we find from (6)

$$q = -20''\!\cdot\!6 \pm 3''\!\cdot\!2. \tag{9}$$

6. As has been remarked by Jeffreys (*loc. cit.* p. 224), it is also possible to obtain q directly from the magnitudes of lunar eclipses. Apart from the small effects due to differences of topocentric libration, the magnitude as seen from anywhere on the Earth depends only on the geocentric co-ordinates of the Moon and Sun. It is thus independent of U.T.

If λ_0, λ'_0 are the ecliptic longitudes of the Moon and Sun, β_0 the latitude of the Moon, and $\dot\lambda$, $\dot\lambda'$, $\dot\beta$ their respective rates of change at an E.T. t_0, say, sufficiently close to opposition, the relative co-ordinates of the Moon and shadow at time t may be written

$$\lambda - \lambda' + 180^\circ = \lambda_0 - \lambda'_0 + 180^\circ + (\dot\lambda - \dot\lambda')(t - t_0),$$
$$\beta = \beta_0 + \dot\beta(t - t_0).$$

The angular distance between the centres of the Moon and shadow is $\{(\lambda - \lambda' + 180^\circ)^2 + \beta^2\}^{1/2}$ which has the minimum value

$$\Delta = \left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} \left|\beta_0 - \frac{\dot\beta}{(\dot\lambda - \dot\lambda')}(\lambda_0 - \lambda'_0 + 180^\circ)\right|. \tag{10}$$

Consider the change in this minimum value corresponding to a change qT^2 in the Moon's mean longitude. We have

$$\Delta + \delta\Delta = \left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} \left|\beta_0 - \frac{\dot\beta}{(\dot\lambda - \dot\lambda')}(\lambda_0 - \lambda'_0 + 180^\circ) + qT^2\left(\frac{\partial\beta}{\partial L} - \frac{\dot\beta}{(\dot\lambda - \dot\lambda')}\frac{\partial\lambda}{\partial L}\right)\right|. \tag{11}$$

Now let t_0 be the actual E.T. of opposition; then (11) and (10) become

$$\Delta + \delta\Delta = \left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} |\beta_0 + \delta\beta_0| \tag{12}$$

$$\Delta = \left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} |\beta_0| \tag{13}$$

where β_0 is now the latitude of the Moon at opposition and

$$\delta\beta_0 = qT^2\left(\frac{\partial\beta}{\partial L} - \frac{\dot\beta}{\dot\lambda - \dot\lambda'}\frac{\partial\lambda}{\partial L}\right). \tag{14}$$

For an eclipse to be partial, $|\beta_0|$ must be at least of the order 26′ whereas, in the eclipses discussed below, the maximum value of $|\delta\beta_0|$ is 6′·2. Accordingly $|\beta_0| > |\delta\beta_0|$ and we can write

$$\delta\Delta = \pm\left\{1 + \frac{\dot\beta^2}{(\dot\lambda - \dot\lambda')^2}\right\}^{-1/2} \delta\beta_0 \tag{15}$$

according as β_0 is positive or negative.

It can readily be verified from numerical formulae given by Cowell (**7**) that

$$\left\{1+\frac{\dot{\beta}^2}{(\dot{\lambda}-\dot{\lambda}')^2}\right\}^{-1/2}\left(\frac{\partial\beta}{\partial L}-\frac{\dot{\beta}}{(\dot{\lambda}-\dot{\lambda}')}\frac{\partial\lambda}{\partial L}\right)\approx \mp 0{\cdot}0076(1-0{\cdot}075\cos g) \qquad (16)$$

where g is the Moon's mean anomaly and the upper or lower sign is to be taken according as the eclipse is at the ascending or descending node. Also, if σ is the Moon's semi-diameter, then

$$\sigma\approx 942''(1+0{\cdot}064\cos g). \qquad (17)$$

If δG is a small change in the magnitude of an eclipse, we have

$$\delta G=-\frac{1}{2\sigma}\delta\Delta \qquad (18)$$

or, combining (14), (15), (16) and (17),

$$\delta G=\pm 4{\cdot}0\times 10^{-6}(1-0{\cdot}14\cos g)qT^2. \qquad (19)$$

The upper or lower sign in (19) is to be taken according as the Moon is eclipsed after or before passing the node.

7. The observed magnitudes of eleven partial lunar eclipses recorded by Ptolemy in the *Almagest* have been compared with tabular magnitudes by Cowell (**8**) who took as first approximations to the secular accelerations $\nu_c=+4''{\cdot}88$, $\nu'_c=+4''{\cdot}11$, or from (6) $q_c=-50''$. The observed *minus* tabular magnitudes are given in Table I, together with the values of g and the epochs T measured in centuries from 1800. These have been taken from the tables on pp. 524-7 of Cowell's paper with the exception of the magnitude of eclipse No. 18, for which Fotheringham's corrected value has been taken (**9**).

TABLE I

Ref. no.	T	g °	$\pm 4\times 10^{-6}\, T^2\,(1-0{\cdot}14\cos g)$	δG	R
2	−25·19	194	+0·00289	+0·21	+0·13
3	−25·18	345	+ 218	+ ·07	+ ·01
4	−24·20	162	− 264	+ ·03	+ ·10
5	−23·21	210	− 241	− ·11	− ·04
6	−23·00	184	− 242	− ·02	+ ·05
7	−22·90	281	− 203	+ ·01	+ ·06
14	−19·73	344	+ 135	+ ·03	− ·01
15	−19·40	359	+ 129	+ ·07	+ ·04
16	−16·75	72	− 108	− ·02	+ ·01
18	−16·65	245	+ 117	+ ·03	·00
19	−16·64	326	−0·00098	·00	+0·03

A solution by least squares, giving unit weight to each observed magnitude, gives a correction of $+27''\pm 6''$ to q_c. The result from the lunar eclipse magnitudes is then

$$q=-23''\pm 6''. \qquad (20)$$

The final column, R, of Table I contains the residuals from this solution. The predominance of positive residuals suggests that the observed magnitudes may be systematically too large by about 0·03; this is in addition to an allowance of 0·02 for increment in the radius of the shadow due to the Earth's atmosphere, which has already been included in δG. The probable error of one magnitude is ±0·04 and the magnitudes were in fact only recorded to the nearest digit (0·08). In view of the indefinite nature of the phenomenon the presence of a small systematic error is hardly surprising. In any case its effect on the final result (20) is inappreciable.

8. The value of q derived from the ancient lunar eclipses (20) is in excellent agreement with that deduced from Hipparchus' observations (9). Combining these two independent determinations we may adopt

$$q = -21'' \pm 3'' \tag{21}$$

as the value which best fits the ancient observations discussed in this paper. This differs from the modern value (4) by nearly three times the sum of their probable errors. There is thus strong evidence that the lunar tidal couple may have changed considerably during the last twenty centuries. This is not improbable. The dissipation is thought to occur in relatively few regions of shallow sea; local changes in coastline and sea level in these regions will therefore have a large effect on the total dissipation.

The latest solution for the secular accelerations involving ancient observations is that by Brouwer (**10**); he, however, imposed the condition that q should have its modern value (4) throughout the whole period covered by his discussion, the validity of which procedure would now appear to be open to doubt.

Apart from changes in the mean longitude at epoch and the mean motion of the Moon, the revision of Brown's mean longitude adopted in constructing the *Improved Lunar Ephemeris* consists of the removal of one empirical term (the "great empirical term"), and the substitution of another (qT^2). This ephemeris is to be used as the standard of ephemeris time, all departures of the Moon's observed position from the ephemeris being attributed to deviations of the rotation of the Earth from uniformity. Any change in q will therefore introduce a systematic departure of E.T. from a truly uniform Newtonian time.

I wish to thank Dr R. d'E. Atkinson for his helpful criticism of this paper.

Royal Greenwich Observatory,
Herstmonceux Castle, Sussex:
1957 *June* 28.

References

(**1**) H. Jeffreys, *The Earth*, 3rd edn., ch. VIII, Cambridge 1952.
(**2**) H. Spencer Jones, *M.N.*, **99**, 541, 1939.
(**3**) *Improved Lunar Ephemeris*, Washington, 1954.
(**4**) G. M. Clemence, *A.J.*, **53**, 169, 1948.
(**5**) J. K. Fotheringham, *M.N.*, **78**, 406, 1918.
(**6**) J. K. Fotheringham, *M.N.*, **69**, 204, 1909.
(**7**) P. H. Cowell, *M.N.*, **65**, 864, 1905.
(**8**) P. H. Cowell, *M.N.*, **66**, 523, 1906.
(**9**) J. K. Fotheringham, *M.N.*, **69**, 666, 1909.
(**10**) D. Brouwer, *A.J.*, **57**, 125, 1952.