INHOMOGENEOUS STELLAR MODELS. V. A SOLAR MODEL WITH CONVECTIVE ENVELOPE AND INHOMOGENEOUS INTERIOR

M. SCHWARZSCHILD,* R. HOWARD, AND R. HÄRM
Princeton University Observatory
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ABSTRACT

A model for the sun has been computed in which account has been taken both of the deep hydrogen convection zone and of the internal inhomogeneity in composition caused by the transmutation of hydrogen during the last five billion years. The model is found to permit good agreement with the analysis of the solar photosphere as regards chemical composition and as regards the depth of the hydrogen convection zone. Furthermore, the model indicates that the sun must have become brighter by nearly $\frac{1}{2}$ mag. during the five billion years.

INTRODUCTION

In recent investigations of the interior of the sun (Epstein and Motz 1953; Naur 1954; Abell 1955) models have been derived under the assumption of radiative equilibrium in the envelope and of homogeneous composition. These models may be improved in two points, by substituting convective equilibrium for radiative equilibrium in the outer parts of the envelope and by taking into account the inhomogeneities caused by the hydrogen burning during the past life of the sun.

The importance of convective envelopes for red dwarfs has been shown by Osterbrock (1953). That the sun also has a convective envelope of appreciable depth follows from an investigation of the turbulent subphotospheric layers by Vitense (1953).

The effect of the hydrogen burning on the composition can be estimated for the sun as a whole directly from its luminosity. One finds for, say, five billion years that 5 per cent of the solar mass has been transmuted from hydrogen to helium. If this transmutation occurred evenly throughout the sun, the effect on the solar model would be small. In fact, however, the local transmutation rate at the center is about ten times higher than the average transmutation rate, and hence a pronounced inhomogeneity in composition must exist in the present sun. The importance of this inhomogeneity for the solar model was recently emphasized by Henyey *et al.* (1955).

It is the purpose of this paper to derive a solar model taking into account the two corrections just indicated. The derivation is carried through in three steps. First, a homogeneous model is determined for the sun in its initial state when the hydrogen burning is just commencing. Second, on the basis of this initial model, the transmutation rates and the present composition are computed for every point in the sun. Finally, with the help of this new inhomogeneous composition, a model is constructed for the sun in its present state.

HOMOGENEOUS MODEL FOR INITIAL SUN

According to the preceding discussion, we require for the sun in its initial state a homogeneous model consisting of a radiative interior and a convective envelope. Thus the problem of the initial sun is exactly the same as that of the red dwarfs which was carried through by Osterbrock (1953). We shall therefore refer to Osterbrock's paper for the definition of all the symbols and for the technique used in the construction of the model and shall discuss here only the main points of interest for the sun.

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The absorption coefficient in the radiative interior of the sun may be represented by the interpolation formula

$$\kappa = 6.52 \times 10^{24} \left(Z + \frac{X+Y}{59.3} \right) (1+X)^{0.75} \rho^{0.75} T^{-3.5} . \tag{1}$$

The exponent of the density as well as the two numerical coefficients in this formula are adjusted so as to fit the formula as closely as possible to the opacity tables by Keller and Meyerott (1955) for the relevant temperatures, densities, and compositions. Equation (1) represents these tables with maximum errors of 40 per cent. Since the tables take account not only of the bound-free absorption by the heavier atoms but also of the free-free absorption by hydrogen and helium and of electron scattering, equation (1) can be taken to represent the combined effect of all these processes. The only essential difference between equation (1) and the corresponding formula appropriate for the red dwarfs is in the exponent of the density. It is this difference which prohibits the use of the red dwarf models for the initial state of the sun with reasonable accuracy.

For all the other physical quantities the same approximations and formulae may be used for the sun as Osterbrock has used for the red dwarfs. In particular, the energy generation by the proton-proton reaction may again be represented by

$$\epsilon = 2.8 \times 10^{-33} X^2 \rho T^{4.5} . \tag{2}$$

The convective envelope is characterized by the parameter K, which is defined by

$$P = K T^{2.5} . \tag{3}$$

It can be transformed into the non-dimensional parameter E by

$$E = 4\pi \left(\frac{H}{k} \right)^{2.5} G^{1.5} \mu^{2.5} M^{0.5} R^{1.5} K . \tag{4}$$

The parameters K and hence E are, in principle, determined by the physical characteristics of the photospheric layers. In practice, however, it is not yet possible to determine K with sufficient accuracy from the photospheric conditions because of the uncertainty in the efficiency of the convective energy transport in the subphotospheric layers. We shall therefore consider K a free parameter here and accordingly derive not one homogeneous model but a whole family of such models, with each particular model corresponding to a particular value of K or E.

The radiative interior of these models is characterized by the two parameters C and D, which are defined by

$$C = \frac{3}{4ac} \left(\frac{k}{HG} \right)^{7.5} \frac{6.52 \times 10^{24}}{(4\pi)^{2.75}} \left(Z + \frac{X+Y}{59.3} \right) \frac{(1+X)^{0.75}}{\mu^{7.5}} \frac{L R^{1.25}}{M^{5.75}} , \tag{5}$$

$$D = \left(\frac{HG}{k} \right)^{4.5} \frac{2.8 \times 10^{-33}}{4\pi} X^2 \mu^{4.5} \frac{M^{6.5}}{L R^{7.5}} , \tag{6}$$

where C is essentially the coefficient in the mass-luminosity relation and D the coefficient in the energy-output relation.

For the convective envelopes the family of numerical integrations previously obtained with the electronic computer of the Institute for Advanced Study could here be used again (Härm and Schwarzschild 1955). For the radiative interior the necessary family of solutions had to be obtained by new numerical integrations. The envelope solutions were fitted to the interior solutions by the method described by Osterbrock.

In this manner eight homogeneous models have been constructed. Their mathematical properties are listed in Table 1. The eight models differ from one another in the value of the parameter E, which governs the depth of the convection zone; the first model has a very deep convection zone, while the eighth model has a rather shallow one.

The main purpose of constructing the homogeneous models is to gain a basis for computing the transmutation rates everywhere in the sun, from which, in turn, we shall derive the present inhomogeneous composition of the sun. Before proceeding with this main purpose, it may be useful to apply the homogeneous model directly to the sun. This direct application is not logically consistent, since the models refer to the homogeneous initial state while the observed luminosity and radius refer to the present state of the sun. This direct application can therefore not yield reliable results. It is useful, however, in showing us to what extent the results for the sun are altered by the introduction of the convection zone without the introduction of the inhomogeneities.

TABLE 1

MATHEMATICAL PROPERTIES OF HOMOGENEOUS MODELS FOR INITIAL SUN

	I	II	III	IV	V	VI	VII	VIII
$(n+1)_c$	2 80	2 88	2 882	2 8821	2 882145	2 8821456	2 8821457	2 88214573
E	44 83	34 73	23 87	19 66	8 20	5 70	4 14	1 68
$\log x_0$	−0 864	−0 956	−1 024	−1 050	− 1 136	− 1 161	− 1 179	− 1 218
$\log p_0$	+1 173	+1 405	+1 633	+1 729	+ 2 057	+ 2.155	+ 2 228	+ 2 381
$\log q_0$	−1 141	−1 210	−1 231	−1 236	− 1 243	− 1 243	− 1 243	− 1 244
$\log f_0$	−0 051	−0 082	−0 084	−0 084	− 0 084	− 0 084	− 0 084	− 0 084
U_1	2 274	1 186	0 660	0 502	0 152	0 092	0 059	0 016
V_1	2 040	5 173	7 059	7 817	10 990	12 472	13 944	19 333
W_1	0 674	0 020	0 001	0 000	0 000	0 000	0 000	0 000
x_1	0 413	0 605	0 675	0 698	0 775	0 801	0 821	0 871
q_1	0 310	0 730	0 877	0 914	0 982	0 991	0 995	0 999
$\log p_1$	+0 564	−0 039	−0 460	−0 647	− 1 433	− 1 754	− 2 038	− 2 845
$\log t_1$	−0 435	−0 632	−0 735	−0 776	− 0 939	− 1 003	− 1 061	− 1 227
f_1	0 887	0 999	1 000	1 000	1 000	1 000	1 000	1 000
$\log p_c$	+1 010	+1 238	+1 466	+1 562	+ 1 890	+ 1 988	+ 2 061	+ 2 214
$\log t_c$	−0 277	−0 254	−0 207	−0 186	− 0 107	− 0 083	− 0 064	− 0 026
$\log C$	−5 430	−5 678	−5 714	−5 708	− 5 641	− 5 611	− 5 590	− 5 539
$\log D$	+0 887	+0 610	+0 239	+0 072	− 0 524	− 0 705	− 0 843	− 1 129
$\log[D p_c t_c^{3\ 5}]$	+0 928	+0 959	+0 980	+0 983	+ 0 992	+ 0 992	+ 0 994	+ 0 994
$\log[C^{1\ 2} D^{0\ 2}]$	−6 339	−6 692	−6 809	−6 835	− 6 874	− 6 874	− 6 877	− 6 873

We may apply the models of Table 1 to the observed mass, luminosity, and radius of the sun by choosing a value for the hydrogen content, X, and then determining the values of the helium content, Y, and the convection parameter, E, with the help of the mass-luminosity relation and the energy-output relation. The results of such a computation are given in Table 2 for three values of X.

The main result shown in Table 2 is the fact that a value for Z of about 0.02 is reached when E has the fairly small value of 0.8. This indicates that the introduction of even a fairly shallow convection zone permits the construction of an interior model with an abundance of the heavier elements, in agreement with the spectroscopic data. Thus it appears that the introduction of a convection zone removes the difficulty encountered with all recent purely radiative models, which persistently gave an abundance of the heavier elements smaller than the spectroscopically observed one.

This encouraging result does not change the fact, however, that the data of Table 2 cannot be accurate, since the sun in its present state cannot be homogeneous. Accordingly, we turn now to the inhomogeneous models.

CHANGES IN COMPOSITION IN FIVE BILLION YEARS

The rate of change with time of the hydrogen content by nuclear transmutation is given by

$$\frac{dX}{d\tau} = -\frac{\epsilon}{6.3 \times 10^{18}}. \tag{7}$$

The numerical coefficient in this equation represents the number of ergs released by the transmutation of 1 gm of hydrogen into helium. If we take the rate of energy generation from the homogeneous models, equations (7) gives us the reduction of the hydrogen content at every point in the sun.

TABLE 2

PRELIMINARY RESULTS FOR SUN FROM HOMOGENEOUS INITIAL MODELS

X	Y	Z	E	x_1	q_1	T_1	T_c	ρ_c
0 60	0 344	0 056	1 02	0 887	0 9997	$0\ 8 \times 10^6$	$15\ 0 \times 10^6$	87
70	276	024	0 86	891	0 9998	7×10^6	$13\ 8 \times 10^6$	88
0 80	0 197	0 003	0 68	0 896	1 0000	$0\ 6 \times 10^6$	$12\ 9 \times 10^6$	90

For convenience we may express the energy generation in terms of the central values by

$$\epsilon = \epsilon_c \frac{p}{p_c} \left(\frac{t}{t_c}\right)^{3\ 5} \tag{8}$$

with

$$\epsilon_c = \frac{L}{M} \times (D p_c t_c^{3\ 5}) . \tag{9}$$

The last equation for the energy generation at the center follows directly from the definition of D by equation (6) and from the usual definitions of the non-dimensional variables p and t. The run of p and t in units of their central values, which occur in equation (8), can be taken from the numerical integrations corresponding to the models of Table 1. Here it matters little exactly which model we use, as long as we restrict ourselves to the last four models, which have relatively shallow convection zones covering, at most, one-quarter of the radius. The close agreement in the distribution of the energy sources between the last four models of Table 1 is physically plausible, since one would not expect the exact extent of a shallow outer convection zone to have a serious influence on the structure of the deep interior, and this close agreement is shown by the minuteness of the differences of $(n+1)_c$ between the four models. Similarly, the non-dimensional quantity which occurs in the parentheses of equation (8) and which is listed in the next to the last line of Table 1 shows no significant variation between the four models in question.

By introducing equations (8) and (9) into equation (7) and by integrating this latter equation over the time interval, τ, under the assumption that the rate of energy generation does not sensibly change during this time interval at any point in the sun, we obtain, for the present hydrogen content,

$$X = X_e - \left(\frac{\tau}{6.3 \times 10^{18}} \frac{L}{M} D p_c t_c^{3\ 5} \right) \frac{p}{p_c} \left(\frac{t}{t_c} \right)^{3\ 5} . \tag{10}$$

Here X_e stands for the original hydrogen content, which still prevails in the envelope. We shall use for X_e in equation (10) the value 0.80. The choice of this value is not critical, however, since the inhomogeneous model will depend on the variation of X rather than on its absolute value. Accordingly, we will be quite free to apply the resulting inhomogeneous models to compositions with X_e-values differing from that used in equation (10). The first parentheses of the second term in equation (10) contain nothing but constants. If we use for τ five billion years, for L and M the solar values, and for the product of the non-dimensional quantities the value for the last model of Table 1, the parentheses take the value 0.50.

After the constants in equation (10) have thus been chosen, this equation can be applied to the interior integration corresponding to the last model of Table 1, and from it the hydrogen content as a function of the mass fraction, q, can be computed for the

TABLE 3

DISTRIBUTION OF HYDROGEN IN PRESENT SUN

log q	X	log q	X	log q	X
−4 888	0 301	−2 853	0 328	−0 808	0 687
−4 289	303	−2 496	347	− 491	766
−3 689	308	−2 142	379	− 376	784
−3 330	314	−1 908	410	− 110	800
−3 091	0 320	−1 336	0 530	−0 014	0 800

sun in its present state after five billion years of hydrogen burning. The results of this computation are given in Table 3. The data of this table represent the only link between the initial solar model and the present solar model. We have seen that these data are quite independent of the uncertain depth of the convection zone in the initial state of the sun.

The hydrogen distribution given in Table 3 was derived under the assumption that at every point in the sun the rate of hydrogen consumption in the initial model represents sufficiently well the average rate during the last five billion years. This assumption could be checked after the final inhomogeneous model was computed. Regarding the distribution of the hydrogen consumption, it was found that the average rate is smaller than the initial rate in the centermost region but somewhat exceeds it farther out; accordingly, the X-values of Table 3 should be increased by about 0.1 for q between 0.00 and 0.02 and decreased by about 0.01 for q between 0.1 and 0.3—a redistribution which can hardly affect the final model seriously. Regarding the total amount of hydrogen consumption, however, the check showed that the use of the present solar luminosity in equation (10) leads to an overestimate of the consumption, since the average luminosity during the last five billion years may have been about 25 per cent smaller than the present luminosity; accordingly, all the differences in this paper between the initial and the final model may be exaggerated by perhaps 25 per cent.

In the numerical integrations for the inhomogeneous model it would be inconvenient to have to take the hydrogen content at every step by interpolation in Table 3. It is more convenient to use the following interpolation formulae which represent the data of Table 3 with maximum deviations of 3 per cent:

$$
\begin{aligned}
&\log\left(\frac{X}{X_e}\right) = -0.416 &&\text{for} && \log q < -3.2436\,, \\
&\log\left(\frac{X}{X_e}\right) = -0.1565 + 0.08 \log q &&\text{for} && -3.2436 < \log q < -2.0436\,, \\
&\log\left(\frac{X}{X_e}\right) = +0.0887 + 0.20 \log q &&\text{for} && -2.0436 < \log q < -0.4436\,, \\
&\log\left(\frac{X}{X_e}\right) = 0.000 &&\text{for} && -0.4436 < \log q\,.
\end{aligned}
\tag{11}
$$

These formulae completely define the composition at every point in the sun, since they give directly the variation of X, while Z is unaffected by the hydrogen burning and hence remains constant throughout the sun.

CONSTRUCTION OF INHOMOGENEOUS MODELS FOR PRESENT SUN

The construction of the inhomogeneous model can be carried out in much the same manner as that of the homogeneous models. The convective envelopes are again characterized by the parameter K or E defined by equations (3) and (4). The available family of integrations for convective envelopes may be used as before.

The radiative interiors are again characterized by the parameters C and D, defined by equations (5) and (6). In these definitions we should now use, for the sake of definiteness, the unaltered composition of the outer layers, designated by a subscript e. The differential equations for the inhomogeneous radiative interior can be written in terms of the usual non-dimensional variables in the following form:

$$
\begin{aligned}
&\frac{dp}{dx} = -l\frac{pq}{tx^2}, \qquad \frac{dq}{dx} = +l\frac{px^2}{t}, \\
&\frac{dt}{dx} = -l j^{0.75} C \frac{p^{1.75} f}{t^{8.25} x^2}, \qquad \frac{df}{dx} = +liDp^2 t^{2.5} x^2 .
\end{aligned}
\tag{12}
$$

Here the three composition functions, l, j, and i, are defined by

$$
l = \frac{\mu}{\mu_e}, \qquad j = \frac{1+X}{1+X_e}\,\frac{\mu}{\mu_e}, \qquad i = \frac{X^2}{X_e^2}\,\frac{\mu}{\mu_e}. \tag{13}
$$

According to these definitions, one can compute the variation of the three composition functions throughout the sun from the variation of composition as derived in the preceding section and given by equation (11). This computation is simplified by replacing the exact definitions (13) by the approximate formulae

$$
l = \left(\frac{X}{X_e}\right)^{-0.46}, \qquad j = \left(\frac{X}{X_e}\right)^{-0.11}, \qquad i = \left(\frac{X}{X_e}\right)^{+1.54}, \tag{14}
$$

which represent the exact relations with maximum errors of less than 3 per cent over the range in composition here in question.

Because of the introduction of the inhomogeneity in composition, new numerical integrations are needed for the radiative interior. These integrations were obtained by the usual method, described by Osterbrock. One new complication appears, however. In the usual method the integrations from the center are reduced to a one-parameter family by applying a simple transformation to all the variables. In particular, the mass fraction, q, is transformed into a new variable by dividing it by a constant q_0. Now, according to equation (11), the composition is known as a function of q. If the value of q_0 were known from the outset, the composition would be known also as a function of the transformed mass variable. In fact, however, q_0 can be determined only after the fitting of the interior solution to the envelope solution. Since the variation of the composition is needed as a function of the transformed variable during the integration, one has to guess at the value of q_0, i.e., q_0 becomes a second trial parameter.

In general, the introduction of a second trial parameter greatly increases the numerical work. In this case, however, the increase in the number of necessary numerical integrations was fortunately found to be small. As a first trial value for q_0 the value of the homogeneous model was used. A short series of trial integrations indicated the necessity of a small correction for q_0. This led to a second trial value, which immediately gave the required accuracy.

TABLE 4

MATHEMATICAL PROPERTIES OF INHOMOGENEOUS MODELS FOR PRESENT SUN

	I	II	III		I	II	III
$(n+1)_c$	4 3596726	4 35967288	4 3596728915	$\log p_1$	− 1 367	− 1 950	− 2 657
E	7 58	3 90	1 76	$\log t_1$	− 0 898	− 1 016	− 1 160
				f_1	1 000	1 000	1 000
$\log x_0$	− 1 140	− 1 188	− 1 227				
$\log p_0$	+ 2 249	+ 2 440	+ 2 594	$\log p_c$	+ 2 237	+ 2 428	+ 2 582
$\log q_0$	− 1 155	− 1 155	− 1 156	$\log t_c$	− 0 015	+ 0 032	+ 0 072
$\log f_0$	+ 0 211	+ 0 211	+ 0 211	$\log C$	− 5 424	− 5 369	− 5 315
				$\log D$	− 0 831	− 1 186	− 1 474
U_1	0 152	0 061	0 020				
V_1	10 22	12 80	16 92	$\log (D p_c t_c^{3\ 5})$	+ 1 353	+ 1 354	+ 1 360
				$\log (C^{1\ 2} D^{0\ 2})$	− 6 675	− 6 680	− 6 673
x_1	0 759	0 805	0 852				
q_1	0 980	0 994	0 999				

The fitting of the interior integrations to the envelope integrations was carried out by the same method as that employed for the homogeneous models. The resulting one-parameter family of inhomogeneous models is represented by the three models listed in Table 4. The differences between these three models are caused by differences in their E-values, which has the main effect of producing differences in the depth of the convective zones.

In the application of the inhomogeneous models of Table 4 to the sun, one encounters exactly the same problem as that Osterbrock found for the red dwarfs. After entering the observed values of M, L, and R into the mass-luminosity relation, equation (5), and the energy-output relation, equation (6), one finds that these two relations contain three unknowns—the hydrogen content, X_e; the helium content, Y_e; and the depth of the convection zone, characterized by E, which enters equations (5) and (6), since C and D are functions of E, as shown by Table 4.

One may be tempted to approach this problem in the following—unsuccessful—manner. One might use a value of E for the sun determined as well as possible from a model of the photosphere. For this value of E one could then determine the corresponding values of C and D by interpolation in Table 4. When these values of C and D are introduced into equations (5) and (6), these two equations contain only the two unknowns X_e and Y_e. In principle, then, one could solve for X_e and Y_e. In practice, however, this is not possible because the composition-dependent factor, $X_e^2\mu_e^{4.5}$, of equation (6) in actuality varies hardly at all with composition. Thus a small error in E and hence in D will cause great errors in the solution for the composition. In fact, an error of only 1.0 in E can change the resulting composition from containing no heavy elements to containing more than 10 per cent, while actually the uncertainty of E is appreciably larger than 1.0. This unfortunate situation is characteristic of stars living on the proton-proton reaction and does not exist for stars living on the carbon cycle, since for the latter the energy-output relation corresponding to equation (6) contains a very high power of μ.

Since we cannot use equations (5) and (6) to solve for X_e and Y_e for an assumed value of E, we have to assume a value for X_e and solve the two equations for Y_e and E. After E is thus determined, the mathematical properties of the corresponding model can be found by interpolation in Table 4. The physical properties then follow in the usual manner. The results of such computations are given in Table 5 for three different values for X_e.

TABLE 5

RESULTS FOR SUN FROM INHOMOGENEOUS MODELS

X_e	Y_e	Z_e	E	x_1	q_1	T_1	ρ_1	T_c	ρ_c
0 60	0 30	0 10	3 61	0 810	0 995	$1\ 46\times10^6$	0 049	$17\ 1\times10^6$	122
70	26	04	3 25	818	996	$1\ 27\times10^6$	041	$15\ 8\times10^6$	127
0 80	0 185	0 015	2 95	0 824	0 997	$1\ 12\times10^6$	0 035	$14\ 8\times10^6$	132

RESULTS FOR PRESENT SUN

The first columns listed in Table 5 refer to the present composition of the envelope of the sun, i.e., the initial composition of the sun. We see that the new models, in contrast to the earlier purely radiative models, permit a comfortably large abundance of the heavier elements Z. If we take it from the spectroscopic observations that the abundance of the heavier elements in the sun must be approximately 2 per cent, then we find from the new models that the helium abundance must be about 20 per cent.

The fourth column of Table 5 shows that the E-value for the sun should be approximately 3. This value has to be compared with the results from the analysis of the solar photosphere. The analysis of Vitense (1953) gives $E = 1.0$ if one assumes the effective mixing length in the subphotospheric layers to be equal to 1 scale height, and $E = 10$ if one assumes the mixing length to be equal to 2 scale heights. Thus the value for E from the new interior model falls happily between the two values from the photospheric analysis. Indeed, one might turn this comparison around and consider it an indirect determination of the subphotospheric mixing length, with the result that this length should be approximately one and a half times the scale height.

The fifth column of Table 5 shows that the radiative interior in the sun covers approximately 82 per cent of the radius, leaving 18 per cent covered by the convection zone. Similarly, the next column shows that the radiative portion contains practically all the mass, leaving only about one-third of 1 per cent to be contained in the convective zone.

The seventh column of Table 5 indicates that the temperature at the bottom of the convection zone is only a little higher than 1000000°. This temperature is a factor of about 2 lower than would be necessary for the transmutation of lithium. It therefore appears impossible to assume any longer that the extraordinary low abundance of lithium observed at the surface of the sun has been caused by the transmutation of lithium at the bottom of the convection zone during recent stages of the sun. It now seems more likely that the reduction in the lithium occurred during the original Kelvin contraction of the sun, when the convection zone may have been much deeper than it appears to be now.

The last two columns of Table 5 give the central temperature and density for the sun. The new density value differs little from those of earlier models. The new central temperature of approximately 15000000° is noticeably higher, however, than the values obtained from the earlier homogeneous models. This increase appears to be caused largely by the low hydrogen abundance of only 30 per cent in the center of the new inhomogeneous model. This increased central temperature suggests that, as a further improvement in the model, it may be necessary to take account of the contribution of the carbon cycle to the energy generation in the centermost region of the sun.

Finally, we may determine the increase in the solar luminosity during the last five billion years by comparing the initial homogeneous model with the present inhomogeneous one. For each of the two models one can derive a formula for the total luminosity as a function of the total mass and the envelope composition (neither of which varies during the evolution) and of C and D (both of which do vary during the evolution) by eliminating R from equations (5) and (6). The resulting formula shows that L varies proportionally to $C^{1.2}D^{0.2}$. This quantity is listed in the bottom line of Table 1 for the homogeneous models and in the bottom line of Table 4 for the inhomogeneous ones. The logarithmic difference between the two types of models—if we here again consider only the cases with rather low E-values—gives 0.20 for the increase in the logarithm of the solar luminosity. We conclude that the sun has become brighter by $\frac{1}{2}$ mag. during the last five billion years.

This result suggests that in the early pre-Cambrian era, two billion years ago, the solar luminosity was about 20 per cent less than now. The average temperature on the earth's surface must then have been just about at the freezing point of water, if we assume that it changes proportionally to the fourth root of the solar luminosity. Would such a low average temperature have been too cool for the algae known to have lived at that time?

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