EQUATIONS OF MOTION FOR AN IDEAL PLASMA

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ABSTRACT

Equations are presented for the motion of a completely ionized gas, or plasma, in a magnetic field sufficiently strong so that during a single gyration around a Larmor circle the relative changes in all macroscopic quantities, such as the electrical and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, the particle density, n, and the kinetic temperature, T, are very small. The ratio of the radius of curvature, a, of a Larmor circle to the mean free path, λ, is also assumed small. Successive approximations are developed in increasing powers of these small quantities.

Since, in the plane perpendicular to $\boldsymbol{B}$, an electrical current arises primarily from a pressure gradient, while an electrical field produces a uniform velocity of the plasma, the current density and the electrical field have no direct relationship. Hence the electrical resistivity is most conveniently defined in terms of the energy dissipated into heat by an electrical current. So defined, the transverse resistivity is 1.96 times the normal resistivity for currents parallel to the magnetic field. While no steady current parallel to $\boldsymbol{E}$ is directly produced by a transverse electrical field, a change in $\boldsymbol{E}$ produces a polarization of the plasma; in interstellar space the transverse dielectric constant is about 10^8. The Hall current, perpendicular to both $\boldsymbol{E}$ and $\boldsymbol{B}$, appears only in the third approximation and is generally negligible.

Approximate equations for the transverse viscosity and heat conductivity are given. Both of these effects are very small.

The motion of a plasma, or ionized gas, in crossed electric and magnetic fields, has been studied by a number of workers, most recently by A. Schlüter[1] and T. G. Cowling [2] This problem is obviously one of great astrophysical interest, since accumulating evidence indicates that electromagnetic effects may be important in the interstellar plasma, as well as in stellar atmospheres. It does not seem to have been generally realized that under certain idealized conditions, which are fulfilled in many astrophysical situations, the motion of a plasma and the electrical currents present may be found from relatively simple equations, which are presented in this paper.

The basic conditions that we shall assume here are threefold: (a) the plasma is completely ionized; (b) the macroscopic density and velocity change only very slightly during the time of one gyration or over the distance of one radius of gyration; (c) the mean free path between collisions is very great, compared to the radius of gyration. A plasma which satisfies these three conditions will be called an "ideal plasma." Condition a is in some ways the most restrictive of the three; this assumption is obviously fulfilled in interstellar H II regions and in the solar corona, although not in H I interstellar clouds or in the solar photosphere.

Condition b is likely to be satisfied in most situations, unless strong plasma oscillations of short wave length are present. In a field of magnetic intensity B, a proton will complete a gyration in about $6\ 6 \times 10^{-4}/B$ seconds. In interstellar space, where a magnetic field of 10^{-5} gauss seems a reasonable[3, 4] value, about a minute is required for a single gyration. Electrons gyrate several thousand times as rapidly. In the sun, where a magnetic field of much more than 10^{-5} gauss seems likely, even more rapid gyration may be assumed. Apart from rapid oscillations, which require separate treatment, no appreciable change in density, velocity, or electric current will occur in so short a time. Similarly,

[1] *Zs. f. Naturforsch.*, **5a**, 72, 1950; **6a**, 73, 1951.

[2] Lectures at Princeton University during the spring of 1951. I am much indebted to Dr. Cowling for a number of stimulating discussions on these problems.

[3] G. K. Batchelor, *Proc. R. Soc. London, A*, **201**, 405, 1950.

[4] L. Biermann and A. Schlüter, *Phys. Rev.*, **82**, 863, 1951.

changes over a distance equal to the radius of gyration—about 10^7 cm for protons at 10,000° in a field of 10^{-5} gauss—should also be small.

Condition *c* is also likely to be satisfied in most situations. In an interstellar H II region, for example, the mean free path of an electron or proton[5] is about 10^{12} cm, some 10^5 times the radius of gyration assumed for these regions. In the solar corona the mean free path is less by a factor of about 10^5, but the radius of gyration will also be decreased, and the ratio of mean free path to radius of gyration should still be large. We may conclude, then, that the ideal plasma considered here does correspond in many cases to actual plasmas in astrophysics.

The system of approximation used for an ideal plasma is described in Section I. Successive approximations are then developed in succeeding sections. Collisions between particles, which are neglected in Sections II, III, and IV, are considered in Sections V and VI. Attention is given chiefly to the equations describing the material motion of the plasma. The actual motion of a plasma in a particular case will also depend on the variation of the electromagnetic field, which depends, in turn, on the currents in the plasma. Variations of density, resulting from divergence of the velocity, must also be taken into account. It is planned to consider such effects in a subsequent paper.

I. SYSTEM OF APPROXIMATION

The motions of particles in an ideal plasma are simplest when the magnetic field, $\boldsymbol{B}$, is constant in space and time, and no external force $\boldsymbol{F}$ is present. Let $\boldsymbol{w}_\perp$ and $w_\parallel$ denote the components of the particle velocity perpendicular and parallel, respectively, to $\boldsymbol{B}$. Then, as is well known, in a plane perpendicular to $\boldsymbol{B}$ the particle gyrates with angular frequency ω in a Larmor circle of radius a, where

$$\omega = \frac{w_\perp}{a} = \frac{|Z|\, eB}{m\,c}. \tag{1}$$

In this equation B and $w_\perp$ denote the scalar magnitudes of the vectors $\boldsymbol{B}$ and $\boldsymbol{w}_\perp$; c is the velocity of light, since e is measured in electrostatic units. In the direction of $\boldsymbol{B}$ the particle moves with the constant velocity $w_\parallel$. Thus the "guiding center" about which the particle gyrates moves with constant velocity along a line of force. In general, we shall denote the velocity of this guiding center by $\boldsymbol{V}$.

In any actual plasma the conditions experienced by a charged particle will change during the course of a single gyration. However, in accordance with condition *b*, such changes will be small in an ideal plasma, and hence such quantities as $dX/X\omega dt$ will be small, where X stands for such physical quantities as the magnetic field $\boldsymbol{B}$, the potential energy, the particle density, etc. We may therefore expand the motion of the particles in an infinite series of such quantities, with the expectation that such a series will converge rapidly. Similarly, the ratio of the radius of gyration, a, to the mean free path, λ, may be assumed small, and we may expand the actual motion in an infinite series in a/λ.

We thus arrive at a twofold system of approximation. In general, we shall let the (n, m) approximation denote the approximation in which terms of the order $(dX/X\omega dt)^n$ and $(a/\lambda)^m$ are considered. Actually, these two expansion parameters do not appear explicitly in the subsequent analysis, but it is readily verified that the ratio of terms in successive approximations is of the order of these expansion parameters.

The properties of the two lowest approximations are relatively obvious. On the basic (0, 0) approximation, already discussed above, the particles simply spiral about the

[5] L. Spitzer, Jr., *Ap. J.*, **93**, 369, 1941.

rectilinear lines of force. In this approximation we must also assume that the particle density is uniform; the distribution of particle velocities is entirely arbitrary but must be the same in all regions. Uniform motion parallel to $\boldsymbol{B}$ may be included in this approximation.

In the (0, 1) approximation, collisions are introduced in a uniform medium. The only effect is to produce a Maxwellian distribution of velocities. If $f^{(0)}(\boldsymbol{w})$ is the density of particles in phase space in this approximation, we have

$$f^{(0)}(\boldsymbol{w}) = \frac{n l^3}{\pi^{3/2}} e^{-l^2 w^2}, \tag{2}$$

where

$$l^2 = \frac{m}{2kT}, \tag{3}$$

n is the particle density, w the total velocity, and T the so-called "kinetic temperature." Equation (2) is given for the frame of reference in which the particles have no systematic velocity.

In the remainder of the paper, approximations as high as (3, 0) and (3, 1) will be considered.

II. MOTIONS IN THE (1, 0) APPROXIMATION

We consider the (1, 0) approximation for particles all of the same mass m and charge Ze, subject to an external force $\boldsymbol{F}$. As before, we denote by $\boldsymbol{w}$ the velocity of a single particle. The mean velocity of all particles in a cubic centimeter will be denoted by $\boldsymbol{v}$. While we shall be concerned primarily with $\boldsymbol{v}$, the macroscopic velocity, the results for the (1, 0) approximation will be derived from consideration of the microscopic velocity, $\boldsymbol{w}$.

If we now neglect collisions, consider only the first derivatives of B, and assume other forces constant, the motion of a charged particle is given by the usual "first-order theory," whose results may be taken from H. Alfvén.[6] Here we shall use electrostatic units for charge and field strength, but electromagnetic units for current and resistivity. In the present notation we have

$$V_{\perp} = -\frac{c\boldsymbol{B}\times\boldsymbol{F}}{Z eB^2} + \frac{m c}{Z eB^3}\left\{\tfrac{1}{2}w_{\perp}^2 \boldsymbol{B}\times\nabla B + w_{\parallel}^2 \boldsymbol{B}\times\left(\boldsymbol{B}\cdot\nabla\frac{\boldsymbol{B}}{B}\right)\right\}, \tag{4}$$

where $V_{\perp}$ represents the two components of $\boldsymbol{V}$ perpendicular to $\boldsymbol{B}$. The first term in braces represents the drift due to the change of B in a direction perpendicular to the lines of force. The second term represents the drift due to the centrifugal force of a particle moving along a line of force whose radius of curvature is R; $\boldsymbol{B}/B$ is the unit vector, $\boldsymbol{q}$, along the line of force, and $\boldsymbol{q}\cdot\nabla\boldsymbol{q}$ equals $\boldsymbol{R}/R^2$. It may be noted that, if no currents are present, so that $\nabla\times\boldsymbol{B}$ vanishes, and if also $\boldsymbol{B}\cdot\nabla B$ vanishes (B constant along a line of force), then

$$\boldsymbol{B}\cdot\nabla\frac{\boldsymbol{B}}{B} = \nabla B. \tag{5}$$

If equation (5) holds, equation (4) can be considerably simplified, and for an isotropic velocity distribution the drift velocities given by the two terms in braces in equation (4) are exactly equal, on the average.

For $V_{\parallel}$ we have the equation

$$\frac{dV_{\parallel}}{dt} = \frac{dw_{\parallel}}{dt} = \frac{\boldsymbol{F}\cdot\boldsymbol{B}}{mB} - \frac{w_{\perp}^2 \boldsymbol{B}\cdot\nabla B}{2B^2}. \tag{6}$$

[6] *Cosmical Electrodynamics* (Oxford: Clarendon Press, 1950), chap. ii.

Equation (6), together with the energy integral, yields the result

$$\frac{d}{dt}\left(\frac{w_\perp^2}{B}\right)=0\,, \tag{7}$$

an equation which is also valid in the first-order theory when B changes with time at each point.

We now consider the macroscopic velocity $\boldsymbol{v}$. One might expect that $\boldsymbol{v}$, the average velocity of all the particles in a small volume, would equal $\overline{\boldsymbol{V}}$, the average velocity of all the guiding centers in the same volume. Actually, this is not the case. There are three effects which must be considered: (1) electrical fields, (2) inhomogeneity of the magnetic field, (3) inhomogeneity of the density distribution. Only the first of these produces the same effect on both $\overline{\boldsymbol{V}}$ and $\boldsymbol{v}$. The second affects $\overline{\boldsymbol{V}}$ but has no direct effect on $\boldsymbol{v}$, while the third affects $\boldsymbol{v}$ but not $\overline{\boldsymbol{V}}$. We consider each of these effects in turn.

The electrical field $\boldsymbol{E}$ produces, as we have seen, a drift velocity V_T and an acceleration $dV_\parallel/dt$. In a homogeneous magnetic field, with a uniform density distribution, all particles will experience the same drift velocity and the same acceleration, and it is obvious that the mean velocity $\boldsymbol{v}$ will in this case equal the average velocity of the guiding centers. Since we are here concerned with the first-order theory, the presence of magnetic inhomogeneities and density gradients will not affect the conclusion that an electrical field produces the same effect on $\boldsymbol{v}$ as it does on $\overline{\boldsymbol{V}}$.

Next we consider the velocity produced by inhomogeneities of the magnetic field. Taking the first term in braces in equation (4), we let $\boldsymbol{B}$ be parallel to the z-axis and let ∇B be directed along the x-axis. Let $\boldsymbol{v}$ be the mean velocity of all the particles in a cube L cm on a side, and let L be large compared to a. We shall compute the difference between $\boldsymbol{v}$ and $\overline{\boldsymbol{V}}$, defined as the mean velocity of all the guiding centers which lie within this volume. If these two averages referred to the same group of particles, $\boldsymbol{v}$ and $\overline{\boldsymbol{V}}$ would be equal. However, some of the particles included in the average for $\boldsymbol{v}$ are not included in the average for $\overline{\boldsymbol{V}}$ and vice versa, and we shall show that the difference exactly cancels any dependence of $\boldsymbol{v}$ on dB/dx.

We denote by the symbol I all particles which lie within the cube we are considering; let there be N such particles. Of these particles, some will have guiding centers outside the cube; we denote this group, a subgroup of I, by the symbol II. There will also be particles which lie outside the cube but whose guiding centers lie inside the cube; we denote these particles by III. Since the mean velocity of each group of particles is the mean velocity of the guiding centers for the group, we have

$$N\overline{\boldsymbol{V}}=\sum_{I}\boldsymbol{w}-\sum_{II}\boldsymbol{w}+\sum_{III}\boldsymbol{w}\,, \tag{8}$$

since the number of particles in II equals that in III, to the first order. Hence we have

$$\boldsymbol{v}=\overline{\boldsymbol{V}}+\frac{1}{N}\Big(\sum_{II}\boldsymbol{w}-\sum_{III}\boldsymbol{w}\Big). \tag{9}$$

In the computation of $\Sigma\boldsymbol{w}$ for groups II and III we need not consider whether particles lie inside or outside of the cube in the y- or z-direction, since conditions in these directions are assumed uniform. As shown in Figure 1, we consider guiding centers at a point P, a distance x from the left-hand side of the cube, such that a fraction θ_1/π of the particles concerned lie outside the cube and are hence members of group III. We shall consider, first, all particles which have particular velocity components $w_\perp$ and

$w_\parallel$. Of the guiding centers at P, the mean velocity in the y-direction of those in group III is evidently given by

$$\bar{w}_y = \frac{1}{\theta_1} \int_0^{\theta_1} w_\perp \cos \theta \, d\theta = \frac{w_\perp \sin \theta_1}{\theta_1} . \tag{10}$$

Since nL^2dx is the number of guiding centers whose distance from the left-hand side of the cube lies between x and $x + dx$, we immediately have

$$\sum_{III} w_y = \int_0^a nL^2 \, dx \, \frac{\theta_1}{\pi} \frac{w_\perp \sin \theta_1}{\theta_1} . \qquad 11)$$

The quantity θ_1 is evidently $\cos^{-1} x/a$, and a simple integration yields

$$\sum_{III} w_y = \tfrac{1}{4} nL^2 a w_\perp . \tag{12}$$

The component Σw_x for group III vanishes by symmetry.

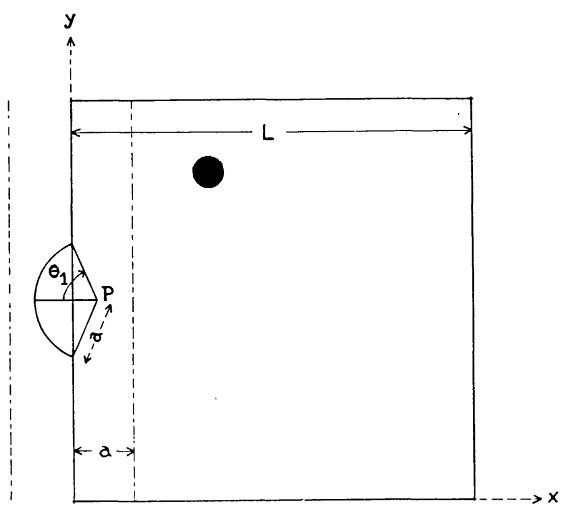

FIG. 1

Consideration of Σw_y for group II indicates that this is equal in magnitude to the corresponding sum for group III, but opposite in sign. The corresponding sums at the right-hand side of the cube in Figure 1 are also given by equation (12), except that now the signs are reversed and a differs by the amount $L da/dx$. Since N also equals nL^3, we find that, for particles with a particular $w_\perp$,

$$\frac{1}{N} \left(\sum_{II} w_y - \sum_{III} w_y \right) = \tfrac{1}{2} w_\perp \frac{da}{dx} . \tag{13}$$

On substituting equation (1) in this expression, we obtain the same expression as is found from equation (4) for $\bar{V}_y$ in the present case, but with the opposite sign. Hence, from equation (9), we see that these particles give no contribution to the mean velocity v_y. Since this result is independent of any particular value of w_y, we conclude that in this case

$$v_\perp = 0 . \tag{14}$$

Similarly, when the second term in braces in equation (4) is considered and a curving magnetic field is assumed, equation (14) is again valid, provided that the particle density and temperature are again uniform. In this case our basic volume is not a cube but a cylinder, whose cross-section is bounded by two circular arcs parallel to the magnetic field, as shown in Figure 2. In this case the difference between the correction terms on the left- and right-hand sides of the figure arises from the different lengths of these two sides, and again cancels the mean value of $\boldsymbol{V}$.

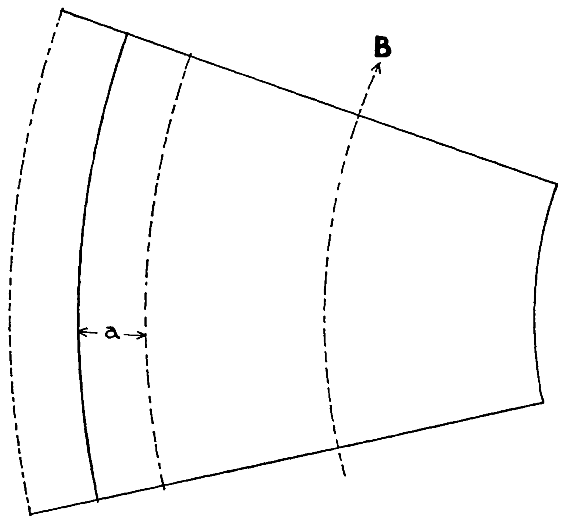

Fig. 2

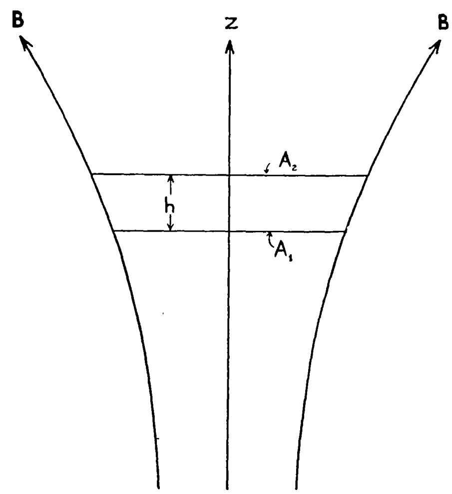

Fig. 3

In addition, the acceleration given in equation (7) cancels out when $dv_{\|}/dt$ is computed. We consider a slab perpendicular to the mean direction of $\boldsymbol{B}$, which is taken to be the direction of the z-axis. The slab is bounded top and bottom by parallel planes and on the side by the lines of magnetic force, as shown in Figure 3. Let the areas of the two faces of the slab be A_1 and A_2, respectively, and let h be the thickness of the slab. We define $\boldsymbol{P}$ to be the change per second of the momentum in the slab. Evidently, if the particle density is uniform,

$$P_z = nAhm\,\frac{dw_z}{dt} + nA_1 m w_{1z}^2 - nA_2 m w_{2z}^2\,, \tag{15}$$

averaged over all particles in the slab or crossing the surfaces; A is the average of A_1 and A_2. If the mean square velocities are assumed the same at faces A_1 and A_2 (constant pressure), if $dw_\parallel/dt$ is obtained from equation (6) (with $\boldsymbol{F}$ set equal to zero), and if $A_2 - A_1$ is determined from the fact that AB is constant, we obtain

$$P_z = \frac{nAhm}{B}\,\frac{dB}{dz}\left(-\tfrac{1}{2}\overline{w_\perp^2} + \overline{w^2}\right), \tag{16}$$

since w_z is $w_\parallel$ in the present case. If the velocity distribution is isotropic, P_z vanishes, and hence, in this case,

$$\frac{dv_\parallel}{dt} = 0\,. \tag{17}$$

Thus we see that in this first-order theory the particle velocities and accelerations induced by the inhomogeneity of the magnetic field have no counterpart in the macroscopic velocity $\boldsymbol{v}$, provided that the density and velocity distribution are constant in space and the velocity distribution is isotropic. From general considerations it may be seen that this result must be true to all orders, since in an inclosure at thermodynamic equilibrium no macroscopic velocities relative to the inclosure can appear. This result has been emphasized by Cowling.[2, 7]

Next we turn to a consideration of the pressure gradient, which, unlike the inhomogeneity in the magnetic field, may have a large effect on $\boldsymbol{v}$. Let us consider, first, a gradient of $n\overline{w_\perp^2}$ in the x-direction, with B assumed uniform and in the z-direction. In this (1, 0) approximation we consider a linear gradient of $nw_\perp^2$. We consider the same cube of length L on a side considered in Figure 1, and determine $\boldsymbol{v}$ from equation (9) as before, although in the present case $\overline{V}$ vanishes. Equation (12) still gives Σw_y for group III and, with a change in sign, for group II also. When the contributions from the left- and right-hand sides of the cube are combined, taking into account the pressure gradient, we have, averaging over all particles present,

$$v_y = \frac{1}{2n}\,\frac{d}{dx}\left(n\,\overline{aw_\perp}\right). \tag{18}$$

More generally, we have

$$\boldsymbol{v}_\perp = \frac{mc}{2ZeB^2 n}\,\boldsymbol{B}\times\nabla\left(n\overline{w_\perp^2}\right). \tag{19}$$

In the presence of a pressure gradient with a component parallel to $\boldsymbol{B}$, we have the familiar result,

$$\frac{dv_\parallel}{dt} = -\frac{\boldsymbol{B}\cdot\nabla\left(n\overline{w_\parallel^2}\right)}{Bn}\,. \tag{20}$$

The above results may be combined to give final equations determining $v_\perp$ and $v_\parallel$. From equations (4) and (19) we have

$$\boldsymbol{v}_\perp = \frac{c}{ZeB^2}\,\boldsymbol{B}\times\left[-\boldsymbol{F} + \frac{1}{n}\nabla p\right], \tag{21}$$

[7] *M.N.*, **90**, 140, 1929; **92**, 407, 1932.

where we have assumed that the pressure distribution is isotropic and that therefore

$$p = \tfrac{1}{2} n m \overline{w_\perp^2} . \tag{22}$$

It may be noted that, if isothermal conditions are assumed and if $\boldsymbol{F}$ is purely electrostatic, then

$$\boldsymbol{F} = -Ze\nabla\phi , \tag{23}$$

and

$$\boldsymbol{v}_\perp = \frac{c}{ZeB^2} \boldsymbol{B} \times \nabla [Ze\phi + kT \log n] . \tag{24}$$

For the velocity parallel to $\boldsymbol{B}$ we have, combining equations (6) and (20),

$$\frac{d v_\parallel}{dt} = -\frac{1}{Bm} \boldsymbol{B} \cdot \left[-\boldsymbol{F} + \frac{1}{n} \nabla p \right] , \tag{25}$$

and, if $\boldsymbol{F}$ is purely electrostatic and T is constant,

$$\frac{d v_\parallel}{dt} = -\frac{1}{Bm} \boldsymbol{B} \cdot \nabla [Ze\phi + kT \log n] . \tag{26}$$

Equations (24) and (26) are implicit in the work of previous investigators, notably L. Tonks and W. P. Allis.[8]

These results may now be compared with the general equation of motion,

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \frac{1}{m} \boldsymbol{F} + \frac{Ze}{mc} \boldsymbol{v} \times \boldsymbol{B} - \frac{1}{n} \nabla \cdot \boldsymbol{\Psi} , \tag{27}$$

where $\boldsymbol{\Psi}$ is the material energy-stress tensor, defined by

$$\boldsymbol{\Psi} = \sum_k \boldsymbol{w}_k \boldsymbol{w}_k , \tag{28}$$

summed over all particles, k; $\boldsymbol{w}_k$ is here the random velocity of particle k, measured in a frame of reference in which the local value of $\boldsymbol{v}$, the mean velocity, vanishes. If in this frame of reference the distribution of velocities is isotropic, then

$$\nabla \cdot \boldsymbol{\Psi} = \frac{1}{m} \nabla p = \frac{1}{m} \nabla (nkT) , \tag{29}$$

where p is the pressure.

If equation (27) is applied to one type of particle, an additional force term must be added to take into account the momentum transferred from one type to another in collisions. When equation (27) is summed over all types of particles, the net transfer of momentum in collisions vanishes, and the equation in its present form is generally valid. Collisions then produce an effect through alteration of the form of $\boldsymbol{\Psi}$.

It is readily seen that, if we substitute equation (29) in equation (27) and neglect $d\boldsymbol{v}_\perp/dt$, we obtain directly equations (21) and (24). It is believed that the derivation of these equations from consideration of individual particle velocities makes the physical situation somewhat more clear. However, for obtaining certain higher approximations, it is most convenient to employ the macroscopic equation of motion. In accordance with the earlier discussion we set

$$\begin{aligned} \boldsymbol{v} = \boldsymbol{v}(0, 0) + \boldsymbol{v}(1, 0) + \boldsymbol{v}(2, 0) + \boldsymbol{v}(3, 0) \ldots \\ + \boldsymbol{v}(1, 1) + \boldsymbol{v}(2, 1) + \boldsymbol{v}(3, 1) \ldots . \end{aligned} \tag{30}$$

[8] *Phys. Rev.*, **52**, 710, 1937.

It is evident from the discussion in Section I that $\boldsymbol{v}(0, 0)$ represents simply the velocity of the frame of reference parallel to $\boldsymbol{B}$ and that $\boldsymbol{v}(0, 1)$ vanishes. The term $\boldsymbol{v}(1, 0)$ is a solution of equation (27) when $d\boldsymbol{v}/dt$ is ignored and equation (29) is assumed. The term $\boldsymbol{v}(2, 0)$ is the additional velocity found from equation (27) when the value of $d\boldsymbol{v}(1, 0)/dt$ obtained from the first approximation is substituted in this equation and when the first-order corrections to equation (29) are considered. This process can, in principle, be extended to any accuracy desired. For the computation of $\boldsymbol{v}(n, 1)$ a consideration of individual particles is again essential. The $(n, 2)$ and higher approximations are not considered in this paper.

As a summary of the results obtained in this section, we now give final equations for the over-all macroscopic velocity $\boldsymbol{v}$ and for the electric current. If $\boldsymbol{v}_i$ is the macroscopic velocity for particles of type i, then we have

$$\boldsymbol{v} = \frac{1}{\rho} \sum_i n_i m_i \boldsymbol{v}_i , \tag{31}$$

where ρ is the density of the plasma, equal to $\sum_i n_i m_i$. Employing equation (21), we find, if we assume that the external force is electrical,

$$\boldsymbol{v}_\perp (1, 0) = c \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} + \frac{c}{eB^2 \rho} \boldsymbol{B} \times \sum_i \frac{m_i \nabla p_i}{Z_i} , \tag{32}$$

where Z_i is -1 for electrons. Evidently, the electrons contribute very little to the motion of the plasma, which is essentially given by the motion of the heavy positive ions.

The current density $\boldsymbol{j}$, in electromagnetic units, may be written

$$\boldsymbol{j} = \frac{e}{c} \sum_i n_i Z_i \boldsymbol{v}_i . \tag{33}$$

As before, we obtain from equation (21),

$$\boldsymbol{j}_\perp (1, 0) = \frac{1}{B^2} \boldsymbol{B} \times \left[- \sum_i n_i \boldsymbol{F}_i + \nabla \sum_i p_i \right] . \tag{34}$$

If the external force is entirely electrical, $\boldsymbol{F}_i$ becomes $Z_i e \boldsymbol{E}$. The first term in brackets then becomes the electrical force on the net charge density in the plasma; since even a small net charge density gives rise to relatively large electrostatic fields, this net force is usually small and may be neglected. Hence in this approximation the component of $\boldsymbol{j}$ perpendicular to the magnetic field is determined entirely by the gradient of the total pressure.

III. THE (2, 0) APPROXIMATION; POLARIZATION

We have seen that the equations for $\boldsymbol{v}$ in the (1, 0) approximation satisfy the general equation of motion—equation (27)—provided that $d\boldsymbol{v}_\perp/dt$ is neglected and an isotropic pressure is assumed. We consider, first, the effect of the acceleration term. To obtain the (2, 0) approximation, we compute $d\boldsymbol{v}_\perp/dt$ from the (1, 0) approximation, substitute in equation (27), and solve for $\boldsymbol{v}$. In this way we obtain

$$\boldsymbol{v}_\perp (2, 0) = \frac{m c}{Z eB^2} \boldsymbol{B} \times \frac{d\boldsymbol{v} (1, 0)}{dt} ; \tag{35}$$

the time derivative is the change of $\boldsymbol{v}$ along a dynamical path in the (1, 0) approximation. The velocity $\boldsymbol{v}_\parallel$, parallel to $\boldsymbol{B}$, is obtained in the conventional way from equation (25), and the only contribution of the (2, 0) approximation to $v_\parallel$ is through the $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ term, which depends on $v_\perp(1, 0)$. Since the emphasis in the present paper is primarily on the determination of $\boldsymbol{v}_\perp$, we shall not consider these additional effects.

The value of $\boldsymbol{v}_\perp(2, 0)$ found from equation (35) will depend on many complicated variations of $\boldsymbol{F}$, $\boldsymbol{B}$, n, and p in time and space. We shall not consider all these dependences in detail, since a number of them lead simply to small changes in the velocity. The most important result of the second approximation is the appearance of space charges produced by secondary currents. When conditions change with time, such space charges may produce a large effect on the velocity found in the first approximation and will therefore be considered in detail here.

We consider, first, acceleration corresponding to the $\partial v/\partial t$ term in equation (27). Of particular importance is the acceleration in the direction of $\boldsymbol{v}_\perp(1, 0)$, since this necessarily produces a component of $\boldsymbol{v}_\perp(2, 0)$ at right angles to $\boldsymbol{v}_\perp(1, 0)$. Let us consider, for example, a situation in which the electrostatic field $\boldsymbol{E}_\perp$ varies with time. Then, from equations (21) and (35), we have

$$\boldsymbol{v}_\perp(2, 0) = \frac{m c^2}{Z e B^2} \frac{d\boldsymbol{E}_\perp}{dt}. \tag{36}$$

Summing over all components, in accordance with equation (33), the current density becomes

$$\boldsymbol{j}_\perp(2, 0) = \frac{\rho c}{B^2} \frac{d\boldsymbol{E}_\perp}{dt}. \tag{37}$$

A current proportional to dE/dt is usually described as a change of the polarization P. Hence, with P in e.s.u., we have

$$\boldsymbol{P}_\perp = \frac{\rho c^2}{B^2} \boldsymbol{E}_\perp. \tag{38}$$

If the dielectric constant κ is defined in the usual way as

$$\kappa = 1 + 4\pi \frac{P_\perp}{E_\perp}, \tag{39}$$

we find

$$\kappa = 1 + \frac{4\pi \rho c^2}{B^2}. \tag{40}$$

Except in very strong fields, κ is half the ratio of the material energy density, including rest mass, to the density of magnetic energy. In interstellar space we may set ρ equal to 10^{-24} gm/cm^3, and B to 10^{-5} gauss, and obtain a value of 10^8 for κ. The velocity of an electromagnetic disturbance, whose period is long compared with the gyration frequency, ω—about 0.1 per second for protons in interstellar space—and in which $\boldsymbol{E}$ is perpendicular to $\boldsymbol{B}$, will therefore be $c/10^4$, or 30 km/sec. The velocity $c/(\kappa)^{1/2}$ is simply the familiar velocity for a magneto-hydrodynamic wave. Such waves in an ideal plasma have been analyzed in some detail by E. Aström,[9] who obtained equation (40) for κ in the special case of a sinusoidal magneto-hydrodynamic wave.

When $\boldsymbol{B}$ varies, a current perpendicular to $\boldsymbol{v}_\perp(1, 0)$ again arises. This current density can still be regarded as a change of polarization, but amounts in this case to only half the value of $d\boldsymbol{P}/dt$ found on differentiating equation (38) with respect to B.

A change of kinetic temperature with time will also produce currents in the (2, 0) approximation. If n is constant in time, then, from equations (21), (33), and (35), we find for the over-all current,

$$i_\perp(2, 0) = \frac{c}{B^2} \sum_i \left\{ n_i m_i \frac{d\boldsymbol{E}_\perp}{dt} - \frac{m_i}{e Z_i} \nabla_\perp \left(n_i k \frac{dT}{dt} \right) \right\}. \tag{41}$$

[9] *Ark. f. Fys.*, **2**, No. 42, 443, 1950.

The electrical field must be included, since the current produced by an increase of T produces, in turn, an electrostatic field. The value of $\boldsymbol{E}$ may be determined from Poisson's equation, which becomes, on partial differentiation with respect to time,

$$\nabla \cdot \frac{\partial \boldsymbol{E}}{\partial t} = -4\pi c \nabla \cdot \boldsymbol{j} . \tag{42}$$

Since the quantity $\boldsymbol{j}$ in equation (42) includes the internal currents in the plasma, no dielectric constant (in these units) is needed in Poisson's equation. The value of κ given in equation (40) is applicable when the internal currents are not considered directly and the response of the plasma to extraneous fields or currents is considered. Equation (42) may be integrated at once to find $\partial \boldsymbol{E}/\partial t$. In general, an additional vector function, with vanishing divergence, must also be added to satisfy the boundary conditions and the condition that $\nabla \times \boldsymbol{E}$ vanishes. In the present case, if the gradients of n_i, T, and B^2 are parallel and if no external electrostatic fields are present, this additional function vanishes. Now $\boldsymbol{j}_\perp$ can be eliminated from equations (41) and (42), and we find

$$\frac{\partial \boldsymbol{E}}{\partial t}\left(\rho + \frac{B^2}{4\pi c^2}\right) = \sum_i \frac{m_i}{eZ_i} \nabla_\perp \left(n_i k \frac{\partial T}{\partial t}\right), \tag{43}$$

provided that we replace $d\boldsymbol{E}/d\tau$ and $dt/d\tau$ by $\partial \boldsymbol{E}/\partial \tau$ and $\partial t/\partial \tau$. If we assume that B^2 and n_i are both constant with time, equation (43) may be integrated at once over time. If the resultant relation between $\boldsymbol{E}$ and T is used to eliminate $\boldsymbol{E}$ from equation (32), we obtain for $\Delta \boldsymbol{v}_\perp(1, 0)$ the increase in the total velocity resulting from the increase of temperature,

$$\Delta \boldsymbol{v}_\perp(1, 0) = \frac{c}{\kappa \rho e B^2} \boldsymbol{B} \times \sum_i \frac{m_i}{Z_i} \nabla_\perp (\Delta p_i) , \tag{44}$$

where κ is defined in equation (40). A comparison of equations (32) and (44) shows that an increase of pressure in this way produces in this case a relatively small increase of the velocity, if κ is large. The electrical currents produced in the (2, 0) approximation give rise to an electrostatic field that nearly cancels the velocity corresponding to the increased pressure gradient. If κ is extremely large, the mean velocities of different types of charged particles will not be reduced by exactly κ; but, when the velocities of all types of atoms are combined, the total macroscopic velocity obeys equation (44). The increase in $\boldsymbol{j}_\perp(1, 0)$ found from equation (34) for an increase in p_i is apparently not counterbalanced by any such secondary effect. In the general case this increase of $\boldsymbol{j}_\perp(1, 0)$ will modify $\boldsymbol{B}$, an effect ignored in equation (44).

We now turn to a consideration of the currents perpendicular to $\boldsymbol{B}$ produced by the $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ term. If the stream lines are curved, this term will yield a centrifugal acceleration, perpendicular to $\boldsymbol{v}$. From equation (35) we see that curvature of $\boldsymbol{v}_\perp(1, 0)$ will produce a secondary velocity parallel to $\boldsymbol{v}_\perp(1, 0)$, and will also change the magnitude of $\boldsymbol{j}_\perp(1, 0)$. These effects do not seem very important. If the magnetic lines of force are curved, with a vector radius of curvature $\boldsymbol{R}$, the motion $v_\parallel$ will produce secondary velocities and currents perpendicular to $\boldsymbol{B}$. For the current we find

$$\boldsymbol{j}_\perp(2, 0) = \frac{\sum_i n_i m_i v_{i\parallel}}{B^2 R^2} \boldsymbol{B} \times \boldsymbol{R} , \tag{45}$$

where the sum extends over all types of charged particles. If $n_i m_i v_i^2/R$ is small compared with the pressure gradient ∇p_i, $\boldsymbol{j}_\perp(2, 0)$ given by equation (45) will be small compared to $j_\perp(1, 0)$. Whether this secondary current produces space charges and electrostatic fields depends on the detailed configuration of $\boldsymbol{B}$ and $v_\parallel^2$ in the plasma.

Finally, the second derivative of p must be included in the (2, 0) approximation. In a plane perpendicular to $\boldsymbol{B}$ the distribution of velocities at any point depends on the density of guiding centers at neighboring points. If second derivatives of this density are considered, the mean square random velocities in different directions become unequal, and the pressure gradient must be replaced by the divergence of the material stress-energy tensor. The effective pressure will now be different in different directions. The magnitude of $\boldsymbol{v}_\perp$ and $\boldsymbol{j}_\perp$ will be altered in this higher approximation, but it would appear that no appreciable velocities or currents in other directions will be produced.

IV. THE (3, 0) APPROXIMATION; HALL CURRENT

Many effects occur in the (3, 0) approximation. We shall here consider only one of these, the currents produced by the acceleration in $\boldsymbol{v}_\perp(2, 0)$. As in equation (35), we have

$$\boldsymbol{v}_\perp(3, 0) = \frac{m c}{Z e B^2} \boldsymbol{B} \times \frac{d \boldsymbol{v}(2, 0)}{d t}. \tag{46}$$

Considering only those velocities due to an electrical field, we now take $\boldsymbol{v}_\perp(2, 0)$ from equation (36). From the definition of $\boldsymbol{j}$ in equation (33) we obtain

$$\boldsymbol{j}_\perp(3, 0) = \frac{c^2}{e B^4} \left(\sum_i \frac{n_i m_i^2}{Z_i} \right) \boldsymbol{B} \times \frac{d^2 \boldsymbol{E}}{d t^2}. \tag{47}$$

The current obtained from equation (47) is perpendicular to $\boldsymbol{B}$ and $\boldsymbol{E}$ and corresponds to the usual Hall current. The occurrence of m_i^2 in equation (47) indicates that the Hall current is carried primarily by the positive ions, whereas usually the Hall current is attributed to the electrons. This apparent contradiction is simply a difference of point of view. When E increases, the electrons rapidly reach the velocity cE/B; the positive ions take longer to reach this same velocity. In the frame of reference in which E is measured, the electrons actually carry the Hall current. In the present instance, however, the velocities obtained from equation (46) measure the deviation from the velocity cE/B. Since this deviation is greatest for the heavy positive ions, the Hall current is therefore attributed to these ions. In any case, it is the difference in mass between the positive ions and electrons that is reponsible for the Hall current.

It may be noted that the ratio of the velocities involved in the Hall current to the velocity cE/B is simply $d^2E/\omega^2 E dt^2$. In an ideal plasma this quantity is, by definition, very small, and the Hall current is virtually negligible.

V. THE (1, 1) APPROXIMATION; ELECTRON-ION COLLISIONS

The previous results describe the motion of a plasma when collisions are entirely absent. The perturbations produced by collisions may be classified into two different types, depending on whether the collisions are between like or unlike particles. Collisions between unlike particles, with different mean motions, may produce an effective force on each group of particles, and this produces effects in the (1, 1) approximation, which may be treated by simply adding this force term to the basic equation of motion. However, if a group of identical particles is considered, mutual collisions obviously produce no net force on the group; in this case, considered in more detail in the following section, the anisotropy of the velocity distribution must be considered, and the resultant effects occur in higher orders of approximation.

We now treat the collisions between positive ions, all assumed to have the same charge Z, and electrons. Let $\boldsymbol{F}_{e,i}$ be the mean force on an electron resulting from collisions with positive ions. Since there are Z times as many electrons as positive ions, the mean force on a positive ion resulting from these collisions is $-ZF_{e,i}$. If in equation (21) we consider the velocity due to $F_{e,i}$ only, we find

$$v_{\perp}(1, 1) = \frac{c}{eB^2} B \times F_{e,i} . \tag{48}$$

Since $F_{e,i}$ is directly proportional both to the collision frequency and to the difference in the mean velocity $v(1, 0)$ between electrons and positive ions, it is clear that equation (48) gives the drift velocity in the (1, 1) approximation. This equation holds for electrons and positive ions separately, and hence also for the macroscopic velocity.

Instead of evaluating $F_{e,i}$ directly, we first relate the quantity to more familiar concepts. As a result of the force $F_{e,i}$, the motion of electrons through ions performs work or, more accurately, dissipates energy into heat. The work done per cubic centimeter per second is equal to $n_e F_{e,i} \cdot (v_i - v_e)$. The power dissipated per cubic centimeter by a current of density j is usually expressed as ηj^2, where η is the resistivity in e.m.u. Hence, using equation (33), we obtain

$$\frac{c}{e} F_{e,i} = \eta j . \tag{49}$$

Equation (49) is generally valid. In the present case, we use $j_{\perp}$ and $\eta_{\perp}$ to denote the current density perpendicular to B and the resistivity for dissipation of this current into heat. Then equation (48) becomes

$$v_{\perp}(1, 1) = \frac{\eta_{\perp}}{B^2} B \times j_{\perp}(1, 0) . \tag{50}$$

If the resistivity is defined in terms of energy dissipation, it is evident that this quantity depends only on microscopic collisions and is not directly dependent on B. In much previous work the transverse conductivity $\sigma_{\perp}$ has been used as the ratio of $j_{\perp}$ to $E_{\perp}$. In an ideal plasma there would seem to be little justification for this procedure. As we have seen, there is no direct connection between $j(1, 0)$ and E; in fact, these vectors may have quite different directions. An electrical field produces primarily a velocity at right angles to E and B and a polarization parallel to $E_{\perp}$. If this velocity gives rise to a pressure gradient, currents will appear, and in certain steady conditions the ratio of j to E will equal $\sigma_{\perp}$. However, the physical situation is clarified if $\sigma_{\perp}$ is defined in terms of energy dissipation and with such a definition it appears more rational to deal with the resistivity instead of the conductivity.[10] It is evident that the primary macroscopic effect of resistivity is the appearance of the drift velocity given in equation (50).

While $\eta_{\perp}$ in an ideal plasma is independent of B and is closely equal to $\eta_{\parallel}$, the normal resistivity in the absence of a magnetic field, these two resistivities are not exactly equal. The value of η depends on how the current j is distributed among electrons of different velocities, i.e., on the detailed form of the velocity-distribution function, $f(v)$. For currents parallel and perpendicular to the magnetic field the details of these distribution functions are quite different.

To evaluate these differences, we express j and $F_{e,i}$ in terms of f. Considering, first, the electron current only, we follow Chapman and Cowling[11] and write, for the electron distribution function,

$$f_e(w) = f_e^{(0)}(w) \{1 + D(lw) \cos\theta\} , \tag{51}$$

[10] I am indebted to Dr. T. G. Cowling for suggesting the use of the resistivity instead of the conductivity in this connection.

[11] *The Mathematical Theory of Non-uniform Gases* (Cambridge: At the University Press, 1939).

where the polar axis is the direction in which the current is flowing, and $f_e^{(0)}(w)$ is the Maxwellian distribution function given by equation (2). The electron current density, j_e, along the polar axis is given by

$$j_e = -\frac{e}{c}\int f_e(w)\, w \cos\theta\, d\boldsymbol{w} = \frac{-4 e n_e I_3(\infty)}{3\pi^{1/2} l_e c}, \tag{52}$$

where, in general,

$$I_n(\infty) \equiv \int_0^\infty x^n D(x)\, e^{-x^2} dx\,. \tag{53}$$

The force $\boldsymbol{F}_{e,i}$ may be found from the coefficient of dynamical friction, introduced by Chandrasekhar.[12] Following the notation of Cohen, Spitzer, and Routly,[13] we let $\langle\Delta_\xi\rangle$ represent the average rate of change of the electron velocity, measured in the direction along which the electron is moving. Then

$$n_e \boldsymbol{F}_{e,i} = m_e \int f_e(w) \langle\Delta_\xi\rangle d\boldsymbol{w}\,. \tag{54}$$

The function $\langle\Delta_\xi\rangle$ may be taken from equations (31), (32), and (35) of Cohen, Spitzer, and Routly.[13] The quantity j in these equations refers to the quantity denoted here by l (to avoid confusion with the current density); for interactions of electrons with positive ions the quantity x occurring in these equations equals $l_i w$, where l_i is the value of l obtained when m_i, the mean mass of the positive ions, is substituted in equation (2). If the velocity of the positive ions is assumed relatively small, x is much greater than unity, and $\langle\Delta_\xi\rangle$ equals $-3L/w^2$, where, for electron-proton encounters, L is one-half its previous value.[13] The integral in equation (54) above is then readily evaluated. If we use equation (49) to express η in terms of $\boldsymbol{F}_{e,i}$, eliminating j_e and l_e by means of equations (52), and (3), we obtain

$$\eta = \frac{3\pi 6^{1/2} Z\, e^2 c^2 \log(qC^2)}{m_e C^3}\, \frac{I_0(\infty)}{I_3(\infty)}, \tag{55}$$

where $I_0(\infty)$ and $I_3(\infty)$ are defined in equation (53); C^2 is the mean-square electron velocity, equal to $3kT/m_e$; and, according to Cohen, Spitzer, and Routly,[13] q is given by

$$q = \frac{m_e}{2 e^3 Z}\left(\frac{kT}{\pi n_e (1+Z)}\right)^{1/2}. \tag{56}$$

The problem of determining the conductivity is therefore reduced to the computation of the ratio of the integrals $I_0(\infty)$ and $I_3(\infty)$. In the general case the computation of $D(x)$ is rather complicated. In the absence of a magnetic field, or for currents parallel to the magnetic field, it may be shown that Cowling's "second approximation"[14] yields

$$\frac{I_0(\infty)}{I_3(\infty)} = 0.383\,. \tag{57}$$

This same value has also been obtained by Spitzer and Härm[15] by the use of a diffusion equation in velocity space; the earlier result by Cohen, Spitzer, and Routly,[13] is modified when all the terms in the diffusion equation are taken into account.

For currents perpendicular to a magnetic field, however, the current perpendicular

[12] *Ap. J.*, **97**, 255, 1943.

[13] *Phys. Rev.*, **80**, 230, 1950.

[14] *Proc. R. Soc. London, A*, **183**, 453, 1945; this paper considers only the first approximation in the velocity-distribution function, but represents a second approximation in the expansion of this function in terms of Laguerre (or Sonine) polynomials.

[15] In preparation.

to $\boldsymbol{B}$ is produced by a pressure gradient and, in this case, $D(x)$ is directly proportional to x. The greater w is, the farther away is the guiding center for the particle passing through a small area; since we are considering velocities on the (1, 0) approximation, we may assume a linear variation with distance for the density of guiding centers. It follows directly that the excess number of particles moving one way will be proportional to the velocity. In this case the integrals can be evaluated at once, and we have

$$\frac{I_0(\infty)}{I_3(\infty)}=\frac{4}{3\pi^{1/2}}=0.752\,. \tag{58}$$

Hence, finally, we have

$$\frac{\eta_\perp}{\eta_\parallel}=1.96\,. \tag{59}$$

This result is derived here only for the electron component of the current, but it is valid also for the current carried by the positive ions. The force on a positive ion moving through a spherical, Maxwellian dlstribution of electrons may again be determined from the coefficient of dynamical friction. The ratio of this force to the positive ion current is the same as in the case in which the electrons carry the current; moreover, this ratio does not depend on the kinetic temperature of the positive ions. Hence the fact that positive ions carry a large fraction of the current does not vitiate the computation of $\eta_\perp/\eta_\parallel$ from the electron current alone, and, in particular, equation (59) is generally valid for the (1, 1) approximation. In the (1, 2) approximation, however, higher terms would require consideration.

It may be noted that encounters between positive ions also produce drifts. If two ions of types 1 and 2 are interacting and each has a Maxwellian velocity distribution, with the perturbation $f^1(\boldsymbol{w})$ resulting from a linear density gradient, an evaluation of equation (54) in the more general case, together with equations (21) and (48), gives

$$\boldsymbol{v}_{1\perp}(1,\,1)=\frac{4\,(6\pi)^{1/2}Z_1^2Z_2^2e^2c^2\,(m_1+m_2)\log\,qC^2}{B^2m_1m_2\,(C_1^2+C_2^2)^{3/2}}\,\frac{n_2}{Z_1}\left[\frac{1}{Z_2n_2}\,\nabla_\perp p_2-\frac{1}{Z_1n_1}\,\nabla_\perp p_1\right], \tag{60}$$

where $\nabla_\perp$ denotes the component of the gradient in the plane perpendicular to $\boldsymbol{B}$. The equation for $\boldsymbol{v}_{2\perp}(1, 1)$ is obtained by interchanging the subscripts 1 and 2 in equation (60). It is evident that the total current produced by these drifts is zero. It may be noted that drifts of positive ions relative to one another will vanish in an isothermal plasma only if

$$\log\,(n_2)^{1/Z_2}=\log\,(n_1)^{1/Z_1}+\text{constant}\,. \tag{61}$$

Equation (61) indicates that the ions of higher Z will tend to be relatively more concentrated in regions of higher particle density.

VI. THE (2, 1) AND (3, 1) APPROXIMATIONS; DIFFUSION

The (2, 1) approximation comprises a large number of different effects. A current will be produced by the acceleration of $\boldsymbol{v}(1, 1)$. Velocities will be produced by interactions between electrons and positive ions moving with the mean velocities obtained in the (2, 0) approximation. These effects may be computed from the equations in the previous sections but will not be considered in detail here. Instead, we shall consider primarily the phenomenon of diffusion. An accurate treatment of this effect, with detailed consideration of the velocity-distribution function, would be complicated and is not attempted here. Instead, we shall compute the rate of diffusion in one extremely simple case and apply this result to obtain the order of magnitude of other diffusion rates.

Let us consider a homogeneous, uniform, infinite plasma. We take rectangular co-ordinates x_1, x_2, and x_3 and assume a uniform magnetic field, parallel to the x_3-axis. In this plasma we shall consider particles, all of the same charge Ze, mass m, and particle density n. Let us suppose, now, that by some method we tag each particle either A or B, with n_A and n_B particles of each type per cubic centimeter. Other ions, of charge $Z_i e$, mass m_i, and particle density n_i will also be assumed. We now consider the rate at which particles of types A and B diffuse into each other.

We shall consider the drifts not of the particles themselves but of the guiding centers about which each particle gyrates. In the absence of encounters, each guiding center would be fixed in position under the present assumptions. If, in an encounter, the velocity of the particle changes by an amount $\Delta \boldsymbol{w}$, the position of the guiding center will also change. The vector displacement from the particle to the guiding center is given by

$$\boldsymbol{a} = \frac{m\,c}{Z\,eB^2} \boldsymbol{w} \times \boldsymbol{B} . \tag{62}$$

If we assume that the encounters occur in a region of space very small compared with the radius of gyration $\boldsymbol{a}$, then we may regard $\Delta \boldsymbol{w}$ as an abrupt change of velocity, which will produce a displacement of the guiding center given by

$$\Delta \boldsymbol{a} = \frac{m\,c}{Z\,eB^2}\, \Delta \boldsymbol{w} \times B . \tag{63}$$

Successive displacements of the guiding center constitute an example of random walk, as in Brownian motion; a thorough survey of such problems has been given by Chandrasekhar.[16] The net flow of guiding centers may be determined from the mean values of $\Delta \boldsymbol{a}$ and $(\Delta \boldsymbol{a})^2$. If we let F_i be the net number of guiding centers per second crossing a square centimeter perpendicular to the x_i-axis, we have, in general, the Fokker-Planck equation,

$$F_i = n \langle \Delta a_i \rangle - \frac{1}{2} \sum_j \frac{\partial}{\partial x_j} \left(n \langle \Delta a_i \Delta a_j \rangle \right) , \tag{64}$$

where n is the number of particles of that type per cubic centimeter and the symbol $\langle\ \rangle$ denotes the sum of all changes in a second, averaged over all such particles, the average extending around the Larmor circle for each particle.

In the general case the computation of the diffusion co-efficients $\langle \Delta a_i \rangle$ and $\langle \Delta a_i\, \Delta a_j \rangle$ is rather involved, as the density of particles and their velocity distribution may both vary with position. Since we have here assumed that collisions with particles of type A are the same as collisions with particles of type B and that $n_A + n_B =$ a constant n, the diffusion coefficients may readily be determined. From symmetry it is evident that the mean value of $\Delta \boldsymbol{w}$, averaged over a Larmor circle, vanishes. Also, from equation (63) we obtain

$$\langle \Delta a_i \Delta a_j \rangle = \begin{cases} \dfrac{m^2 c^2}{2Z^2 e^2 B^2} \overline{\langle | \Delta \boldsymbol{w}_\perp |^2 \rangle} & \text{if } i = j = 1 \text{ or } 2 , \\ 0 & \text{if } i \neq j , \end{cases} \tag{65}$$

in which the bar denotes that an average is to be taken over all particle velocities. The velocity change $\Delta \boldsymbol{w}$ has Cartesian components Δ_ξ, Δ_η, Δ_ζ, where the ξ-axis is parallel to w, while the η-axis lies in the plane defined by $\boldsymbol{B}$ and the ξ-axis. If θ denotes the angle between $\boldsymbol{w}$ and $\boldsymbol{B}$, then

$$(\Delta w_\perp)^2 = (\Delta_\xi)^2 \sin^2 \theta + (\Delta_\eta)^2 \cos^2 \theta + (\Delta_\zeta)^2 . \tag{66}$$

[16] *Rev. Mod. Phys.*, **15**, 1, 1943.

Since the distribution of velocities is isotropic, equation (65) yields, on substitution from equation (1),

$$\langle (\Delta a_1)^2 \rangle = \frac{1}{2\omega^2} \left\{ \tfrac{2}{3} \overline{\langle \Delta_\xi^2 \rangle} + \tfrac{1}{3} \overline{\langle \Delta_\eta^2 \rangle} + \overline{\langle \Delta_\zeta^2 \rangle} \right\}. \tag{67}$$

The coefficient $\langle (\Delta a_2)^2 \rangle$ equals $\langle (\Delta a_1)^2 \rangle$.

The diffusion coefficients in equation (67) may be found from Chandrasekhar[12] or from Cohen, Spitzer, and Routly.[13] Since we are here considering diffusion in physical space, the diffusion coefficients, given as a function of velocity by the above authors,[12, 13] must be averaged over a Maxwellian velocity distribution. The coefficients $\langle \Delta_\eta^2 \rangle$ and $\langle \Delta_\zeta^2 \rangle$ are equal (their sum is denoted by $\Sigma[\Delta v_1^2]$ by Chandrasekhar),[12] and we obtain, finally,

$$K \equiv 2 \langle (\Delta a_1)^2 \rangle = \frac{4 (6\pi)^{1/2} e^2 c^2 \log (q C^2)}{3 C B^2} \sum_i \frac{n_i Z_i^2}{(1 + C_i^2 / C^2)^{1/2}}, \tag{68}$$

where C_i^2 is the mean-square velocity for ions of type i. Evidently, collisions with more slowly moving particles produce more rapid diffusion than do collisions with more rapid particles. Diffusion of protons, for example, is produced primarily by proton-proton collisions, with proton-electron collisions relatively unimportant. Because of the factor $1/C$ in equation (68), the heavier particles tend to diffuse more rapidly than do the lighter ones. In the argument of the logarithm in equation (68) we may take a mean value of C for all ions, since the value of this argument makes little difference. The quantity q is again determined from equation (56).

If we now assume that n_A and n_B have equal but opposite gradients along the x-axis, the rate of diffusion is given by

$$F_A = -K \frac{\partial n_A}{\partial x} = -F_B. \tag{69}$$

If this rate of flow is used to yield the mean velocity with which atoms of type A move in the x-direction, we arrive at exactly the same velocity as that obtained from equation (60), if we let atoms of types A and B correspond to types 1 and 2 in that equation and if T is assumed constant. Hence this process of autodiffusion considered here is simply another way of looking at the drift velocity in the (1, 1) approximation.

Next we may consider what happens when atoms all of a single type interact, but a density gradient is present. Since no systematic drift appears in this case in the (1, 1) approximation, one would expect autodiffusion to vanish in this same approximation. A computation of F_i in this case shows that the term in $\langle \Delta a_1 \rangle$ in equation (64) cancels the term in $\langle (\Delta a_1)^2 \rangle$, and no net drift appears. As we shall see below, collisions between identical atoms produce a net drift in the direction of the pressure gradient only in the (3, 1) approximation.

While the consideration of autodiffusion does not directly yield any new results, the diffusion coefficient K may be used to give approximate results in other problems. Let us consider the transfer of heat by diffusion. Suppose that in individual collisions each particle retained its same energy but that the position of its guiding center changed in accordance with the above results. Then the flow of heat should be given by an equation similar to equation (69), but with nkT replacing n_A. Since, as we have seen, the gradient of particle density produces no effect in this approximation when autodiffusion is considered, we may take n outside the derivative. The increase of $3nkT/2$, the internal energy of the plasma per cubic centimeter, is simply equal to minus the divergence of the heat flow, and we have, if changes in the x-direction only are considered,

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(K \frac{\partial T}{\partial x} \right). \tag{70}$$

Equation (70) is only approximate. To obtain an accurate expression, one must employ the detailed velocity-distribution function. When a temperature gradient is present but nT is constant, a flow of heat occurs in the direction perpendicular to ∇T and $\boldsymbol{B}$, with the more rapid atoms moving in one direction, the slower ones in the other.The effect of encounters on these two streams then produces a flow of heat in the direction of decreasing T.

Finally, we consider the viscous forces present. The transfer of lateral momentum may be treated in the same way as the transfer of heat, with the difference that the divergence of the flow of lateral momentum gives a force per unit volume. By equation (34) this force gives a current orthogonal to $\boldsymbol{B}$ and to the force. We find

$$j_{\perp} = -\frac{1}{B^2}\boldsymbol{B}\times\sum_i n_i m_i \frac{\partial}{\partial x}\left(K\frac{\partial \boldsymbol{v}_{\perp}}{\partial x}\right). \tag{71}$$

Since $v_{\perp}$ is found from the (1, 0) approximation, $j_{\perp}$ obtained from equation (71) is of the (3, 1) approximation and is very small. This term tends to eliminate velocity gradients between different parts of the plasma, since the current will produce space charges that will tend to reduce $\partial v_{\perp}/\partial x$. Again, to improve the accuracy of equation (71), a detailed examination of the velocity-distribution function would be required.