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THE INSTABILITY OF THE EINSTEIN UNIVERSE

W. B. Bonnor

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Summary

The stability of the Einstein universe is investigated using the new boundary conditions at surfaces of discontinuity, discovered by O'Brien and Synge. On certain assumptions it is shown that the Einstein model may represent a universe with inhomogeneities, but that, in general, these must be static otherwise the model is unstable.

In the case of an instantaneous disturbance in the Einstein universe, the processes which cause instability must involve discontinuities in the pressure or its time derivatives. If the matter present satisfies an equation of state, no physically plausible disturbance of the static condition is possible, and the model is stable. If there is no equation of state, small disturbances involving changes in the pressure or its derivatives can occur, and will, in general, lead to instability, though in certain special cases condensations may begin to form without initial instability.

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1. *Introduction.*—Between the years 1930 and 1935 there appeared several papers on the stability of the Einstein model of the universe. The interest in this problem arose mainly because it was thought that the present expansion of the universe might have started from a perturbation of the Einstein equilibrium state. Although the writers all agreed that the Einstein model is unstable in some way or other, there was no agreement on the nature of the process in the model which would cause instability, or on the conditions in which an expansion, rather than a contraction, would begin to take place. These questions were not settled, and for various reasons cosmologists turned their attention to the point-source models of general relativity. However, these models are not without their difficulties—such, for example, as that of the formation of condensations in them—which make it premature to rule out the disturbed Einstein model. It therefore seems worth while to reconsider the stability of the Einstein universe in the light of certain recent developments in the theory of relativity.

A study of the previous work on the stability of the Einstein model clearly shows that much of the discordance in the results is due to the use of different boundary conditions at surfaces of discontinuity. The usual procedure was to suppose some disturbance to occur in the model at a definite time, say $t=0$, and to try to find, under certain assumed conditions of discontinuity of the metric and of the pressure and density, what happens for $t>0$. As twenty years ago the question of boundary conditions to be used for such problems in general relativity had received no systematic attention, it is not surprising that different sets of conditions were used, and different conclusions reached.

Quite recently this subject of boundary conditions at surfaces of discontinuity in general relativity has been studied by O'Brien and Synge (1), and by

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Lichnerowicz (**2**). By the use of boundary conditions obtained by these authors it is possible to give a rather more definite answer than formerly to the question of the stability of the Einstein universe. Of course, in a subject like general relativity, where the fundamental equations are so far removed from physical intuition, sets of boundary conditions must depend on certain mathematical assumptions, so that in the absence of direct experimental confirmation they are arbitrary to a certain extent, and must be used tentatively. However, since essentially the same sets of conditions have been reached by O'Brien and Synge and by Lichnerowicz, starting from plausible but different assumptions, and since, as I have pointed out elsewhere (**3, 4**), the results of applying them to certain problems seem highly realistic, it is reasonable to use them in studying the stability of the Einstein universe.

In this paper I shall use these conditions in two problems connected with the stability of the Einstein model. The first of these, treated in Section 3, is the one stated above, of finding what instantaneous disturbances are possible, and what effects they would have if they occurred. The second problem, dealt with in Section 4, is that of the existence of models similar to Einstein's in being on a large scale homogeneous and static, but containing local inhomogeneities and motions. Before discussing these problems I give a short historical survey of the subject in Section 2. The conclusions which I reach, given in Section 5, are different from widely expressed views on the subject. They are, roughly, that the Einstein model is not necessarily unstable, nor *essentially* homogeneous, but that instability would result from certain disturbances in it which involve changes in the pressure.

2. *Historical survey.*—Using co-moving coordinates, the line-element of the homogeneous cosmological models of general relativity may be written

$$ds^2 = -e^{g(t)}\left[\left(1-\frac{r^2}{R^2}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2)\right] + dt^2, \tag{2.1}$$

where R is a constant which may be real, imaginary or infinite, and $g(t)$ is an undetermined function of the time in terms of which the proper-density ρ and the proper-pressure p of the matter in the model are given by

$$8\pi\rho = 3R^{-2}e^{-g} + \tfrac{3}{4}\dot{g}^2 - \Lambda, \tag{2.2}$$

$$8\pi p = -R^{-2}e^{-g} - \ddot{g} - \tfrac{3}{4}\dot{g}^2 + \Lambda. \tag{2.3}$$

Here $\cdot$ denotes differentiation with respect to t, and Λ is the cosmological constant. The special case, corresponding to a static model, in which $g(t)\equiv 0$ and R is real, is called the Einstein universe, and for this the line-element may be put in either of the forms:

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2) + dt^2, \tag{2.4}$$

$$ds^2 = -\left(1+\frac{r^2}{4R^2}\right)^{-2}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) + dt^2. \tag{2.5}$$

The statement that the Einstein universe is unstable was first made in 1930 by Eddington (**5**), whose argument was as follows. If R is real then $Re^{1/2g}(=a)$ is the radius of the model (2.1), and (2.2) and (2.3) give

$$\frac{3}{a}\frac{d^2a}{dt^2} = \Lambda - 4\pi(\rho+3p). \tag{2.6}$$

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For an Einstein universe with $p=0$ we have $4\pi\rho=\Lambda$; but if in such a model there is a slight disturbance causing a decrease in density, it is clear from (2.6) that d^2a/dt^2 is positive, so that the radius starts to increase. Moreover, since this expansion will cause a further decrease in density, it will continue, so that the model is unstable. A similar argument may be used, with the same result, if the pressure is not zero; and evidently increases in p or ρ will, on this reasoning, cause instability due to contraction. Eddington did not give any clear indication how the initial change in the density would occur, but considered that the formation of condensations might be a cause of the instability.

Shortly afterwards the instability of the Einstein universe was studied in some detail by Lemaître (**6, 7**), who tried to find, in particular, the nature of the process which would cause instability. He concluded that the process required to start an expansion from the Einstein equilibrium state was a reduction of the pressure in the model, which he called a "stagnation". The formation of condensations, except in so far as it is accompanied by a change in pressure, has, according to Lemaître, no effect on stability.

At about the same time McCrea and McVittie (**8, 9, 10**) attacked the problem by considering the differences between the volume of the completely homogeneous Einstein universe and of various static models similar to the Einstein universe but containing condensations represented as point singularities. They eventually concluded that, to the first order of approximation, the presence of condensations does not affect the total volume of the model, and that their method was not suitable for judging the effect of condensations on stability.

Subsequent studies of the instability were made by Dingle (**11**), Tolman (**12**) and Sen (**13, 14**), with conflicting results. The method of these authors was of the type referred to in Section 1, and it is at this stage that it becomes clear that the results obtained must depend to a large extent on the boundary conditions used. It should, however, be remarked that the importance of such conditions had evidently been realized in the earlier work of Lemaître.

3. *The conditions for instability.*—In this section we shall consider the stability of the Einstein universe to a disturbance which takes place at a definite instant $t=0$, and which preserves isotropy of pressure and also spherical symmetry about the origin of coordinates. By a disturbance is meant an event which corresponds to discontinuities in the functions which characterize the model; that is to say, in p, ρ and the metric tensor g_{ik} (but not in Λ). I suppose that the discontinuities must conform with the boundary conditions of O'Brien and Synge (**1**), which are that at the boundary $t(\equiv x_4)=0$, the following quantities are continuous:

$$g_{ik},\ \partial g_{\mu\nu}/\partial x_4,\ T_k^4,\ g_{i\mu}T_k^{\mu}-g_{k\mu}T_i^{\mu}, \tag{3.1}$$

where $i, k=1, 2, 3, 4$ and $\mu, \nu=1, 2, 3$, and T_k^i is the usual energy tensor of general relativity.

The most general expression for a non-static universe, using co-moving coordinates and supposing spherical symmetry about the origin, but not isotropy for all observers, is

$$ds^2=-e^{\lambda}dr^2-e^{\omega}(d\theta^2+\sin^2\theta d\phi^2)+e^{\nu}dt^2, \tag{3.2}$$

where λ, ω and ν are functions of r and t. In accordance with the assumptions of the previous paragraph, we suppose that the line-element has form (2.4) for $t<0$, and form (3.2) for $t>0$, and that the conditions (3.1) apply at $t=0$. The

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problem is then to find the restrictions imposed on λ, ω and ν by the boundary conditions and by the field equations

$$G_k^i - \tfrac{1}{2}\delta_k^i(G-2\Lambda) = -8\pi T_k^i. \tag{3.3}$$

Since co-moving coordinates are being used, and the pressure is supposed isotropic, the only non-zero components of T_k^i are

$$T_1^1 = T_2^2 = T_3^3 = -p, \quad T_4^4 = \rho,$$

where p and ρ are the proper-pressure and proper-density.

In a previous paper (3) (hereafter referred to as I) I considered this problem for similar disturbances in a model which for $t<0$ is the non-static homogeneous universe (2.1), and I proved two theorems which, adapted to the special case of an Einstein universe, are as follows.

Theorem 1. If for $t>0$ the matter present satisfies an equation of state $p=f(\rho)$, where f is a single-valued function, then (3.2) reduces to (2.1).

Theorem 2. If the pressure is continuous at $t=0$ and if all its time derivatives at $t=0+$ are zero, then for $t>0$ (3.2) reduces to (2.4).

I shall refer to (2.1) as a Friedmann universe. Certain differentiability conditions are required in the proof of these theorems. Sufficient conditions are that p, ρ and g_{ik} possess time derivatives of all orders at $t=0+$, and that these derivatives be at least twice-differentiable functions of r. We shall assume throughout this section that these conditions are satisfied.

From Theorem 2 it follows that *disturbances of the type envisaged here can arise only if there is some change in the Einstein model which affects the pressure or its time derivatives: otherwise, the model is completely stable.* Theorem 1 shows that *if such a change takes place, and if there is an equation of state for $t>0$, the model will start expanding or contracting, and will, in fact, become a Friedmann universe.*

If there are discontinuities in the pressure and its derivatives, the stability may be studied, as was done in I for Friedmann models, by finding the initial rate of change of an element of proper volume dv of (3.2), given by $dv = e^{\frac{1}{2}\lambda+\omega}\sin\theta\, d\theta d\phi dr$. From I it follows that

$$\left[\frac{\partial(\log dv)}{\partial s}\right]_+ = 0,$$

$$\left[\frac{\partial^2(\log dv)}{\partial s^2}\right]_+ = -12\pi(p_+ - p_-), \tag{3.4}$$

$$\left[\frac{\partial^3(\log dv)}{\partial s^3}\right]_+ = -12\pi\left(\frac{\partial p}{\partial t}\right)_+, \tag{3.5}$$

where s is the proper time of an observer at rest in the coordinate system of (3.2), and $+$ and $-$ refer to $t=0+$ and $t=0-$. From (3.4) it follows that an instantaneous decrease of pressure at $t=0$ causes an expansion. This was the result found by Lemaître, who called the decrease of pressure a "stagnation". We see also from (3.5) that the expansion will be reinforced if $\partial p/\partial t$ is initially negative. If we suppose that the matter in the model is such that the pressure will be further diminished by an expansion, (3.5) indicates the instability of the model to reductions in the pressure. Evidently, a similar argument indicates that an increase of pressure will cause the Einstein model to contract.

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In reaching his result Lemaître used arguments of questionable physical significance, and his methods were strongly criticized by McCrea and McVittie (**10**). It is therefore interesting that it is possible to arrive at his result by a method which uses as its only essential assumption (apart from the spherical symmetry and isotropy of pressure) the boundary conditions (3.1).

It was shown in I that p_+ is not a function of r, but $\dot{p}_+$ may be. This implies that if there is a discontinuity in the pressure, it must be universal, but that the pressure could begin to change in some isolated region. If there is an equation of state for $t>0$, the universal discontinuity in pressure is necessary for instability to develop, because of the following argument. If there is instability, the result, by Theorem 1, is the Friedmann model (2.1); now the boundary conditions require

$$g_+=\dot{g}_+=0, \qquad \rho_+=\rho_-,$$

so that from (2.3)

$$8\pi(p_+-p_-)=-\ddot{g}_+.$$

If the pressure is continuous, $p_+=p_-$ so that $\ddot{g}_+=0$. But owing to the equation of state, we then have

$$8\pi\left(\frac{\partial p}{\partial t}\right)_+ = 8\pi\frac{df}{d\rho}\left(\frac{\partial \rho}{\partial t}\right)_+ = \frac{df}{d\rho}[-3R^{-2}e^{-g_+}\dot{g}_+ + \tfrac{3}{2}\dot{g}_+\ddot{g}_+]=0. \tag{3.6}$$

Further, differentiating (2.3) we obtain

$$8\pi\left(\frac{\partial p}{\partial t}\right)_+ = R^{-2}e^{-g_+}\dot{g}_+ - \dddot{g}_+ - \tfrac{3}{2}\dot{g}_+\ddot{g}_+. \tag{3.7}$$

Equations (3.6) and (3.7) give

$$\dddot{g}_+=0.$$

In this way it may be shown that $(d^ng/dt^n)_+=0$ for all n, so that (2.1) reduces to the Einstein model and there is no instability.

Unless, therefore, we are prepared to suppose that there is a universal discontinuity in the pressure, we have to rule out instability when there is an equation of state for $t>0$. The most realistic possibility for the development of instability seems to be the case where there is no such equation, and where the pressure is continuous, but $\partial p/\partial t$ (or a higher derivative) is discontinuous in some small region of space. In this case expansion or contraction starts according to (3.5). This case will be studied in the next section.

4. *Approximate Einstein models.*—In the previous section I have given a solution of the problem of the stability to certain types of disturbance of the Einstein model considered as completely homogeneous and static. One would naturally wish to interpret the homogeneity as the result of averaging the density and pressure over regions of a certain size, because inhomogeneities might be expected in a physical realization of the model. But for this interpretation to be permissible one must be sure that inhomogeneities could in fact exist in the Einstein universe. Stated rather more precisely, the question is whether there are solutions of the field equations representing static, inhomogeneous models which are similar to the Einstein model on a sufficiently large scale. If such solutions do not exist, then we are not entitled to regard the Einstein universe as a convenient model for something more complicated than it actually represents: in this case we must appreciate that any physical realization of the Einstein model must be strictly homogeneous, at least on a macroscopic scale.

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Such static, inhomogeneous solutions do indeed exist; some have been published already, and I shall prove the existence of others later. We are therefore allowed to think of condensations in the Einstein universe without being in danger of any logical contradiction.

I have dealt with this point at some length because some writers on the Einstein universe seem to think that it does not allow the presence of condensations at all. There has been a tendency to confuse the existence of condensations with the formation of condensations. The latter is a different question, involving non-static solutions of the field equations.

The formation of condensations may be regarded from two points of view. First, we may consider processes in the strict Einstein model involving changes of the pressure and its derivatives, discussed in Section 3, which lead to changes in density and metric. Secondly, we may suppose that we *start* with a non-static model which resembles the Einstein model in the large, but which includes regions where density changes are taking place. In both cases we want to know whether the process of condensation can continue in bounded regions without causing the model to expand or contract as a whole. But in the second case we do not have to satisfy initial conditions at $t=0$, because the model is not supposed to start from the strict Einstein state.

To study these problems I consider what I call "approximate Einstein models". In these the variations from the Einstein metric are either confined to small regions of space, or (if not) are small except over limited regions.

Several approximate Einstein models were obtained by McCrea and McVittie (**8, 9, 10**), who found line-elements corresponding to Einstein universes containing particles represented as point-singularities. As an example, a special case of a solution of McVittie (**15**) may be quoted:

$$ds^2=\left\{\frac{1-(m/2r)[1+(r^2/4R^2)]^{1/2}}{1+(m/2r)[1+(r^2/4R^2)]^{1/2}}\right\}^2 dt^2-\frac{\{1+(m/2r)[1+(r^2/4R^2)]^{1/2}\}^4}{[1+(r^2/4R^2)]^2}$$
$$\times(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2). \qquad (4.1)$$

If $m=0$ this reduces to the Einstein model in the form (2.5), and if $R=\infty$ it gives the Schwarzschild metric in isotropic form. The pressure and density, obtainable from expressions given by McVittie, are functions of Λ, R, m and r, and we may not put $p\equiv 0$ without abolishing m.

Models such as (4.1) suggest that inhomogeneities could exist in an Einstein universe. This is the conclusion also from the work of Raychaudhuri (**16**), who found that there are exact static solutions of the field equations corresponding to an Einstein model containing spherical pockets of radiation of variable density. Indeed, as we shall see later, models can be constructed in which static, spherically symmetric regions of arbitrary density are embedded in an Einstein model.

In the approximate Einstein models which we shall consider, we suppose that the inhomogeneity, in which there may be motion, has spherical symmetry about the origin of coordinates, and lies within the sphere $r=a$ (constant). We assume that co-moving coordinates may be used so that the line-element has the form (3.2) inside $r=a$, and also that the pressure is isotropic. Outside $r=a$ the line-element is supposed to be that of the Einstein model (2.4), and across the boundary $r=a$, we suppose that the conditions (3.1) are satisfied with the obvious change that since the boundary is now $x_1=$ constant, the quantities which must be continuous are

$$g_{ik},\ \partial g_{\mu\nu}/\partial x_1,\ T_k^{\ i},\ g_{i\mu}T_k^{\ \mu}-g_{k\mu}T_i^{\ \mu}, \qquad (4.2)$$

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where $i, k = 1, 2, 3, 4$ and $\mu, \nu = 2, 3, 4$. Evidently the model may contain any number of non-overlapping spheres of this type, each corresponding to an inhomogeneity. This model is somewhat similar to one used by Lemaître to study condensations in the expanding universe.

The conditions (4.2) give the following restrictions on $\lambda(r, t)$, $\omega(r, t)$ and $\nu(r, t)$ in (3.2):

$$\left.\begin{aligned} &\lambda(a, t) = -\log\left(1 - \frac{a^2}{R^2}\right), \quad \omega(a, t) = 2\log a, \quad \nu(a, t) = 0, \\ &\frac{\partial\omega}{\partial r}(a, t) = \frac{2}{a}, \qquad\qquad \frac{\partial\nu}{\partial r}(a, t) = 0. \end{aligned}\right\} \tag{4.3}$$

The continuity of p, required by (4.2), is secured by (4.3).

Let us consider first the case of a purely static inhomogeneity. Then we may take (3.2) as

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\nu}dt^2,$$

where λ and ν are functions of r only. The field equations (3.3) give (Tolman (17))

$$8\pi T_1^{\,1} = -8\pi p = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda, \tag{4.4}$$

$$8\pi T_2^{\,2} = -8\pi p = -e^{-\lambda}\left(\frac{1}{2}\nu'' - \frac{1}{4}\nu'\lambda' + \frac{1}{4}\nu'^2 + \frac{\nu' - \lambda'}{2r}\right) - \Lambda, \tag{4.5}$$

$$8\pi T_4^{\,4} = 8\pi\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda, \tag{4.6}$$

where $'$ denotes d/dr. Equation (4.6) may be integrated to give

$$e^{-\lambda} = 1 - \frac{1}{3}\Lambda r^2 - \frac{8\pi}{r}g(r), \tag{4.7}$$

where

$$g(r) = \int^r \rho r^2 dr + K,$$

the density ρ being an arbitrary function of r, and the constant of integration K being chosen so that g satisfies the following, which is required by (4.3):

$$g(a) = \frac{a^3}{8\pi}\left(\frac{1}{R^2} - \frac{1}{3}\Lambda\right).$$

Equating the right-hand sides of (4.4) and (4.5), and substituting for λ from (4.7), we find a second-order differential equation for ν. It is clear that this admits a solution $\nu(r)$ containing two arbitrary constants which may be chosen to satisfy

$$\nu(a) = \nu'(a) = 0$$

without further restriction on $g(r)$. Hence for an inhomogeneity with an arbitrary density distribution there exists a solution satisfying the boundary conditions at $r = a$. *This leaves no doubt that static inhomogeneities can exist in the Einstein universe*, though they may not all be non-singular.

One further point about the static solutions is worth noticing. If p is constant for $r < a$, it follows from the identity

$$dp/dr = -\tfrac{1}{2}\nu'(p + \rho)$$

that $\nu' \equiv 0$ and the solution reduces to the homogeneous Einstein metric. In particular, in a model with $p \equiv 0$, static inhomogeneities cannot exist.

Turning to non-static inhomogeneities, we face a much more difficult problem because it is necessary to integrate the field equations obtained from (3.2).

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I shall not attempt the complete solution of this, but confine attention to the approximate solution which arises by supposing that the metric (3.2) inside the inhomogeneity differs only slightly from the Einstein metric (2.4).

First, let us suppose that ν, $\partial\lambda/\partial t$, $\partial\omega/\partial t$ and all their derivatives are small quantities of the first order. Then, neglecting small quantities of order higher than the first, the field equations for (3.2) (which are given in I) reduce to the following:

$$-8\pi T_1^{\ 1}=8\pi p=-e^{-\omega}+e^{-\lambda}(\tfrac{1}{2}\omega'\nu'+\tfrac{1}{4}\omega'^2)-\ddot{\omega}+\Lambda, \tag{4.8}$$

$$-8\pi T_2^{\ 2}=8\pi p=e^{-\lambda}(\tfrac{1}{2}\omega''+\tfrac{1}{4}\omega'^2+\tfrac{1}{2}\nu''-\tfrac{1}{4}\omega'\lambda'-\tfrac{1}{4}\nu'\lambda'+\tfrac{1}{4}\nu'\omega')-\tfrac{1}{2}(\dot{\lambda}+\ddot{\omega})+\Lambda, \tag{4.9}$$

$$-8\pi T_4^{\ 4}=-8\pi\rho=-e^{-\omega}+e^{-\lambda}(\omega''+\tfrac{3}{4}\omega'^2-\tfrac{1}{2}\omega'\lambda')+\Lambda, \tag{4.10}$$

$$-8\pi T_4^{\ 1}=0=-\dot{\omega}'-\tfrac{1}{2}\omega'(\dot{\omega}-\dot{\lambda}), \tag{4.11}$$

where $\cdot$ denotes $\partial/\partial t$, and $'$ denotes $\partial/\partial r$.

Integrating (4.11), we have

$$e^{\lambda}=e^{\omega}\omega'^2h(r), \tag{4.12}$$

where $h(r)$ is an arbitrary function, which for convenience we give the value

$$h(r)=\frac{1}{4}\left(1-\frac{r^2}{R^2}\right)^{-1}e^{k(r)} \tag{4.13}$$

where $k(r)$ is another arbitrary function.

We now suppose further that

$$e^{\omega}=r^2[1+\alpha(r,t)], \tag{4.14}$$

where α and its derivatives are small quantities of the first order. Then (4.12), (4.13) and (4.14) give, to first order,

$$e^{\lambda}=\left(1-\frac{r^2}{R^2}\right)^{-1}(1+\alpha+r\alpha'+k). \tag{4.15}$$

With (4.14) and (4.15), equations (4.8), (4.9) and (4.10) give after some calculation

$$-8\pi T_1^{\ 1}=8\pi p=\Lambda-\frac{1}{R^2}-\ddot{\alpha}+\frac{\alpha}{R^2}+\frac{\nu'}{r}\left(1-\frac{r^2}{R^2}\right)-\frac{k}{r^2}\left(1-\frac{r^2}{R^2}\right), \tag{4.16}$$

$$-8\pi T_2^{\ 2}=8\pi p=\Lambda-\frac{1}{R^2}-\ddot{\alpha}+\frac{\alpha}{R^2}+\frac{1}{2}\left(1-\frac{r^2}{R^2}\right)\left(\nu''+\frac{\nu'}{r}\right)+\frac{1}{2}\frac{r}{R^2}(\alpha'-\nu')$$
$$+\frac{k}{R^2}-\frac{1}{2}\left(1-\frac{r^2}{R^2}\right)\frac{k'}{r}-\frac{1}{2}r\ddot{\alpha}', \tag{4.17}$$

$$-8\pi T_4^{\ 4}=-8\pi\rho=\Lambda-\frac{3}{R^2}+\frac{3\alpha}{R^2}+\frac{r\alpha'}{R^2}-\frac{k}{r^2}\left(1-\frac{3r^2}{R^2}\right)-\frac{k'}{r}\left(1-\frac{r^2}{R^2}\right). \tag{4.18}$$

Equating the right-hand sides of (4.16) and (4.17), we have

$$\ddot{\alpha}'+\frac{\nu'}{r^2}-\frac{\nu''}{r}\left(1-\frac{r^2}{R^2}\right)-\frac{\alpha'}{R^2}-\frac{2k}{r^3}+\frac{k'}{r^2}\left(1-\frac{r^2}{R^2}\right)=0,$$

which gives, on integration,

$$\ddot{\alpha}-\frac{\alpha}{R^2}-\frac{\nu'}{r}\left(1-\frac{r^2}{R^2}\right)-\frac{\nu}{R^2}+\frac{k}{r^2}\left(1-\frac{r^2}{R^2}\right)=p(t). \tag{4.19}$$

The boundary conditions (4.3) require that α, ν and ν' vanish at $r=a$ for all t, and as the continuity of λ requires $k(a)=0$, we deduce that $p(t)$ in (4.19) must be identically zero. If we put

$$\beta(r,t)=\alpha-\frac{kR^2}{r^2}\left(1-\frac{r^2}{R^2}\right), \tag{4.20}$$

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(4.19) may then be written

$$\ddot{\beta} - \frac{\beta}{R^2} = \frac{\nu}{R^2} + \frac{\nu'}{r}\left(1 - \frac{r^2}{R^2}\right), \tag{4.21}$$

and we find from (4.16) and (4.18)

$$8\pi(p - p_0) = -\frac{\nu}{R^2}, \tag{4.22}$$

$$8\pi(\rho - \rho_0) = -\frac{3\beta}{R^2} - \frac{r\beta'}{R^2}, \tag{4.23}$$

where p_0, ρ_0 are the values of p and ρ in the homogeneous part of the model.

Equations (4.22) and (4.23) give the pressure and density inside the inhomogeneity in terms of two functions ν and β, which have to satisfy (4.21) and also the following conditions at $r = a$:

$$\nu(a, t) = \nu'(a, t) = 0, \tag{4.24}$$

$$\beta(a, t) = 0, \quad \beta'(a, t) = \frac{R^2}{a^2}\left(1 - \frac{a^2}{R^2}\right) k'(a) = \text{constant}. \tag{4.25}$$

In deriving (4.25) we remember that $k(a) = 0$, but $k'(a)$ is an arbitrary constant. The conditions (4.24) and (4.25) require

$$p(a, t) = p_0, \qquad p'(a, t) = 0, \tag{4.26}$$

$$\rho(a, t) - \rho_0 = \text{constant}.$$

To investigate the solution further, additional assumptions are necessary. Let us suppose that inside the inhomogeneity matter obeys a single-valued equation of state $p = f(\rho)$ (excluding, for the moment, $p = $ const. and $\rho = $ const.), that p and ρ have derivatives with respect to r of all orders for $r < a$ and left derivatives with respect to r of all orders at $r = a$, and that p, ρ and these derivatives are at least twice-differentiable functions of t. Using (4.22) and (4.23), we find from (4.21)

$$\ddot{\rho} - \frac{\rho - \rho_0}{R^2} = \frac{3(p - p_0)}{R^2} + \frac{p'}{r}\left(2 - \frac{3r^2}{R^2}\right) + p''\left(1 - \frac{r^2}{R^2}\right);$$

and from the equation of state we may write this as a partial differential equation for p in the form

$$p'' = F(r, p, p', \dot{p}, \ddot{p}), \tag{4.27}$$

where F is a known function. If in this we put $r = a$ and use (4.26) we find that p'' is constant at $r = a$; and differentiating (4.27) with respect to r, it then follows easily that p''' also is constant at $r = a$. In this way it may be shown that all the derivatives of p with respect to r are constant at $r = a$, so that p is not a function of the time immediately within $r = a$. Using the differentiability condition on p, we can use (4.27) to extend this result to any point inside $r = a$. Thus p does not depend on t for $r < a$, and it is easy to see that ν, ρ and β do not depend on t either. It can also be shown that if $\rho = $ const. the solution is not a function of t. *Thus given an equation of state (other than $p = $ constant) and appropriate differentiability conditions of p and ρ, the inhomogeneity must be static.* This result would be expected from Section 3, in which it was shown that in an Einstein model with an equation of state any disturbance must be universal.

If there is no equation of state, we may suppose that we are given $\nu(r, t)$ (i.e. the pressure) for $r < a$. In this case the right-hand side of (4.21) is known,

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and the equation may be integrated to give

$$\beta = A(r)e^{t/R} + B(r)e^{-t/R} + H(r, t). \tag{4.28}$$

where $A(r)$ and $B(r)$ are arbitrary subject to conditions at $r=a$, and $H(r, t)$ is a particular integral, known if ν is given. (This form of β applies also in the special case where $p=\text{const.}=p_0$, which gives $H(r, t)=0$.) The functions A and B would be determined if we were given appropriate initial conditions (e.g. $\rho(r, 0)$ and $\dot{\rho}(r, 0)$).

From (4.28) it is clear that β will increase or decrease exponentially with time except in special cases, such as, for example, that in which A and H vanish. This means that the metric (3.2) for $r<a$ will depart more and more from the Einstein state, and the assumptions of our method of approximation will eventually cease to apply. Before, however, we conclude that this means instability, we must examine a little more closely what is implied if β is an increasing or decreasing function of the time.

From the assumption that for $r>a$ the metric is that of the Einstein universe, it follows that the inhomogeneity is necessarily confined within $r=a$. The effect on stability of the motion inside $r=a$ is to be judged by the changes in the proper radius and proper volume of the inhomogeneity, which will affect the proper radius and proper volume of the whole model. If the proper radius and volume of the inhomogeneity tend to infinity with time, then the whole model is unstable because of expansion; if these quantities tend to zero, the whole universe contracts, though in this case we cannot be certain that this contraction will continue after the volume of the inhomogeneity becomes zero. If, however, the change in proper radius and proper volume of the inhomogeneity is zero, or remains small, then there is no reason why the motion inside $r=a$ should cause instability.

The proper radius l and the proper volume v of the inhomogeneity are, to the first order of approximation,

$$l = \int_0^a e^{\lambda/2}\,dr = \int_0^a \left(1 - \frac{r^2}{R^2}\right)^{-1/2} (1 + \tfrac{1}{2}\beta + \tfrac{1}{2}r\beta')\,dr, \tag{4.29}$$

$$v = \int_0^a \int_0^{\pi} \int_0^{2\pi} e^{\lambda/2+\omega} \sin\theta\,d\phi\,d\theta\,dr = 4\pi \int_0^a r^2 \left(1 - \frac{r^2}{R^2}\right)^{-1/2} (1 + \tfrac{3}{2}\beta + \tfrac{1}{2}r\beta')\,dr. \tag{4.30}$$

In these expressions $k(r)$ does not appear because the terms which involve it may be integrated, and then vanish on the assumption that $k(a)=0$ (already made), and that $k(r)$ tends to zero with r more rapidly than r. It may be stated here that the physical significance of $k(r)$ is connected with the spatial geometry of the model for $r<a$. When the other unknown functions, corresponding to the density and pressure, are determined, this geometry is not completely fixed, being unspecified to the extent of the arbitrary function $k(r)$. The arbitrariness disappears if appropriate initial conditions are given, as we shall see later in an example.

The contributions to the integrals (4.29) and (4.30) which could make l and v change rapidly with time are those involving, through β and β', $A(r)e^{t/R}$ and $H(r, t)$. In general, whatever the form of H, the dependence on $Ae^{t/R}$ will make l and v change rapidly and cause instability. However, in certain special cases A and H may be such that l and v remain nearly constant in time, even though at certain points in $r<a$, β may become large. For example, this will be so if

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$H\equiv 0$ (which implies $p=p_0$) and A satisfies

$$\int_0^a \left(1-\frac{r^2}{R^2}\right)^{-1/2}(A+rA')\,dr=0, \qquad \int_0^a \left(1-\frac{r^2}{R^2}\right)^{-1/2} r^2(3A+rA')\,dr=0,$$

with $A \not\equiv 0$. In other words, the inhomogeneity may remain roughly the same size, though the density (which depends on β) changes considerably at some points in it. Thus condensation, or rarefaction, may begin without causing the whole model to contract or expand, though this will be so only for these special cases, and even in these we cannot be sure (since our method of approximation breaks down when β becomes large) that the integrals will not eventually begin to change rapidly, thus making the model unstable.

The solution, which is given essentially by (4.20) and (4.28), depends on a function $\nu(r, t)$, whose prescription amounts to the specification of the pressure $p(r, t)$; on the functions $A(r)$, $B(r)$ which may be considered as determined by the specification of ρ and $\partial\rho/\partial t$ as functions of r at $t=0$; and also on $k(r)$, determined by the metric at $t=0$. We have not so far supposed the model to have originated in any particular way: we may indeed regard it simply as a refinement of the Einstein model which represents, in so far as it is stable, a universe static and homogeneous on a large scale, but containing local irregularities and motions. On the other hand, we may ask whether a model of this type could have originated from the strict Einstein universe by a perturbation of the kind described at the end of Section 3. The result of this enquiry, to which we now turn, may be expected to restrict the arbitrariness of the functions A, B and k.

In addition to the conditions (4.3) at $r=a$, we must now impose the following at $t=0$ (which result from (3.1)):

$$\left.\begin{aligned} &\lambda(r, 0)=-\log\left(1-\frac{r^2}{R^2}\right), && \dot\lambda(r, 0)=0,\\ &\omega(r, 0)=2\log r, && \dot\omega(r, 0)=0,\\ &\nu(r, 0)=0. \end{aligned}\right\} \tag{4.31}$$

Equations (4.3), (4.14), (4.15), (4.20) and (4.31) together give

$$\left.\begin{aligned} &\alpha(r, 0)=\dot\alpha(r, 0)=0,\\ &k(r)=0,\\ &\alpha(a, t)=\alpha'(a, t)=0,\\ &\nu(a, t)=\nu'(a, t)=0,\\ &\beta(r, t)=\alpha(r, t). \end{aligned}\right\} \tag{4.32}$$

Using (4.28) for β, and putting $t=0$, we see that from (4.32)

$$\left.\begin{aligned} &\beta(r, 0)=A(r)+B(r)+H(r, 0)=0,\\ &\dot\beta(r, 0)=R^{-1}[A(r)-B(r)]+\dot H(r, 0)=0, \end{aligned}\right\} \tag{4.33}$$

which determine A and B since $H(r, t)$ may be supposed known as $\nu(r, t)$ is given. In addition, (4.32) requires that

$$\left.\begin{aligned} &A(a)e^{t/R}+B(a)e^{-t/R}+H(a, t)=0,\\ &A'(a)e^{t/R}+B'(a)e^{-t/R}+H'(a, t)=0. \end{aligned}\right\} \tag{4.34}$$

It is easy to see that a function $\nu(r, t)$ may be chosen so that the problem has a solution. For example, if we take

$$\nu=r^2(r-a)^3q(r)e^{-\alpha t},$$

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where α is a positive constant and $q(r)$ is a polynomial, we get a non-singular solution satisfying all the conditions (4.33) and (4.34). The solution corresponds to a disturbance in the Einstein model which takes place inside $r=a$, and in which the pressure begins to change from the equilibrium value at $t=0$. Although the pressure difference dies away eventually and may be arbitrarily small at all times, it causes instability in the general case because (4.33) requires $A(r)\not\equiv 0$ so that the integrals (4.29) and (4.30) will eventually change rapidly, causing the whole model to contract or expand. However, in special cases this model can correspond to condensation or rarefaction inside $r=a$ without initial instability because the constants in $q(r)$ can evidently be chosen so that l and v do not change rapidly with time, even though $A(r)\not\equiv 0$.

I shall now summarize the results of this section. We have seen that exact static solutions of the field equations exist corresponding to an Einstein universe with inhomogeneities. If there is an equation of state (other than $p=\text{const.}$) no non-static approximate models with inhomogeneities are possible, but if there is no such equation, non-static models exist and are in general unstable, though there are special cases in which motion inside $r=a$ can occur without instability, including some in which condensations may begin to form. If a non-static approximate model results at $t=0$ from some change in the time derivatives of p inside a region $r=a$ of the strict Einstein model, its future is completely determined given $p(r, t)$ for $t>0$, $r<a$, and in general it will be unstable no matter how small is the pressure difference in the homogeneous and inhomogeneous parts of the model; but in special cases, condensations can begin to form without instability.

5. *Conclusion.*—From Section 4 it seems that we are justified in regarding the homogeneity of the strict Einstein model (2.4) as a smoothed out representation of a static universe with inhomogeneities (except in the case of a completely pressure-free model). Its static character must, however, be interpreted more strictly because it appears that, except in particular cases, systematic motions are accompanied by instability. Speaking roughly, we may say that the Einstein universe is not *essentially* homogeneous, but it *is* static*, except for the special motions mentioned in Section 4, and also, possibly, for rotational motions, which have been excluded from this work by the assumption of spherical symmetry, and which may not be accompanied by instability.

If we accept this view of the Einstein universe, it is clear that instability will result from any process which introduces motion, subject to the exceptions referred to above. Except in the case $p\equiv 0$, such a process will not follow from the mere presence of condensations of matter, because static solutions corresponding to them exist. From Section 3 it appears that the processes concerned are those which involve discontinuities in the pressure or its derivatives, since in the absence of these no instantaneous disturbance can occur. However, if there is an equation of state after the disturbance, the only permitted discontinuity of the type we have considered is a universal discontinuity in the pressure itself, and this is not likely to occur. (If it did, a Friedmann model would result.) Thus a model with an equation of state would probably be stable because no change whatever could take place in it.

* I think it would be wrong to conclude from this that molecular motions contributing to the pressure are themselves sufficient to cause instability. In all ordinary cosmological investigations, including this one, p and ρ are macroscypic quantities, and the theory is not intended to apply at the molecular level.

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In a model without an equation of state, instability could arise through a gradual pressure change (i.e. a discontinuity in the time derivatives of p at $t=0$) inside a limited region; such a disturbance,e ven if small, will in general lead to condensation or rarefaction, and will cause instability through an expansion or contraction of the region in which it occurs. However, in special cases, the disturbance could die out or lead to condensation or rarefaction without immediate instability.

A universal decrease in the pressure in an Einstein model (if such could occur) would initiate an expansion, and an increase would start a contraction. For a gradual pressure change in a limited region, whether there is an expansion or contraction depends on the sign of dl/dt and dv/dt, which in turn depends on the pressure distribution, and the initial distributions of ρ and $\partial\rho/\partial t$.

If the Einstein universe contained evolving stars, changes of the pressure of radiation would presumably be sufficient to cause instability. If, however, one pictures it as filled with uniform gas at low temperatures (Bondi, **18**), or as a static mixture of gas and condensed, non-radiating matter, this work suggests that it would be stable.

As the boundary conditions (3.1) are still to some extent tentative, it is worth while emphasizing how convincingly they seem to work in the problems dealt with here. All the functions which they leave arbitrary have a reasonable physical interpretation, and the data required to determine the solutions are very much what one would expect on classical grounds. This has certainly not been true of sets of boundary conditions used in some of the previous work on this subject.

Department of Applied Mathematics,
The University,
Liverpool:
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