JEANS' FORMULA FOR GRAVITATIONAL INSTABILITY

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Summary

Jeans' formula gives the condition that a gravitating mass of gas shall be stable to small fluctuations in the density. It was originally obtained for a static mass of uniform density, and the proof is questionable. In this paper the formula is derived by studying small perturbations of the density and velocity in an expanding Newtonian world-model. It turns out that, given a suitable interpretation of the quantities involved, the formula is valid in all cosmologies except the steady-state theory.

Although instability occurs in the models as suggested by Jeans' formula, the condensation process is too slow to account for the formation of the nebulae—a result which has now been reached in several ways, and which may be regarded as well-established.

In the Newtonian version of the steady-state theory small perturbations of the average density die out, and so Jeans' formula does not apply. The relevance of this to the work of Sciama, who assumes the contrary, is discussed. It seems that the claim that the steady-state theory explains the formation of the nebulae requires further substantiation.

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1. *Introduction.*—Jeans' formula (**4, 5**) gives the maximum size consistent with stability of a cloud of gas whose density ρ, and equation of state $p(\rho)$, are known. It states that a cloud with linear dimensions greater than

$$\left(\frac{\pi}{G\rho}\frac{dp}{d\rho}\right)^{1/2} \tag{1.1}$$

(where G is the constant of gravitation) is unstable to slight fluctuations in the density. This formula is often used in astrophysical work, and important deductions are made from it.

In (1.1) the density and pressure are supposed to be uniform throughout the gas. In fact, however, if gravitation is taken into account, the equation of hydrostatic equilibrium has no solution for a finite uniform mass. This is disturbing, and one's uneasiness is enhanced if one remembers that certain of the polytropic gas spheres (which *do* satisfy the hydrostatic equation) are mechanically stable, so far as is known, whatever their size. On the other hand, as I have recently shown (**3**), an isothermal gas sphere of sufficient size is unstable, and the condition for instability is similar to (1.1).

Since Jeans' work appeared, uniform distributions of matter have been studied in great detail in cosmology. For the most part, this work has been based on general relativity, in which uniform distributions are quite unobjectionable as there are rigorous solutions of the field equations referring to them. However, the Newtonian cosmological models of Milne and McCrea (**9, 10**) also use uniform distributions of matter, and these seem to be a natural vehicle for a further study of Jeans' type of gravitational instability. As a result of recent work of McCrea (**8**), the assumptions on which these models are based are now well understood, and they can be used with confidence as a guide to the models of general relativity, and also of the steady-state theory.

My intention in this paper is to use the Newtonian models to study the gravitational instability of a uniform material filling an expanding universe. I write with full awareness of the extremely beautiful and powerful work of Lifshitz (**6**) on gravitational instability in relativistic cosmology, and my results, as far as they are comparable, agree with his. However, the extra simplicity gained by using the Newtonian models, apart from giving formulae which are more tractable and readily comprehensible, enables one to extend the investigation to a wider class of models—namely, those with cosmological constant, and that representing the steady-state universe.

The result of my work, very briefly, is that something very like Jeans' instability occurs in all the cosmologies, *except* the steady-state theory. This latter conclusion is unfortunate because the steady-state theory is the only one which has at present an explanation of the formation of the nebulae (**11**), and it now seems that this explanation is made doubtful by an unjustified use of Jeans' formula. However, it may be possible to retrieve the situation, as will be explained later.

The plan of the paper is as follows: in Section 2, I obtain Jeans' formula in its original form, and I follow this in Section 3 by a derivation of the corresponding results in the expanding universe; this is followed in Sections 4–6 by a discussion of the results for certain special models, and in Section 7 by the application of the theory to the steady-state universe; in the Conclusion, Section 8, I give a brief summary and general discussion of the results of the earlier sections.

2. *Jeans' formula.*—To derive Jeans' formula we use the theory of sound in a large mass of gas, taking gravitation into account. Let us suppose that the gas satisfies an equation of state $p \equiv g(\rho)$, and is at rest except for a small velocity $\boldsymbol{u}$ $(\boldsymbol{r}, t)$ due to the wave motion. Let $\rho(\boldsymbol{r}, t)$ and $p(\boldsymbol{r}, t)$ be the density and pressure; the space derivatives of p and ρ are not at the moment supposed to be small, but we shall assume that the deviations of p and ρ from their equilibrium values are of the same order of smallness as $\boldsymbol{u}$. Squares and products of small quantities (and of their derivatives) are to be neglected.

On these assumptions the hydrodynamical equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{F} - \frac{1}{\rho}\nabla p, \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{2.2}$$

become

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F} - \frac{1}{\rho}\nabla p, \tag{2.3}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{2.4}$$

where $\boldsymbol{F}$ is the body force per unit mass, consisting solely of gravitation. Since the gas is in equilibrium except for the small perturbations, we have

$$\boldsymbol{F}_0 = \frac{1}{\rho_0}\nabla p_0, \tag{2.5}$$

where a suffix 0 refers to the equilibrium value. Put

$$\boldsymbol{F} = \boldsymbol{F}_0 + \boldsymbol{F}_1, \qquad \rho = \rho_0 + \omega,$$

where $\boldsymbol{F}_1$ and ω are small; then Poisson's equation gives

$$\nabla . \boldsymbol{F}_0 = -4\pi G\rho_0, \qquad \nabla . \boldsymbol{F}_1 = -4\pi G\omega. \tag{2.6}$$

Equations (2.3) and (2.5) give

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F}_1 - \nabla \left\{ \frac{\omega}{\rho_0} \frac{dp}{d\rho} \right\}, \tag{2.7}$$

where the equilibrium value of $dp/d\rho$ is to be taken. From (2.6) and (2.7) we have

$$\begin{aligned} \frac{\partial}{\partial t} \nabla . \boldsymbol{u} &= \nabla . \boldsymbol{F}_1 - \nabla^2 \left\{ \frac{\omega}{\rho_0} \frac{dp}{d\rho} \right\} \\ &= -4\pi G\omega - \nabla^2 \left\{ \frac{\omega}{\rho_0} \frac{dp}{d\rho} \right\}. \end{aligned} \tag{2.8}$$

Equation (2.4) may be written, after differentiation and neglect of second order terms,

$$\frac{\partial^2 \omega}{\partial t^2} + \rho_0 \frac{\partial}{\partial t} (\nabla . \boldsymbol{u}) + (\nabla \rho_0) . \frac{\partial \boldsymbol{u}}{\partial t} = 0. \tag{2.9}$$

From (2.8) and (2.9) we find

$$\frac{\partial^2 s}{\partial t^2} + \frac{(\nabla \rho_0)}{\rho_0} . \frac{\partial \boldsymbol{u}}{\partial t} = 4\pi G s \rho_0 + \nabla^2 \left(s \frac{dp}{d\rho} \right), \tag{2.10}$$

where we have introduced the condensation s defined by

$$s = \omega/\rho_0.$$

If now we suppose that ρ_0 is constant, (2.10) reduces to the formula given by Jeans'

$$\frac{\partial^2 s}{\partial t^2} = 4\pi G \rho_0 s + \nabla^2 \left(s \frac{dp}{d\rho} \right). \tag{2.11}$$

Jeans (4) is not quite explicit on this point, and his method is somewhat different, but careful study of his work shows that in order to get (2.11) he also is required to take ρ_0 as constant. If ρ_0 (and therefore p_0) is constant, (2.5) gives $\boldsymbol{F}_0$ zero throughout the gas, and one is at first inclined to think that the gravitation has been removed from the problem. However, one can retrieve the situation, after a fashion, by supposing that one has an infinite mass of gas so that, as there can be no preferred direction for $\boldsymbol{F}_0$ it must be zero. (Strictly, $\boldsymbol{F}_0$ is in this case not zero, but undefined.) As, however, Jeans goes on to apply the investigation to a finite mass of gas, the argument is, to say the least, precarious.

The deduction of Jeans' formula from (2.11) is simple. Consider a wave of length λ propagated along Ox and suppose a solution of the form

$$s = h(t) \cos 2\pi x/\lambda.$$

Since $dp/d\rho$ is constant, we find

$$\frac{d^2 h}{dt^2} = \left[4\pi G \rho_0 - \frac{4\pi^2}{\lambda^2} \frac{dp}{d\rho} \right] h. \tag{2.12}$$

Hence the disturbance increases exponentially with time if

$$\lambda > \left(\frac{\pi}{G\rho_0} \frac{dp}{d\rho} \right)^{1/2}. \tag{2.13}$$

A mass of gas with linear dimensions greater than this will allow such disturbances, and will therefore be unstable.

As it will be convenient in the following sections to study spherical waves, I shall show that (2.13) holds also for such waves. If we suppose that the disturbance has spherical symmetry, we may write

$$s = h(t) r^{-1} \sin 2\pi r/\lambda.$$

Equation (2.11) then gives (2.12), and (2.13) follows as before.

3. *Gravitational stability of Newtonian world-models.*—Let us consider small perturbations in a Newtonian world-model. Suppose that a definite observer O sees the motion of the world-fluid as purely radial, and that the velocity, density and pressure have spherical symmetry about O. The imposition of this symmetry on the perturbations is not essential to the analysis, which can be followed through in a similar way if small non-radial motion and deviations from spherical symmetry are allowed. These latter perturbations make little difference to the final result, and are not important from the point of view of stability, so for simplicity I confine the argument to the use of radial perturbations with spherical symmetry.

The hydrodynamical equations (2.1) and (2.2) give

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = F - \frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0, \tag{3.2}$$

where u is the radial velocity measured by O, and p, ρ and F are the pressure, density and radial force per unit mass. As the motion is supposed to be that of a slightly perturbed Newtonian world-model, we may write

$$\left.\begin{aligned} u &= rf(t) + v(r, t), \\ \rho &= \rho_0(t) + \omega(r, t), \\ p &= p_0(t) + q(r, t), \end{aligned}\right\} \tag{3.3}$$

where v, ω and q are small functions whose squares and products (and those of their derivatives) may be neglected, and where $f(t)$, $\rho_0(t)$, $p_0(t)$ are the functions which occur in the unperturbed model. The body force per unit mass $\boldsymbol{F}$ satisfies

$$\nabla . \boldsymbol{F} \equiv \frac{\partial F}{\partial r} + \frac{2F}{r} = -4\pi G\rho + \Lambda, \tag{3.4}$$

where Λ is the cosmological constant. In the homogeneous model v, ω and q are zero, and the functions ρ_0 and f satisfy

$$\dot{\rho}_0/\rho_0 = -3f, \qquad 3\dot{f} + 3f^2 = -4\pi G\rho_0 + \Lambda, \tag{3.5}$$

where $\dot{}$ denotes $\partial/\partial t$. Later we shall need the function $R(t)$ given in terms of f by

$$f = \dot{R}/R. \tag{3.6}$$

From (3.5) and (3.6) we find

$$\rho_0 = \alpha R^{-3},$$

where α is an arbitrary constant. Equations (3.5) follow from (3.1)–(3.4) if v, ω and q are zero. The pressure does not appear in the theory of the homogeneous model since it enters (3.1) only through its gradient. We shall suppose that there is an equation of state of the form

$$p \equiv g(\rho)$$

so that

$$q(r, t) = \omega(r, t)\frac{dg}{d\rho}. \tag{3.7}$$

Here and elsewhere it is to be understood that $dg/d\rho$ is to be calculated at $\rho=\rho_0$.

If we substitute (3.3) and (3.7) into (3.1) and (3.2), and neglect small quantities of the second order, we find

$$r\dot{f}+\dot{v}+rf^2+rfv'+fv=F-\frac{\omega'}{\rho_0}\frac{dg}{d\rho}, \tag{3.8}$$

$$\dot{\rho}_0+\dot{\omega}+rf\omega'+3\rho_0 f+2\rho_0 vr^{-1}+\rho_0 v'+3\omega f=0, \tag{3.9}$$

where $'$ means $\partial/\partial r$. If from (3.8) we form $\partial F/\partial r+2F/r$, and use (3.4) and (3.5), we have

$$\dot{v}'+rfv''+4fv'+2\frac{\dot{v}}{r}+\frac{2fv}{r}=-4\pi G\omega-\frac{1}{\rho_0}\frac{dg}{d\rho}\left(\omega''+\frac{2\omega'}{r}\right), \tag{3.10}$$

where once again products of small quantities have been neglected. Equations (3.9) and (3.5) together give

$$\dot{\omega}+rf\omega'+2\rho_0 vr^{-1}+\rho_0 v'+3\omega f=0,$$

which may be written

$$\frac{\partial}{\partial r}(r^2 v)=-\frac{r^2}{\rho_0}(\dot{\omega}+rf\omega'+3\omega f)\equiv X, \text{ say.} \tag{3.11}$$

Equation (3.10) may be written

$$\frac{1}{r^2}\frac{\partial^2}{\partial r\partial t}(r^2 v)+\frac{f}{r}\frac{\partial^2}{\partial r^2}(r^2 v)=-4\pi G\omega-\frac{1}{\rho_0}\frac{dg}{d\rho}\left(\omega''+\frac{2\omega'}{r}\right); \tag{3.12}$$

then (3.11) and (3.12) together give

$$\dot{X}+rfX'=-4\pi G\omega r^2-\frac{r^2}{\rho_0}\frac{dg}{d\rho}\left(\omega''+\frac{2\omega'}{r}\right).$$

Introducing for X its value in (3.11), we find, after a rather long calculation in which use is made of (3.5),

$$\ddot{\omega}+2rf\dot{\omega}'+r^2f^2\omega''+8f\dot{\omega}+r\omega'(9f^2+\dot{f})+\omega(18f^2+6\dot{f}-\Lambda)=\frac{dg}{d\rho}\left(\omega''+\frac{2\omega'}{r}\right). \tag{3.13}$$

This is a differential equation for the perturbation in the density, ω, in terms of the functions f and $dg/d\rho$, supposed given.

To simplify (3.13) we change to a new independent variable x by the transformation

$$r=xR(t)$$

where R is the function introduced in (3.6). A long calculation shows that (3.13) becomes

$$\frac{\partial^2\omega}{\partial x^2}+\frac{2}{x}\frac{\partial\omega}{\partial x}=\left(\frac{dg}{d\rho}\right)^{-1}R^2\left[\frac{\partial^2\omega}{\partial t^2}+8\frac{\dot{R}}{R}\frac{\partial\omega}{\partial t}+\omega\left(12\frac{\dot{R}^2}{R^2}+6\frac{\ddot{R}}{R}-\Lambda\right)\right]. \tag{3.14}$$

Finally, we introduce the condensation s by

$$s=\omega/\rho_0=\alpha^{-1}\omega R^3,$$

which reduces (3.14) to

$$\frac{\partial^2 s}{\partial x^2}+\frac{2}{x}\frac{\partial s}{\partial x}=\left(\frac{dg}{d\rho}\right)^{-1}R^2\left[\ddot{s}+\frac{2\dot{R}}{R}\dot{s}+s\left(3\frac{\ddot{R}}{R}-\Lambda\right)\right]. \tag{3.15}$$

Equation (3.14) or (3.15) is the equation representing the propagation of spherically symmetric sound waves through the Newtonian world-model. Let us study solutions of the form

$$s=\frac{\sin kx}{kx}h(t), \tag{3.16}$$

where k is a real constant and where $h(t)$ is a function to be determined which will show how the amplitude of the wave changes with time. Substituting (3.16) into (3.15) we find

$$\ddot{h}+2\frac{\dot{R}}{R}\dot{h}+h\left(\frac{3\ddot{R}}{R}-\Lambda+k^2R^{-2}\frac{dg}{d\rho}\right)=0,$$

which with the help of (3.5) and (3.6) gives

$$\ddot{h}+2\frac{\dot{R}}{R}\dot{h}+h\left(k^2R^{-2}\frac{dg}{d\rho}-4\pi G\rho_0\right)=0. \tag{3.17}$$

In (3.16) sin kx/kx is a bounded oscillating function of r with order of magnitude 1 near the origin for all time, so the growth of the condensation is measured by $h(t)$. Thus $h(t)$ is strictly comparable with the function h in Section 2, and formula (3.17) corresponds to equation (2.12). The similarity between the two equations is obviously close, but in (3.17) there is a secular change in the coefficients owing to the expansion.

It can be shown from (3.17) that condensations can begin to form in favourable circumstances. Suppose that there is a perturbation at time t_1 such that

$$h=h_1>0, \quad \frac{dh}{dt}=\dot{h}_1>0 \quad \text{at} \quad t=t_1, \tag{3.18}$$

where h_1, $\dot{h}_1$ are assumed given; suppose also that

$$k^2R^{-2}\frac{dg}{d\rho}-4\pi G\rho_0<0 \quad \text{for} \quad t_1\leqslant t\leqslant t_2. \tag{3.19}$$

Immediately after t_1, $\dot{h}$ is positive so that h increases; and h will stop increasing only if it reaches a maximum, for which the conditions are

$$\dot{h}=0, \quad \ddot{h}<0.$$

Now from (3.17) and (3.19) it follows that, when $\dot{h}=0$ and $h>0$, $\ddot{h}>0$ so that h does not achieve a maximum in $t_1\leqslant t\leqslant t_2$, and it must go on increasing. Hence, given the initial conditions (3.18) and also the condition (3.19), a condensation begins to form.

We may express (3.19) in a form closer to Jeans' formula if we introduce the wave-length λ of the disturbance by

$$\lambda=2\pi R/k. \tag{3.20}$$

Substituting (3.20) into (3.19) we find

$$\lambda^2>\frac{\pi}{G\rho_0}\frac{dg}{d\rho}. \tag{3.21}$$

Here λ is a function of the time as expressed by (3.20), and so are ρ_0 and $dg/d\rho$. The condensation process continues so long as (3.21) is satisfied. Equation (3.21) is formally the same as (2.13), so, *with the appropriate interpretation of the quantities involved, Jeans' formula is true in the expanding universe.* In terms of the initial wave-length λ_1, (3.21) may be written

$$\lambda_1^2>\frac{\pi}{G\rho_0}\frac{dg}{d\rho}\frac{R_1^2}{R^2}, \tag{3.22}$$

where R_1 is the value of R at time t_1. Supposing this inequality satisfied at $t=t_1$, the condensation process will continue until such time t_2 as it ceases to hold.

Let us consider the special case in which the equation of state is

$$p \equiv g(\rho) = \kappa\rho^{4/3}. \tag{3.23}$$

The formulae (3.21) and (3.22) then simplify, and give as condition for instability

$$\lambda_1^2 > \frac{4\pi\kappa}{3G\rho_1^{2/3}}, \tag{3.24}$$

where ρ_1 means the unperturbed density at time t_1; if this condition is satisfied at t_1, condensation will proceed indefinitely. *Equation* (3.24) *is precisely Jeans' condition for a gas with equation of state* (3.23). The equation of state for diatomic hydrogen ($p = \kappa\rho^{1.4}$) is not very different from (3.23), and in Section 5 I shall use the latter equation to obtain an estimate of the behaviour of an expanding model filled with hydrogen.

If we had not confined attention to spherically symmetric perturbations we should have found an equation similar to (3.15) but containing angular terms on the left-hand side, making up the full Laplacian. The solution (3.16) would have contained a surface harmonic, and a spherical Bessel function would have replaced sin kx/kx. The results of this and the subsequent sections would not have been basically different.

The foregoing results will now be applied to certain special models.

4. *The Einstein universe.*—In the Einstein universe we have

$$R = \text{const.} \qquad \Lambda = 4\pi G\rho_0,$$

so that (3.17) becomes

$$\ddot{h} + h\left(\frac{4\pi^2}{\lambda^2}\frac{dg}{d\rho} - 4\pi G\rho_0\right) = 0,$$

where
$$\lambda = 2\pi R/k.$$

Thus h depends exponentially on the time if

$$\lambda > \left(\frac{\pi}{G\rho_0}\frac{dg}{d\rho}\right)^{1/2}.$$

This is exactly the same as (2.13) and gives the maximum size of a stable sphere. It follows that *Jeans' formula is true without modification for the Einstein universe.* It should be noted, however, that the constant density of the unstable gas cloud must be that appropriate to the Einstein universe, i.e. $\Lambda/4\pi G$. The theory would not apply to a uniform mass of gas of different density.

5. *The expanding world-models with* $\Lambda = 0$.—The main interest in the present theory is in what it predicts about the formation of the nebulae in an initially homogeneous universe. Except for the steady-state theory, which will be dealt with later, most of the plausible models start from a singular state of infinite density. For some time after the singular state the pressure would be large, and for this period the Newtonian models do not give a satisfactory representation because Newtonian theory does not allow for the gravitational effects of the pressure. The theory constructed here will therefore apply only to the later stages in the history of the universe when the pressure is small. From Lifshitz's work it seems that the later period is the most favourable for the formation of the nebulae, so the inapplicability of the theory to the earlier stages is not very serious.

Equations (3.5) and (3.6) for the unperturbed model may be integrated to give

$$\rho_0 = \frac{\alpha}{R^3}, \quad \dot{R}^2 = \frac{8\pi G\alpha}{3R} + \frac{1}{3}\Lambda R^2 + \epsilon, \tag{5.1}$$

where α and ϵ are constants of integration. Let us now consider a definite model, obtained by taking

$$\Lambda = 0, \qquad \epsilon = -\beta^2,$$

where β is a real constant. This model corresponds to one of general relativity which is spatially closed, and has zero cosmological constant and zero pressure. Of the pressure-free models with zero cosmological constant, it seems that this one is the most favourable to the formation of condensation (2). We can see from the Newtonian analogue why this should be so. It is easily shown (9) that the total energy of any one of the fundamental particles of the Newtonian model has the sign of ϵ, so if ϵ is negative all these particles have velocities less than that required to escape from the matter within the spheres on whose boundaries they lie. This makes the model more favourable to the growth of condensations.

The second of equations (5.1) gives on integration

$$R = \frac{8\pi G\alpha}{3\beta^2} \sin^2 \tfrac{1}{2}\theta, \tag{5.2}$$

$$t + t_0 = \frac{4\pi G\alpha}{3\beta^3} (\theta - \sin\theta), \tag{5.3}$$

where t_0 is a constant of integration which we may take to be zero by choosing the origin of θ at $t=0$. The observations which the model is required to fit are the present values of Hubble's constant, $\dot{R}/R$, and the average density. Taking these as

$$\rho = 2 \times 10^{-28}\ \mathrm{g/cm^3}, \qquad \dot{R}/R = 2{\cdot}8 \times 10^{-10}\ (\mathrm{yrs.})^{-1},$$

we find (2)

$$K \equiv \frac{4\pi G\alpha}{3\beta^3} = 9{\cdot}3 \times 10^9\ \mathrm{yrs.},$$

and the value of θ referring to the present time is

$$\theta = 1{\cdot}15\ \mathrm{rad}.$$

We may now return to (3·17) which governs the growth of a condensation. If we transform the independent variable from t to θ by (5.3) and express ρ_0 as a function of θ we find, using (5.1) and (5.2), that (3.17) becomes

$$\frac{d^2h}{d\theta^2} + \cot \tfrac{1}{2}\theta \frac{dh}{d\theta} - h\left[\frac{3}{2}\operatorname{cosec}^2 \tfrac{1}{2}\theta - \frac{k^2}{\beta^2}\frac{dg}{d\rho}\right] = 0. \tag{5.4}$$

To proceed further we must make some assumptions about the equation of state. For the reasons explained at the end of Section 3, let us suppose that

$$p \equiv g(\rho) = \kappa\rho^{4/3};$$

then proceeding as in Section 3 we find that (5.4) becomes

$$\frac{d^2h}{d\theta^2} + \cot \tfrac{1}{2}\theta \frac{dh}{d\theta} - h \operatorname{cosec}^2 \tfrac{1}{2}\theta \left[\frac{3}{2} - \frac{2\pi\kappa}{G{\rho_1}^{2/3}{\lambda_1}^2}\right] = 0, \tag{5.5}$$

where a perturbation of wave-length λ_1 originated the disturbance when the density was ρ_1. Suppose that this was at time $t = t_1$ when $\theta = \theta_1$. As in Section 3, it follows that the disturbance can continue to grow if (3.24) is satisfied.

Let us get an idea of the behaviour of the solution of (5.5) by using the approximations

$$\cos\theta \sim 1, \qquad \sin\theta \sim \theta.$$

The equation then becomes

$$\theta^2 \frac{d^2h}{d\theta^2} + 2\theta \frac{dh}{d\theta} - 4n^2 h = 0, \tag{5.6}$$

where

$$n^2 = \frac{3}{2} - \frac{2\pi\kappa}{G\rho_1^{2/3}\lambda_1^{2}}.$$

The solution of (5.6) is

$$h = A\theta^{m_1} + B\theta^{m_2},$$

where A and B are arbitrary constants, and where

$$m_1, m_2 = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+16n^2}.$$

As expected, the disturbance will grow if $n^2 > 0$.

From this approximate solution it seems that the most favourable case for condensation is that when n^2 has its maximum value, i.e. $3/2$. This is obtained by taking $p=0$, and supposing the model filled with dust. Condensations will then form for perturbations of any wave-length. If $n^2 = 3/2$, (5.5) can be solved exactly in terms of elementary functions. A particular solution is

$$h = \operatorname{cosec}^2 \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta,$$

and the general one can be found by substituting in (5.5)

$$h = v(\theta) \operatorname{cosec}^2 \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta.$$

This eventually leads to the solution

$$h = -\tfrac{3}{2}A\theta \operatorname{cosec}^2 \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta + A(2 + 3\cot^2 \tfrac{1}{2}\theta) + B \operatorname{cosec}^2 \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta, \tag{5.7}$$

where A and B are arbitrary constants.

We now wish to find out how quickly a small perturbation can grow. Suppose that the time t_1 is early enough for us to put

$$\cos\theta_1 \sim 1, \qquad \sin\theta_1 \sim \theta_1.$$

Then (5.7) gives

$$h_1 \sim 8B\theta_1^{-3} + \tfrac{1}{10}A\theta_1^{2}. \tag{5.8}$$

If we assume that h_1 is initially small, we must have B very small since θ_1 is small; however, A need not be as small as h_1.* At the present time, say when $\theta \sim 1$, we have from (5.7)

$$h \sim A;$$

therefore from (5.8)

$$h \sim 10 h_1 \theta_1^{-2}. \tag{5.9}$$

It is now easy to see that the condensation process does not take place quickly enough to account for the formation of the nebulae. For a collection of N molecules of an ideal gas the fluctuation in the density is

$$\frac{\delta\rho}{\rho} = N^{-1/2},$$

and for a nebula of hydrogen $N \sim 3 \times 10^{67}$. Hence we may take

$$h_1 \sim 10^{-34}.$$

Supposing the perturbation to have occurred, say, 1000 years after the singular state, we have

$$\theta_1 \sim 10^{-2},$$

and from (5.9)

$$h \sim 10^{-29}.$$

Thus although the disturbance has grown in magnitude, it is still quite negligible,

* One might at first think that, since A and B are independent constants, one could satisfy (5.8) with h_1 small and yet have A and B large. However, it must be remembered that dh/dt must also be small at time t_1, and this leads to the conclusion given.

and there seems no hope whatever of accounting for the formation of the nebulae by this mechanism if one assumes fluctuations of the values to be expected on ordinary statistical theory. The difficulty cannot be overcome by taking a sufficiently small value for θ_1 because at such times the gravitational effect of the pressure could not be ignored and the model (5.2)–(5.3) would not apply. (For a fuller discussion, see (**2**)).

The foregoing analysis applies to the "closed" model with $\Lambda=0$. The "open" models with $\Lambda=0$, obtained by putting $\epsilon\geqslant 0$ in (5.1), give similar results; that is to say, small perturbations cannot grow into nebulae in the time available.*

6. *Models with* $\Lambda\neq 0$.—The foregoing analysis is not easy to apply to models with non-zero cosmological constant, because, in general, elliptic functions are needed to integrate the second of equations (5.1). I shall not deal with this problem in detail here. I have previously discussed the formation of condensations in Lemaître's model (**2**), and shown that this model seems hardly more favourable to the condensation process than those with $\Lambda=0$. Considered in the light of equation (3.17), the main difference between Lemaître's model and those with $\Lambda=0$ is that in the former $\dot{R}/R$ is nearly zero for a certain period, and at this time the model resembles the Einstein universe. If this period were long enough it would certainly allow the formation of the nebulae provided $dg/d\rho$ were small. However, if one supposes, as Lemaître did, that the model started about 5×10^9 years ago, this state of affairs cannot go on for very long. It might be worth while to investigate this problem further, but I have little doubt that detailed calculations would show that the condensations form too slowly to account for the formation of the nebulae.

It is clear that if one allows Λ to be non-zero, models exist in which the nebulae could have formed. If the pressure is small enough to be neglected, and if $\dot{R}/R$ is small for a long period, (3.17) shows that h will increase very rapidly provided the density is not too small. A model with these characteristics is the one investigated by Eddington and Lemaître which expands from the Einstein state, though this is not the only one.

7. *Condensations in the steady-state theory*.—The only plausible cosmology which has a theory of the formation of nebulae is the steady-state theory of Bondi, Gold and Hoyle. Sciama (**11**) has worked out the mechanism in some detail, and in his work Jeans' formula plays a fundamental part. It is therefore important to see whether the use of Jeans' formula in the steady-state theory can be justified.

Here, however, one meets a difficulty. The steady-state theory, though an interesting and imaginative cosmology, has not yet developed either a dynamics or a gravitation theory which carries any conviction. There is, though, a Newtonian model of the theory, developed with much ingenuity by McCrea (**7**). I shall try to adapt McCrea's work to the problem in hand.

According to McCrea, the Newtonian world models will accommodate the steady-state universe if one adds two postulates: (*a*) that mass and energy are convertible; (*b*) that the density of gravitational mass, instead of being ρ, is

$$\sigma=\rho+3pc^{-2}, \tag{7.1}$$

* Attempts have been made to overcome the difficulty by postulating turbulence in the primordial medium; this gives rise to large fluctuations from which the nebulae could have formed. However, this proposal can hardly be regarded as a satisfactory solution of the problem unless it can be shown first how the turbulence might have arisen, and secondly that it would have survived the period of high pressure. Neither of these questions has yet been answered.

c being the velocity of light. The equations of the unperturbed model, after allowing for McCrea's two extra assumptions, are

$$3\dot{f}+3f^2=-4\pi G(\rho_0+3p_0c^{-2}), \tag{7.2}$$

$$\dot{\rho}_0+3\rho_0 f+3p_0 fc^{-2}=0. \tag{7.3}$$

These differ from the equation (3.5) of the standard theory in the presence of the terms involving p_0. The solution of (7.2) and (7.3) which gives the Newtonian analogue of the steady-state theory is

$$f=\dot{R}/R, \qquad R=Ae^{ct/a}, \tag{7.4}$$

$$\rho_0=3c^2/8\pi Ga^2, \qquad p_0=-3c^4/3\pi Ga^2, \tag{7.5}$$

where A and a are arbitrary constants.

To extend this work to the perturbed Newtonian model considered in Section 2, let us make two further assumptions:

(*c*) equation (7.1) holds even in the non-uniform model;

(*d*) the rate of creation of matter is independent of the density, and is

$$9c^3/8\pi Ga^3 \quad \text{g/cm}^3\ \text{sec}.$$

The latter is the rate in the model (7.4) and (7.5). Although the theory is not quite explicit about the dependence of the creation rate on the local density, postulate (*d*) seems a reasonable interpretation of the authors' intentions (**1**). The last term on the left in (7.3) represents the creation of matter, which, of course, causes an alteration in the equation of continuity.

With these assumptions the equations for the model subject to spherically symmetric perturbations are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}=F-\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{7.6}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}+\rho\frac{\partial u}{\partial r}+\frac{2\rho u}{r}+\frac{3p_0}{c^2}f=0, \tag{7.7}$$

$$\nabla\cdot\boldsymbol{F}\equiv\frac{\partial F}{\partial r}+\frac{2F}{r}=-4\pi G\left(\rho+\frac{3p}{c^2}\right). \tag{7.8}$$

These correspond to (3.1), (3.2) and (3.4), and in the unperturbed model are equivalent to (7.2)–(7.5).*

If, following McCrea, we regard the negative pressure p_0 of the homogeneous model as a "zero-point" pressure, we may suppose that the equation of state of the gas of the model takes the form

$$p=p_0+g(\rho),$$

where $g(\rho_0)=0$. Using the notation of (3.3), we find that for small perturbations

$$p=p_0+\omega\frac{dg}{d\rho}, \tag{7.9}$$

$$\frac{\partial p}{\partial r}=\omega'\frac{dg}{d\rho}, \tag{7.10}$$

where, as usual, $dg/d\rho$ is to be calculated at $\rho=\rho_0$.

* A difficulty should be pointed out here. In (7.6) F is the force per unit inertial mass, whereas in (7.8) F is the force on unit "passive" gravitational mass (i.e. mass on which gravitation acts). One cannot be sure that these F's are really the same, especially as we are taking the density of "active" (gravitation-producing) mass as $\rho+3pc^{-2}$, whereas the density of inertial mass is ρ. My justification for equating the two F's lies in McCrea's demonstration of the self-consistency of his use of gravitational mass, which is similar to that here.

We may now treat (7.6)–(7.8) as we treated the corresponding equations in Section 3, using (7.2), (7.3), (7.9) and (7.10) where necessary. The result, corresponding to (3.13) is

$$\ddot{\omega} + 2rf\dot{\omega}' + r^2f^2\omega'' + \dot{\omega}f(8 + 3p_0c^{-2}\rho_0^{-1})$$
$$+ r\omega'f^2\left(9 + \frac{\dot{f}}{f^2} + \frac{3p_0}{c^2\rho_0}\right) + \omega f^2\left(18 + \frac{6\dot{f}}{f^2} + \frac{9p_0}{c^2\rho_0} + \frac{12\pi G p_0}{c^2f^2}\right)$$
$$= \frac{dg}{d\rho}\left(\omega'' + \frac{2\omega'}{r}\right) - \frac{\omega}{c^2}\frac{dg}{d\rho}\left(9f^2 + 9\dot{f} + \frac{36\pi G p_0}{c^2}\right).$$

We can simplify this by using (7.4) and (7.5), which reduces it to

$$\ddot{\omega} + 2r\frac{c}{a}\dot{\omega}' + r^2\frac{c^2}{a^2}\omega'' + \frac{5c}{a}\dot{\omega} + 6r\frac{c^2}{a^2}\omega' + \frac{9c^2}{2a^2}\omega\left(1 - \frac{1}{c^2}\frac{dg}{d\rho}\right)$$
$$= \frac{dg}{d\rho}\left(\omega'' + \frac{2\omega'}{r}\right). \tag{7.11}$$

Now change to a new variable x by the transformation

$$r = xR(t);$$

this changes (7.11) to

$$\frac{\partial^2\omega}{\partial t^2} + \frac{5c}{a}\frac{\partial\omega}{\partial t} + \frac{9c^2}{2a^2}\omega\left(1 - \frac{1}{c^2}\frac{dg}{d\rho}\right) = \frac{dg}{d\rho}R^{-2}\left(\frac{\partial^2\omega}{\partial x^2} + \frac{2}{x}\frac{\partial\omega}{\partial x}\right).$$

If we assume a solution of the form

$$\omega = \frac{\sin kx}{kx}h(t),$$

we find the following equation for h:

$$\ddot{h} + \frac{5c}{a}\dot{h} + \frac{9c^2}{2a^2}h\left(1 - \frac{1}{c^2}\frac{dg}{d\rho} + \frac{k^2}{R^2}\frac{dg}{d\rho}\right) = 0. \tag{7.12}$$

Evidently we may ignore the term $c^{-2}dg/d\rho$ in the bracket, which in the case of a gas at ordinary temperatures will be small compared with unity, and it is then clear that h cannot increase with time. In the most favourable case, in which $dg/d\rho = 0$, the solution of the equation (7.12) then is

$$h = Ae^{m_1t} + Be^{m_2t},$$

where

$$m_1, m_2 = \frac{1}{2}\frac{c}{a}[-5 \pm \sqrt{7}].$$

Since the condensation $\delta\rho/\rho_0$ is proportional to ω (because ρ_0 is constant) it follows that *in the steady-state theory condensations cannot form as a result of small perturbations in the average density.*

In Sciama's work the density ρ of the condensing gas-cloud is taken to be equal to the average density ρ_0, and from the foregoing analysis it seems that no condensation would in fact take place if ρ and ρ_0 were equal, or differed by a small quantity. Strictly, therefore, Sciama's theory of the formation of the nebulae is not valid. However, in the steady-state theory, large perturbations—of the order of several times the average density—are possible as a result of clouds of gas breaking away from the accretions of already-existing nebulae. It may be that with such large perturbations new nebulae can form in the time allowed; but this is not obvious, and until it has been proved one cannot say with any confidence that the steady-state theory explains the formation of the nebulae.

8. *Conclusion.*—In this paper I have been studying the applicability of Jeans' formula to the expanding universe. Only small perturbations of the average cosmic density ρ_0 have been investigated: this is the only case to which Jeans' work strictly applies because in no other circumstances is one allowed to treat the unperturbed gas as uniform.

The result of this work is that, with a suitable interpretation of the quantities involved, Jeans' formula applies in the present phase of the expansion (except in the steady-state theory). Thus disturbances of sufficiently large wave-lengths result in instability of the cosmic gas, which causes it to start condensing. If one supposes that independent perturbations take place in different regions of the universe, then matter will condense into separate conglomerations. The size and mass of the condensing regions are connected by (3.21), in which the epoch of the perturbation implicitly enters through ρ_0, which, of course, depends on the epoch.

Where Jeans' original work is misleading (though through no fault of Jeans) is in the suggestion (from the solution of (2.12)) that the condensations begin to form at an exponential rate. It is only in the Einstein universe that this actually happens. When the expansion is taken into account one finds that the condensation process takes place very slowly—too slowly to account for the formation of the nebulae unless one supposes either that the perturbations are much larger than those to be expected on ordinary statistical theory, or that the universe has a longer time-scale than that provided by the point-source models.

Since so much of the matter in the universe is now condensed into nebulae, clouds of gas with density very nearly equal to the average density ρ_0 must be rare at the present time. Thus Jeans' formula as originally derived and as studied in this paper, though it may prove to be important in explaining the growth of the nebulae, has little practical application to contemporary astrophysics.

Naturally one would like to know whether Jeans' type of instability operates for clouds of gas with densities larger than ρ_0. This represents an unsolved problem, even in the spherically-symmetric case, but the stability will presumably depend markedly on the equation of state; this follows from the fact, mentioned in the introduction, that an isothermal gas sphere of sufficient size is unstable whereas adiabatic spheres with $\gamma > 4/3$ are stable so far as is known. I hope to deal with this question in another paper.

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Note added by D. W. Sciama on "Jeans' Formula for Gravitational Instability".—Bonnor's paper demonstrates in a very clear way what has been generally believed for some time, namely that infinitesimal density fluctuations will not give rise to galaxies in the times available. This difficulty led Gamow to suggest that large density fluctuations would be produced by the high Mach number (~ 10) compressible turbulence that he invoked. Unfortunately the analogue of the Jeans' criterion is not known for this type of motion.

At first sight the situation is even worse for the steady-state theory. For the material density must increase by a factor three in order that self-gravitation should exactly compensate the cosmical repulsive forces that the steady-state theory must invoke. However, the situation is saved by the fact that these large density fluctuations do not have to form in an initially homogeneous universe. Indeed, all we have to show is that galaxies are self-propagating. We can hence use the fact that a galaxy will produce a large density fluctuation in its wake in a *systematic* way, so that the region of large density will not be highly turbulent, and Jeans' criterion can be applied to it.

In my paper I gave some sketchy calculations of the process of galaxy formation in a steady-state universe. Further details are given in my thesis (*On the Origin of Inertia*, Chapter V, Cambridge University 1953). In my paper I glossed over the cosmological aspects of the problem in order to concentrate on the astrophysical ones. That is why I assumed that the system was Newtonian in the region of interest. In fact one must consider only that part of the wake in which the density exceeds 3ρ. If we go further and require the density to exceed 4ρ, then the net gravitating density to be used in the Jeans formula is of order ρ (because of the cosmical repulsion). From the formula for ρ_p given on page 5 of my paper we see that $\rho_p \geqslant 4\rho$ when

$$r_p \sin\theta_p \leqslant \frac{1}{16}\left(\frac{3M_1}{8\pi\rho}\right)^{1/3}$$

where ρ_p is the density at the point p, r_p is the distance of p from the parent galaxy P, θ_p is the angle Pp makes with the peculiar velocity vector of P, and M_1 is the mass of P. Gravitational instability only occurs in the part of the wake which satisfies this equation. This restriction does not alter the order of magnitude estimates made in my paper.

I should add that various other points are considered in the thesis, e.g. the effect of the Hubble velocities initially present in the inter-galactic gas, and the stress produced by newly-created matter which is presumably under the gravitational influence of the parent galaxy only after it appears. Again these effects do not alter the order of magnitude estimates. I conclude then that Bonnor's calculations do not affect my theory of galaxy formation.