ON THE THEORY OF WHITE DWARF STARS

I. The Energy Sources of White Dwarfs

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Summary

Present theories of the origin of white dwarfs are discussed; it is shown that all theories imply that there can be no effective energy sources present in a white dwarf at the time of its birth. The temperature distribution of a white dwarf is then discussed on the assumption that no energy liberation occurs within the star, and that it radiates at the expense of the thermal energy of the heavy particles present. In the resulting picture, a white dwarf consists of a degenerate core containing the bulk of the mass, surrounded by a thin, non-degenerate envelope. The energy flow in the core is due to the large conductivity of the degenerate electrons, while the high opacity of the outer layer keeps down the luminosity to a low level. Estimates of the ages of observed white dwarfs are given and interpreted. Finally, it is shown that white dwarfs may accrete energy sources and yet continue to cool off, provided the temperature at the time of accretion is not too high; this suggests a possible model for Sirius B.

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1. *The origin of white dwarfs.*—The problem of the structure of white dwarfs was solved essentially by Fowler (**1**). He showed that under white dwarf conditions, the electron gas within these stars is "degenerate", and exerts a huge pressure due to its zero-point "exclusion" energy. The dominant term in the pressure formula depends strongly on the density, but is independent of the temperature, so the star is able to withstand its own gravitation without possessing a far higher temperature than the observed luminosities would warrant. The theory determines the radius of a homogeneous degenerate sphere in terms of its mass and the parameter μ_e, the number of electrons per proton mass.

Stoner and Anderson, and later Chandrasekhar (**2**), modified the theory by using the relativistic Hamiltonian in the Fermi–Dirac partition function, and so deduced a more complicated pressure–density relation. The new mass–radius relation predicts smaller radii for stars of mass below $5{\cdot}75\, M_\odot/\mu_e^2$ (Chandrasekhar's limit), while for masses above this limit no static spherically symmetric states exist. As all observed white dwarfs are below the limit, the existence of the limit is not serious for them, but the result has important evolutionary consequences.

In order to understand how white dwarfs can come about, it is convenient to consider the behaviour of a mass of gas, obeying the usual pressure law, as it contracts under its own gravitation. The condition of hydrostatic support demands a central temperature varying roughly as the inverse of the radius (**3**), and, once the material has become opaque to radiation, the rate of contraction will be determined by the rate at which energy flows down the temperature gradient from the centre to the surface. This is the Kelvin–Helmholtz mechanism of stellar energy generation. If the mass contains nuclear energy sources, however, the contraction is halted when the internal temperatures are sufficiently high for the energy liberated to balance the surface loss. (The possibility that rotational instability may set in before this stage is reached is neglected.)

Within stars containing hydrogen and the merest trace of carbon and nitrogen, the Bethe cycle liberates large quantities of energy per second at temperatures above 10^7 deg. K. There is also the direct synthesis of helium from hydrogen via the proton–proton reaction, which, if quantum-mechanically "allowed", is about as powerful as the Bethe cycle at 2×10^7 deg. K. The rate of energy liberation by both processes increases rapidly with temperature. It at once follows that a quantity of gas of the same mass as the observed white dwarfs cannot condense to a degenerate state while it contains energy sources; long before degeneracy is reached, the central temperature will be high enough for rapid energy liberation, and so the star takes up the normal equilibrium state. Thus any theory of the origin of white dwarfs must postulate the existence of stellar masses containing no energy sources throughout the bulk. As all observations of interstellar matter confirm that only the minutest fraction of it differs from hydrogen, the direct condensation of hydrogen-free material into a degenerate state can be ruled out.

The first possibility that presents itself is that a white dwarf is the result of complete hydrogen-exhaustion in a main-sequence star of the same mass. Once all the energy sources have been exhausted, the star again contracts according to the Kelvin–Helmholtz theory. As the central density ρ_c increases as $1/R^3$ while the central temperature T_c only as $1/R$, the criterion for degeneracy (**4**), $10^8\rho \gg T^{3/2}$, must ultimately be fulfilled, and the star becomes a white dwarf. However, this line of evolution has against it one insuperable difficulty. Except for the white dwarfs, and a few sub-giants (discussed in the following paper), all stars of mass comparable with or less than the Sun's are in the very early stages of their evolution (**5**), as is shown by the very small scatter about the lower end of the main sequence in the Hertzsprung–Russell diagram (**6**). It is quite unsatisfactory to imagine that the initial and final stages in the evolution of small stars are present in the heavens today, yet without any of the intermediate stages. We are driven to conclude that all the main-sequence dwarf stars are still in their youth, and that the white dwarfs, sub-giants and perhaps some sub-dwarfs have arisen through indirect lines of evolution. The upper limit thus found for the galactic age agrees well with estimates from independent evidence, such as the age of the Earth, the rate of disintegration of galactic clusters, etc. (**5**).

For stars rather more massive than the Sun, this objection disappears because of the strong dependence of luminosity on mass: a star of mass $5M_\odot$, say, can consume its hydrogen within about 10^9 years, well within the galactic lifetime. Such a mass, however, exceeds Chandrasekhar's limit for hydrogen-free material and so cannot attain equilibrium even when degeneracy has set in. The subsequent behaviour has been described in detail by Hoyle (**7, 8**). As the contraction proceeds, the conservation of the star's angular momentum causes its angular velocity to increase, and sooner or later rotational instability sets in.

For stars with low angular momentum, this state will not be reached before densities and temperatures are so high that heavy elements are synthesized within the highly collapsed star. When enough material has been expelled by rotational instability for the remaining mass to be below Chandrasekhar's limit, the star can settle down into a white dwarf state. The expulsion of matter can be either spasmodic or catastrophic, according to the density when instability sets in; these two cases are identified with novae and supernovae respectively. Observational support for the theory comes from the work of Baade and Minkowski on the Crab Nebula (**9, 10**).

This provides a mechanism by which a hydrogen-free star of solar mass can be produced within the galactic lifetime. It is to be noted that the theory predicts the presence in some of these white dwarfs of considerable quantities of elements heavier than helium, owing to the synthesis of the heavier elements from the helium in the highly collapsed state reached by stars with low angular momentum.

A rather different theory was put forward by Eddington in 1939 (**11**). He was impressed by the fact that the observations of Sirius B seemed to demand a high hydrogen content for the star in order to account for its large radius. In order that the Bethe cycle should not start up during the contraction of a mass of gas, rich in hydrogen, it is necessary that carbon and nitrogen be less abundant than one part in 10^{16}; as there is no reason for other heavy elements to be present in large quantities if carbon and nitrogen are absent, Eddington concluded that the observations were very slightly in error, and that the true hydrogen content is 100 per cent.

A further necessary hypothesis is that Fermi and not Gamow–Teller selection rules apply to the proton–proton reaction, which is therefore "forbidden"; otherwise, this reaction will liberate enough energy per second when the central temperature is about 10^7 deg. K to bring the contraction to a halt. If this is the case, then "Eddington" white dwarfs may be expected to condense, provided that for some period of the galactic lifetime the interstellar gas was effectively pure hydrogen. The relevance of Eddington's theory depends therefore on the original composition of the galaxy. Hoyle (**12**) has discussed the early evolution of the galaxy on the alternative assumptions (*a*) that the galaxy originally consisted of pure hydrogen and that the Fermi selection rules apply to the proton–proton reaction, (*b*) the same with Gamow–Teller rules, (*c*) that the primeval galactic matter was, as today, mainly hydrogen but with small quantities of other elements present. He shows that on each hypothesis the theory can account for the existence of white dwarfs, nova and supernova outbursts, and the subsequent distribution of heavy elements in space. The result of interest for the present work is that case (*a*) predicts the existence of "Eddington" white dwarfs, which should condense in large numbers early on in the galactic lifetime, before substantial quantities of heavy elements have been distributed in space by supernova processes and so adulterated the original hydrogen.

It cannot be claimed that we have as yet sufficient evidence to decide between the alternatives. The problem of the constitution of the primeval galactic matter depends partly on the particular cosmology adopted; for example, the steady-state theory of the expanding universe, involving continuous creation, implies that galaxies condense from partially adulterated hydrogen, and so again rules out "Eddington" white dwarfs.

The problem is a real one for cosmogony, in that the number of white dwarfs observed is very large, in spite of their faintness, and the indications are that their number is only less than that of the red dwarfs. But the proportion of massive stars in the solar neighbourhood is very much less, and the number of novae and supernovae per millennium observed in our own and neighbouring galaxies does not appear sufficient to account for the high proportion of white dwarfs. Thus if "Eddington" white dwarfs are ruled out for one reason or another, we are forced to assume that *all* white dwarfs arise through the gravitational collapse of burnt-out massive stars. This implies that earlier in the galactic lifetime there existed a much higher proportion of massive stars, at least in the solar neighbourhood.

One further possibility might appear open; this is that a small mass of gas could contract to a degenerate state, without the internal temperature rising high enough for energy liberation to start. The mean temperature T of a quasi-static sphere of gas, of mass M, radius R, and mean molecular weight μ must (**2**) be greater than $G\mu M/5\mathscr{R}R$, where G is the gravitational constant and $\mathscr{R}$ the gas constant. In order that the Bethe cycle should not start up, this temperature must not rise beyond 10^7 deg. K. Degeneracy has set in when $10^8\bar{\rho}\sim 10\bar{T}^{3/2}$, where $\bar{\rho}$ is the mean density $M/\frac{4}{3}\pi R^3$. Together, these values give about 2×10^{32} g for the upper limit to the mass, which is about one-tenth of the solar mass. The time taken for such a mass to contract to a degenerate state is determined by the rate of outflow of energy from the centre. The opacity of a small dense star is given by Kramers' law, giving for the luminosity $4\times 10^{33}(M/M_\odot)^{5\cdot 5}/(R/R_\odot)^{1/2}$. (The conductivity of the non-degenerate electron gas is negligible (**4**).) The energy (thermal and gravitational) of a star is of order $-GM^2/R$ (**2**). The resulting time-scale for contraction to a degenerate state is therefore about 10^{12} years. Hence, even if such a small mass could subsequently accrete enough matter to become a white dwarf of observable size, it seems unlikely that the galactic time-scale will allow this theory.

2. *The thermal properties of white dwarfs.*—One fact stands out from the above discussion; all the suggested mechanisms for the production of white dwarfs demand that such stars are effectively without energy sources at the time when they become degenerate. Thus, during their subsequent life one would expect them to radiate energy without replenishment. In spite of this, previous treatments of the temperature distribution within white dwarfs (**13, 14**) have assumed that the star is in a state of thermal equilibrium, with energy liberation balancing the surface loss. One reason for this is perhaps the high hydrogen content which the observations demanded for Sirius B. This hydrogen, however, must have been accreted by the star after its birth as a white dwarf; this is discussed in detail in Section 4 and in the following paper. In any case, it is more logical to discuss the star as a cooling system, as it must begin its life as such, and then to investigate the effect of subsequent accretion.

Many of the published treatments of the problem are misleading in that they seem to imply that there *must* be energy liberation within a white dwarf in order that it may shine at all. Thus, having built a heterogeneous white dwarf model in thermal equilibrium, Schatzman assumes that all white dwarfs must exist in this state—he refers in one paper (**15**) to "l'état final hétérogène existant, comme je l'ai montré, dans les naines blanches". But in fact neither a white dwarf nor a normal star needs nuclear energy sources to make it shine. The Kelvin–Helmholtz theory for a normal star is thermodynamically perfectly sound, and predicts a luminosity fixed (at least to order of magnitude) by the mass, composition and central temperature of the star; it is only the time-scale difficulty which makes us look for nuclear energy sources that will keep the star in a state of thermal equilibrium. No similar time-scale difficulty arises for a white dwarf, as it can supply its very low luminosity from its internal heat for a period comparable with 10^{10} years. It will be shown that, as for normal stars, the luminosity of a white dwarf is fixed by its mass, composition and internal temperature.

A normal star, in which the ordinary gas laws apply, and which possesses nuclear energy sources, always tends to approach a state in which the energy generation balances the surface loss. For by the virial theorem (**2**)

$$3(\gamma-1)U+\Omega=0, \tag{1}$$

where U is the total thermal energy and Ω the total gravitational energy. The total energy E is $U+\Omega$, whence

$$E=-(3\gamma-4)U=(3\gamma-4)\Omega/3(\gamma-1). \tag{2}$$

Hence, provided $\gamma>\frac{4}{3}$, as is the case for highly ionized matter within stars, the thermal energy decreases as the total energy increases, and vice versa. Thus, if, say, the energy liberation exceeds the surface loss, the resulting increase in energy leads to an expansion and to a decrease in temperature throughout the body of the star. Since the energy liberation from nuclear sources is highly sensitive to temperature variation, after a small decrease in temperature thermal equilibrium will hold. Similar considerations apply if the star is radiating more energy than the sources supply—a slight contraction increases the temperature to secure thermal balance. A detailed treatment, taking account of the variation with temperature of energy outflow as well as of energy liberation has been given by Jeans (**16**).

Thus normal stars have a "safety valve" mechanism which prevents secular changes in the state of the star. White dwarfs have no such mechanism. The pressure of a degenerate gas depends almost entirely on the density, and is almost independent of the temperature. This means that a white dwarf whose energy sources are insufficient to supply its surface radiation cannot restore the balance by contracting; contraction is prevented by the strong pressure, independent of the temperature, of the degenerate electrons. Whereas a normal star heats up as it loses energy, a white dwarf cools down, and its energy sources become still more inadequate to provide its radiation. Conversely, a white dwarf whose energy generation exceeds its surface radiation cannot restore the balance by expanding and cooling; it heats up, continually generating more and more energy, until it ceases to exist as a white dwarf. Thus even if a white dwarf has, subsequent to its birth, acquired energy sources, *there is no reason to equate its energy liberation to its radiation.* At one internal temperature only does thermal equilibrium for a particular star hold, but at this temperature the star is "balancing on a razor's edge"; as soon as any departure from equilibrium occurs, there is no automatic adjustment, and the star departs further and further from this unique state.

A permanent white dwarf must therefore be considered as supplying much of its energy-emission from its internal heat. It is convenient first to consider a model free of energy sources, and then discuss the effect of accretion.

3. *The internal temperature of a white dwarf.*—All save the outer fringe of a white dwarf consists of matter in a highly degenerate state (2). The degree of degeneracy is measured by the value of the parameter, λ, defined (4) by

$$\frac{\rho h^3}{8\pi\mu_e m_H(2mkT)^{3/2}} = \int_0^\infty \frac{x^2\,dx}{(1+e^x/\lambda)}, \tag{3}$$

where h is Planck's constant, k Boltzmann's constant, and m, m_H the electron and proton masses respectively. The effects of degeneracy begin to be felt when $\lambda \sim 1$, and $\rho/T^{3/2} \sim 10^{-8}$. When $\lambda \sim 300$, $\rho/T^{3/2} \sim 10^{-7}$; the pressure law $K\rho^{5/3}$ then holds very accurately. Thus degeneracy may be said to be "complete" after $\lambda = 300$. Bars will be used to denote values at this point.

Radiative transfer of energy is negligible compared with thermal conduction in the highly degenerate core, in spite of the reduced opacity of a degenerate gas. It will be shown that the thermal conduction is so great that, during the cooling of a white dwarf, the temperature is always very nearly uniform in the core.

The thermal conductivity of degenerate matter is very nearly $\alpha\rho T$, where $\alpha = 2{\cdot}44 \times 10^3\, A/Z^2$ c.g.s. units, and Z and A are the mean atomic number and weight respectively of the heavy particles (4). As the energy of a degenerate gas depends on the temperature to the second order only, the electronic contribution to the heat content is small, as in the theory of metals. Hence the specific heat is essentially that of the non-degenerate nuclei; it can be taken as $\frac{3}{2}k/Am_H$, where A is the mean atomic weight of the nuclei. The equation of cooling thus becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\alpha\rho T r^2 \frac{\partial T}{\partial r}\right) = \tfrac{3}{2}k\frac{\rho}{Am_H}\frac{\partial T}{\partial t}, \tag{4}$$

where r is the distance from the centre, t the time.

The density distribution does not vary appreciably as the star cools; it is given in the non-relativistic approximation by the polytropic function for index $n = \frac{3}{2}$, or in the relativistic case by the Chandrasekhar function corresponding to the mass (2). To simplify the problem, we shall use the polytropic function for ρ; it is easy to verify that the conclusions are substantially the same if Chandrasekhar functions are used.

It is impossible to find a general solution for the non-linear equation (4). However, we are not interested in a general solution; we are interested in the form to which the solution approximates after a time sufficiently long for irregularities in the original temperature distribution to be smoothed out.

Integrating equation (4) with respect to r,

$$\alpha\rho T r^2 \frac{\partial T}{\partial r} = \int_0^r \frac{3}{2}\frac{k\rho}{Am_H}\frac{\partial T}{\partial t} r^2\,dr. \tag{5}$$

Thus if $\dot{T}_m$ is the maximum value of $|\partial T/\partial t|$,

$$-\alpha\rho T r^2 \frac{\partial T}{\partial r} \leqslant \frac{3}{2}\frac{k\dot{T}_m}{Am_H}\int_0^r \rho r^2\,dr. \tag{6}$$

Integrating again

$$T_c - T \leqslant \frac{3}{2}\frac{k\dot{T}_m}{\alpha Am_H}\int_0^r \frac{1}{\rho r^2 T}\left\{\int_0^r \rho r^2\,dr\right\}dr. \tag{7}$$

Since $1/T$ steadily increases with r, this gives

$$T_c - T \leqslant \frac{3}{2}\frac{k\dot{T}_m}{\alpha Am_H T}\int_0^r \left(\int_0^r \rho r^2\,dr\right)\frac{dr}{\rho r^2}. \tag{8}$$

The integral can be evaluated in terms of polytropic functions; substitute

$$r = \epsilon z, \quad \rho = \rho_c u^{3/2}, \tag{9}$$

where ρ_c is the central value and

$$\epsilon = \left(\frac{5}{2} \cdot \frac{K}{4\pi G} \cdot \rho_c^{-1/3} \right)^{1/2}. \tag{10}$$

Here K is the constant of proportionality in the non-relativistic pressure law $K\rho^{5/3}$; from statistical mechanics (4), its value is known to be $3 \cdot 2 \times (2/\mu_e)^{2/3} \times 10^{12}$ c.g.s. units, where μ_e is as previously defined. The variable u satisfies Emden's equation for index $n = 3/2$, and can be found from British Association Tables. In terms of u and z,

$$\int_0^r \left(\int_0^r \rho r^2 \, dr \right) \frac{dr}{\rho r^2} = 2\epsilon^2 \{ u^{-1/2} - 1 \}. \tag{11}$$

Hence

$$T_c - T \leqslant \frac{3k\dot{T}_m \epsilon^2}{\alpha A m_H T} (u^{-1/2} - 1). \tag{12}$$

The order of magnitude of $\dot{T}_m$ can be inferred from the observed luminosity L. The mass of the outer fringe being negligibly small, its contribution to the cooling may be disregarded. Thus L arises from the cooling of the degenerate part; its value is given by

$$L = - \int_0^{\bar{r}} \frac{3}{2} \frac{k\rho}{A m_H} \frac{\partial T}{\partial t} 4\pi r^2 \, dr, \tag{13}$$

where $\bar{r}$ corresponds to the boundary of the degenerate matter. Thus if $\dot{T}_0$ is a mean value of $-\partial T/\partial t$,

$$\begin{aligned} L &= \frac{3}{2} \frac{k\dot{T}_0}{A m_H} \int_0^{\bar{r}} 4\pi \rho r^2 \, dr \\ &= \frac{3}{2} \frac{k\dot{T}_0}{A m_H} M, \end{aligned} \tag{14}$$

where M is the total mass. Using this equation in conjunction with (12),

$$T_c - T \leqslant \frac{2L\dot{T}_m \epsilon^2}{\alpha T M \dot{T}_0} (u^{-1/2} - 1). \tag{15}$$

The maximum value of $T_c - T$ comes at the boundary $r = \bar{r}$ of degeneracy, where we can put $T = \bar{T}$, $u = \bar{u}$. Here

$$T_c - \bar{T} \leqslant \frac{2L\dot{T}_m \epsilon^2}{\alpha \bar{T} M \dot{T}_0} (\bar{u}^{-1/2} - 1). \tag{16}$$

As is shown in Section 4, the temperature at the onset of degeneracy in white dwarfs of observed luminosities is about 10^7 deg. K. The corresponding density $\bar{\rho}$ is given (4) by $10^8 \bar{\rho} / \mu_e \bar{T}^{3/2} = 8 \cdot 615$, and so $\bar{u}$ can be found. Thus taking $M = \frac{1}{2} M_\odot$, $L = L_\odot / 400$ as typical white dwarf values, and assuming that $\dot{T}_m / \dot{T}_0$ is not much greater than unity, it is found that

$$T_c - \bar{T} \leqslant 0 \cdot 75 \cdot \frac{Z^2}{A} \left\{ \frac{1}{(\bar{\rho})^{1/3}} - \frac{1}{(\rho_c)^{1/3}} \right\} \cdot 10^7. \tag{17}$$

Hence, taking the mean atomic weight to be about 10, the relative temperature drop in the core is not more than about $\frac{1}{8}$ and this drop decreases rapidly inwards. Hence the core is nearly uniform in temperature. (If $\dot{T}_m / \dot{T}_0$ is much greater than unity, this simply means that near the centre of the star, where $|\partial T/\partial t|$ is greatest in magnitude, the temperature is rapidly approaching equality with that farther out.)

4. *The rate of cooling of a white dwarf.*—The thermal energy of the heavy particles in the bulk of a white dwarf is given by

$$\frac{3}{2} \cdot \frac{kT}{Am_H} \cdot M, \tag{18}$$

where T is the temperature of the core. The rate of cooling is therefore given by

$$L = -\frac{3}{2} \frac{\mathscr{R} M}{A} \cdot \frac{dT}{dt}. \tag{19}$$

The dependence of L on T is found by integrating the equations to the non-degenerate outer regions. Before $\lambda = 1$, conduction is found to be negligible, and the equations of support and energy flow are

$$\frac{\mathscr{R}}{\mu(T)} \frac{d}{dr} (\rho T) = -\frac{GM\rho}{r^2}, \tag{20}$$

$$\tfrac{4}{3} ac \frac{T^3}{\rho} \frac{dT}{dr} = -\frac{\kappa_0 \rho}{T^{3.5}} \cdot \frac{1}{f(T)} \cdot \frac{L}{4\pi r^2}. \tag{21}$$

Here $f(T)$ is the guillotine factor, and $\mu(T)$ the effective molecular weight, both functions of the degree of ionization of the material. Dividing (20) by (21), and integrating

$$(\rho T)^2 = \frac{32\pi acGM}{3\kappa_0 L\mathscr{R}} \int_0^T T^{7.5} \mu(T) f(T)\, dT. \tag{22}$$

Both $\mu(T)$ and $f(T)$ are slowly varying functions of T (**2**), and so they may be taken from under the integral sign and replaced by their values μ_1, f_1 at the strong maximum of $T^{7.5}$. Then (22) reduces to

$$\frac{\rho}{T^{3/2}} = T^{7/4} \left(\frac{32\pi acGM\mu_1 f_1}{25.5\kappa_0 L\mathscr{R}} \right)^{1/2}. \tag{23}$$

This solution holds until degeneracy comes into play, i.e. up to $\lambda \sim 1$. Then

$$L = kT_1^{7/2}, \tag{24}$$

where k is constant, and T_1 is the temperature at $\lambda = 1$.

If this solution is continued until $\lambda = 300$, the temperature increases only by a factor of about 2, and this is a gross overestimate, as conduction dominates after $\lambda \sim 10$. Hence the final equation to the cooling of a white dwarf is

$$-\frac{d}{dt} \left(\frac{3}{2} \frac{\mathscr{R} M}{A} T \right) = kT^{7/2}, \tag{25}$$

where T is the internal temperature, given by (23). Numerically, taking the guillotine factor to be 10, as estimated by Chandrasekhar (**2**), T is given by

$$T = \left(\frac{L}{M} \right)^{2/7} \times 10^8 \text{ deg. K.} \tag{26}$$

Hence the final model for a white dwarf is as follows: a degenerate, nearly isothermal core, containing the bulk of the mass, is surrounded by a non-degenerate radiative envelope with a high temperature gradient. The luminosity of the star, given by (24), is determined by the high opacity of the envelope, which acts as a blanket limiting the rate of cooling.

These results hold as long as most of the star is degenerate and the mass in the radiative envelope is small. It will be shown that this is the case over most of the star's lifetime. The difference between the temperature at the onset of degeneracy ($\lambda = 1$) and that of the core will alter estimates of age by at most a factor 2, whereas only factors of 10 or more are of interest.

Equation (25) integrates immediately to give:

$$0{\cdot}6\,\frac{\mathscr{R}M}{A}\cdot\frac{1}{kT^{5/2}} = t + \text{const.} \tag{27}$$

The time for the star to cool down from T_1 to T_2, with corresponding luminosities L_1, L_2, is therefore:

$$0{\cdot}6\,\frac{\mathscr{R}M}{A}\left(\frac{T_2}{L_2} - \frac{T_1}{L_1}\right). \tag{28}$$

If T_1 and T_2 differ by a factor only slightly different from 1, the term T_1/L_1 is negligible. From this, it is easy to estimate the time the star will continue to have a luminosity above a certain level, or emit light of a temperature above a given minimum. By inserting present values of T_2 and L_2, the time that the star has been degenerate is found.

For Sirius B and 40 Eridani B, the only two stars for which we have reliable data for both mass and luminosity, the results show that the times taken to reduce the present luminosity by a magnitude are about $10^{10}/2A$ years and $10^{10}/3A$ years respectively, while Sirius B will have the same colour as the Sun today in about $2{\cdot}5\times 10^{10}/A$ years. By then, however, the luminosity will have dropped by a factor 20, and the star will be very difficult to observe.

Large numbers of other white dwarfs have been observed, notably by Kuiper, but while their magnitudes and effective temperatures are fairly accurately known, the masses are seldom easily determinable. However, from L and T_e, R can be estimated, and M then read off from Chandrasekhar's graph (**2**). The values of M so found should be accurate enough for rough estimates of the ages of the stars to be made.

Schatzman (**17**) gives a table of visual magnitudes; these must be corrected by means of Kuiper's bolometric table (**18**). The following table gives the visual magnitude, internal temperature and age of a representative selection of white dwarfs. The internal temperature depends on the composition of the star, in that the opacity of "Russell mixture" is about six times as great as that of hydrogen; thus the internal temperature of an "Eddington" white dwarf should be slightly lower than one of the same mass consisting of helium and heavy elements. However, this error has only a small effect on the age of a star, which is influenced much more by the factor A in the last column.

TABLE I

Star	Vis. Mag.	Int. Temp. (deg. K)	Age (years)
Sirius B	11·4	2×10^7	$3\times10^9/A$
40 Eridani B	11·1	3×10^7	$2\times10^9/A$
Van Maanen 2	14·3	6×10^6	$10^{11}/A$
Wolf 1346	9·8	3×10^7	$10^8/A$
Wolf 457	15·1	10^7	$10^{11}/A$
Wolf 489	15·1	$8{\cdot}5\times10^6$	$10^{11}/A$
2 of Hyades	11·3	3×10^7	$10^9/A$
Ross 627	13·6	2×10^7	$10^{10}/A$

The computed ages are really the times that the stars have been in white dwarf states; however, it can be shown that the time interval between the onset of degeneracy at the centre of the star and the appearance of degeneracy throughout most of the star is very short compared with the time taken by the star to cool. For the above analysis will hold as long as most of the star is degenerate, so that the mass in the radiative zone makes a small contribution to the cooling. If the mass of the degenerate core at any time be M_1, then from the equation of support for the outer zone we have

$$\rho_1 T_1 = \frac{G\mu}{4\pi\mathscr{R}} \int_{M_1}^{M} \frac{M\,dM}{r^4} > \frac{G\mu}{4\pi\mathscr{R}} \cdot \frac{1}{R^4} \cdot \frac{(M^2 - M_1^2)}{2}. \tag{29}$$

The suffix 1 refers to conditions at the onset of degeneracy, and so $10^8 \rho_1 \sim T_1^{3/2}$; also, the radius of the star must be less than the radius of the star at the time of onset of degeneracy at the centre of the star. Hence (29) gives a lower limit to the temperature of the star when a certain fraction of its mass is degenerate. Assuming a star of solar mass to contract homologously, it is found that degeneracy sets in when $T_c = 2 \times 10^8$ deg. K, with the star's radius then about 3×10^9 cm. The available energy of the star in this state is $\sim 2\mathscr{R} M T_c / A$ (a factor of 2 being introduced to take account of the gravitational energy that will be released during the further contraction). With a temperature at the edge of the core given by (29), the temperature gradient in the outer zone will be of order

$$\frac{dT}{dr} = -\frac{1}{R} \left\{ \frac{G\mu}{4\pi\mathscr{R}} \cdot \frac{1}{R^4} \cdot (M^2 - M_1^2) \times 10^8 \right\}^{2/5} \tag{30}$$

with a corresponding energy flow

$$-\frac{4}{3} \frac{ac}{\kappa_0} \cdot \frac{T_1^{13/2}}{\rho_1^2} \cdot \frac{dT}{dr} \cdot 4\pi R^2. \tag{31}$$

When the mass of the core is about three-quarters the total mass, T (core) is about 10^8 deg. K, and the consequent flux at least 10^{35} erg/s. Therefore the time for the star to become degenerate throughout most of its mass is at most 2×10^7 years, and this is a huge exaggeration, as the luminosity is higher at higher temperatures. Thus it is justifiable to assert that the star spends most of its lifetime in a degenerate state, and that the ages in Table I give the time since the star reached white dwarf dimensions.

The tabulated lifetimes of the stars are satisfactory for the cooling theory. The results show that white dwarfs exist of all ages up to the age of the galaxy; this is to be expected, for the processes which give birth to white dwarfs—novae and supernovae—should occur at all times from 10^9 years after the birth of the galaxy (7). But, in addition to this, it is seen that some of the tabulated stars have calculated ages longer than the age of the galaxy if the material present does not contain substantial quantities of heavy elements. Helium is not sufficient to reduce the star's calculated age satisfactorily. In this way the theory links up well with that of the synthesis of elements within collapsed massive stars.

Finally, it is to be noted that stars in the neighbourhood of Ross 627 in the Hertzsprung–Russell diagram have ages of about $10^{10}/A$ years. This would fit in with the Eddington theory, which requires $A = 1$, and the age to be not very different from the galactic age (*cf.* Section 1). If the Eddington theory is not accepted, then Ross 627 must be the core of a nova or supernova, which occurred about 10^9 years ago.

5. *The Hertzsprung–Russell diagram.*—As a white dwarf cools, its radius will not alter by a sensible factor and so the surface temperatures decrease according to

$$L \propto T_e^4. \tag{32}$$

Thus the path of a white dwarf in the Hertzsprung–Russell diagram is given by

$$\text{Mag.} = -10 \ln T_e + c, \tag{33}$$

the constant c depending on mass and composition. On this theory, therefore, there is no question of arranging the stars in a regular sequence depending on mass and composition, as with normal stars in thermal equilibrium. The position of a white dwarf of given mass and composition depends on its temperature, and hence on its age. The radius of a white dwarf which has not accreted any hydrogen is a function of the mass alone, and a star of given mass describes, as it cools off, an individual straight line in the Hertzsprung–Russell diagram. The greater the mass, the further to the left in the diagram is its cooling path. Thus the irregular spread of the white dwarfs over their part of the diagram is no reason for surprise.

6. *A white dwarf model with energy sources.*—The next problem to be tackled is the effect of accretion on a white dwarf. A general discussion is given in the following paper; this section deals simply with accretion by a star cool enough for energy liberation in the accreted material to be less than the radiation from the star.

The author was led originally to investigate this model by the well-known discrepancy in the radius of Sirius B. From its orbital motion relative to Sirius A, this star is known to have a mass nearly equal to the Sun's. The radius was first determined by the effective-temperature method; later measurements of the Einstein shift were claimed to support not only the order of magnitude of the previously found M/R, but also its precise value. Errors of a factor 2 were declared very unlikely. A homogeneous star of this radius demands a hydrogen content of about 50 per cent if the number of electrons per unit mass is to be high enough to provide the necessary pressure. But if the opacity is due to heavy elements, the temperature gradient in the radiative envelope is high and leads to an internal temperature of about 2×10^7 deg. K, with a consequent energy liberation far above the surface loss.

A possible way out of the difficulty is suggested as follows. As discussed in Section 1, the hydrogen in Sirius B must have been accreted, and will therefore tend to lie on the surface. Hydrogen is less opaque than "Russell mixture" by a factor 6, and so a lower temperature gradient is required to derive the observed luminosity. It will be shown that a composite model can be built, consisting of a hydrogen-free core ($\mu_e = 2$) surrounded by a hydrogen layer, such that the internal temperatures are low enough for energy liberation to be less than the surface loss. A necessary condition is that the proton–proton reaction be forbidden.

The first task is to estimate the mass of the hydrogen layer in order that the observed and theoretical radii should agree.

The pressure of a relativistically degenerate gas is given (2) by

$$p = Af(x); \; \rho = Bx^3, \tag{34}$$

where

$$A = 6{\cdot}01 \times 10^{22} \text{ c.g.s. units; } B = 9{\cdot}82 \times 10^5 \mu_e \text{ c.g.s. units} \tag{35}$$

and

$$f(x) = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x. \tag{36}$$

Chandrasekhar reduces the equation of hydrostatic support

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{37}$$

to

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\phi}{d\eta}\right) = -\left(\phi^2 - \frac{1}{y_0^2}\right)^{3/2} \tag{38}$$

by making the following substitutions:

$$\left.\begin{aligned} y^2 &= x^2 + 1, \\ r &= \left(\frac{2A}{\pi G}\right)^{1/2} \cdot \frac{1}{By_0} \cdot \eta, \\ y &= y_0\phi, \\ y_0^2 &= x_0^2 + 1. \end{aligned}\right\} \tag{39}$$

The suffix 0 refers to conditions at the centre. In this notation, the mass interior to η is given by

$$M(\eta) = 4\pi\left(\frac{2A}{\pi G}\right)^{3/2} \cdot \frac{1}{B^2} \cdot (-\eta^2\phi'). \tag{40}$$

Chandrasekhar has tabulated, for different values of y_0, a set of solutions of (38), which start at the origin. A homogeneous star of given mass and composition has a density distribution given by one of these solutions continued outwards until its first zero, thus fixing the boundary of the star. In a non-homogeneous model, the distribution within the core will again be given by a particular one of Chandrasekhar's solutions of (38) but, after the discontinuity in μ_e, further integrations must be performed. To simplify the problem, the gravitational effect of the hydrogen in the hydrostatic equation will be neglected; this is justified by the final result that the mass of the hydrogen is well below that of the whole star.

Then

$$\frac{dp}{dr} = -\frac{G\rho\overline{M}}{r^2} \tag{41}$$

holds in the hydrogen zone; $\overline{M}$ is the mass of the core. Substituting from (34–36) this equation integrates to

$$(1 + x^2)^{1/2} - 1 = 2{\cdot}66 \times 10^8 \cdot \frac{\overline{M}}{M} \cdot \left(\frac{1}{r} - \frac{1}{R}\right), \tag{42}$$

where the condition of the vanishing of ρ at the boundary R has been used.

In equilibrium the pressure, and hence the value of x, must be continuous across the interface between the two zones. As $\mu_e = 2$ in the core and 1 in the layer, this condition leads to a drop by a factor 2 in the density and density gradient, thus distending the radius.

Let $\bar{\eta}$ be the radial parameter at the interface. Then $\bar{r}$ is given by (39) and $\overline{M}$ by (40). To find $\overline{M}$ and y_0 a trial and error method is used. Assuming a particular value of $\bar{M}$, $(\overline{-\eta^2\phi'})$ is found from the mass relation (40). If then a particular y_0 is assumed, Chandrasekhar's tables supply $\bar{\eta}$ and ϕ, whence x, $\bar{\rho}$, and r are found with the help of (34) and (39). The correct y_0 for the assumed value of $\bar{M}$ is fixed by the relation (42) between r and x. As it is unlikely that the correct y_0 will be among Chandrasekhar's tabulated set, it is necessary to interpolate between the tables to find the appropriate y_0.

Thus to each assumed value of $\bar{M}$, there can be found a unique set of parameters. The final test, to fix the correct $\bar{M}$, and so determine the problem, is the equality of $(M-\bar{M})$ and

$$4\pi R^3 \int_{\bar{r}/R}^{1} \rho z^2 \, dz. \tag{43}$$

The function ρ is known from (34) and (42), and this integral can be computed in each case.

An accurate set of results (within the limit of the approximation of neglecting the hydrogen mass in (41)) is

$$\bar{M}=0{\cdot}73M, \qquad \bar{r}=0{\cdot}36R, \qquad \bar{\rho}(\text{core})=8{\cdot}3\times10^5 \text{ g/cm}^3 \tag{44}$$

(using $M=1{\cdot}95\times10^{33}$ g, $R=1{\cdot}36\times10^9$ cm).

Interstellar matter consists of hydrogen with small quantities of other elements intermingled; Dunham (**19**) gives a proportion of one part in 10^4, and certainly 1 per cent is an overestimate. Such small quantities will not affect either the opacity or the conductivity, but any carbon and nitrogen present will start up the Bethe cycle at sufficiently high temperatures. It is therefore important to estimate an upper limit to the temperature at the interface between the core and the hydrogen layer.

The temperature at the point $\lambda=1$ is given by (28) of Section 3; the factor K_0 is one-sixth of that for "Russell mixture", while the guillotine factor may be put equal to 1, as pressure ionization sets in early. The value found is $T=2{\cdot}91\times10^6$ deg. K.

Between $\lambda=1$ and $\lambda=300$, account must be taken of incipient degeneracy and thermal conduction. The equations are

$$\frac{d}{dr}(p_a+p_e)=-\frac{GM\rho}{r^2}, \tag{45}$$

$$\left(\nu+\frac{4}{3}\frac{ac}{K}\frac{T^3}{\rho}\right)\frac{dT}{dr}=-\frac{L}{4\pi r^2}, \tag{46}$$

where ν is the thermal conductivity, K the opacity and p_a and p_e the nuclear and electron pressures respectively. The equations must be integrated numerically, using previously tabulated values of p_e/p_a and ν (**4**), and Marshak's expression for the opacity of incipiently degenerate matter (**13**). Between $\lambda=1$ and $\lambda=10$, conduction dominates throughout the star. The temperature at $\lambda=300$ is found to be at most $6{\cdot}9\times10^6$ deg. K.

An upper limit to the temperature at the interface is found by neglecting the cooling of the hydrogen and integrating through the degenerate hydrogen the equation

$$\frac{d}{dr}\left(r^2\alpha\rho T\frac{dT}{dr}\right)=0, \tag{47}$$

using for ρ the function given by (34) and (42). The resulting maximum temperature is $8{\cdot}4 \times 10^6$ deg. K.

Energy will be liberated by both the Bethe cycle and the proton–proton reaction. On Bethe and Critchfield's theory (**20**), the proton–proton reaction, if allowed, liberates 2 erg g^{-1} s^{-1} at the centre of the Sun, the conditions assumed being $\rho = 80$ g/cm^3, $T = 2 \times 10^7$ deg. K, and hydrogen 35 per cent by mass. The energy liberation varies as ρT^4. From this, a minimum for the energy liberation by the proton–proton reaction in Sirius B can be calculated, and the result exceeds the surface loss by a factor of about 100. Hence this model demands that Fermi and not Gamow–Teller selection rules apply to the proton–proton reaction, in which case the energy liberation is reduced by at least 10^4. The evidence from nuclear theory is indecisive as to what interaction to assume, and the reaction is far too slow to be measured at terrestrial temperatures. Bondi (**21**) suggests that the shape of the lower end of the main sequence in the $L - T_e$ diagram is best explained by assuming the reaction to be allowed, but before this can be decided with certainty, further detailed integrations of stellar models of small mass are needed, taking into account incipient degeneracy and thermal conduction.

The Bethe cycle generates 100 erg g^{-1} s^{-1} at the Sun's centre, assuming the previous data, and a 10 per cent nitrogen concentration (**20**). The reaction rate varies roughly as ρT^{18}. Taking 1 per cent to be the proportion of nitrogen present in the accreted material, the maximum amount of energy produced per second in the zone is only $\frac{1}{1000}$ of the luminosity. Hence on the previously stated assumption, the model satisfies the theoretical mass–radius relation and is stable.

After this work was done a report appeared (**20**, p. 293) that the generally accepted radius for Sirius B is wrong, and that the correct value of the Einstein shift is such as to give a radius requiring almost zero hydrogen content. (The hydrogen lines visible on the surfaces of Sirius B and 40 Eridani B require only a surface layer of negligible mass to account for them.) This would mean, of course, that the effective temperature of the star is wrong by a factor $\sqrt{2}$, in spite of what the spectroscopists have been reporting during the last thirty years. Other workers are of the opinion that the probable errors in such delicate spectroscopic measurements are too great for reliance on the values of the radii determined from the Einstein shift. The question, therefore, is still open, and it seems worth while publishing the details of the above model in order to illustrate what happens to a cool white dwarf which accretes hydrogen, even if the model should turn out to have no relevance to Sirius B.

The models considered by Schatzman are similar to this one in structure, but he assumes that the temperature is always at the right level for the energy liberation in the hydrogen to balance the surface loss. The objections to this assumption have already been stated. That Schatzman's calculated luminosities agree with the observed magnitudes is no reason for surprise; for as the energy liberation varies as T^{18}, while the surface loss only as $T^{7/2}$, a slightly higher temperature would be sufficient for the energy liberation to balance the luminosity, without the luminosity being increased very much.

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