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## A STUDY OF THE CHANGES IN THE RATE OF ROTATION OF THE EARTH

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*Summary.* It is shown that the observed fluctuations in the moon's mean longitude are compatible with the hypothesis that the rate of rotation of the earth is affected by cumulative random changes. A solution of the secular accelerations based on this assumption yields $(+2''.2 \pm 3''.8)\ T^2$ and $(+1''.01 \pm 0''.28)\ T^2$ for the $T^2$ terms in the mean longitudes of the moon and the sun, respectively. The resulting value for the secular increase in the length of the day is $+0^s.00135 \pm ^s.00038$ per century. The principal uncertainty in these evaluations is due to the random process which causes the amplitude of the fluctuations to increase proportionally to the power $\frac{3}{2}$ of the time and produces a spurious quadratic term with mean coefficient inversely proportional to the square root of the interval of time covered by the observations.

1. The first systematic study of the so-called fluctuations in the moon's mean longitude was made by Simon Newcomb,[1,2] stimulated by the deviations that were found to exist between the observed positions of the moon and Hansen's tables. Subsequent discussion, due primarily to the introduction of Brown's new lunar theory and tables, established beyond question that we are dealing with irregular changes in the earth's rate of rotation. This second phase was brought to a close by two publications by Sir Harold Spencer Jones in 1932 and 1939, respectively.[3,4] In the first of these papers, Newcomb's occultation memoir[2] was revised. The deviations of the moon's motion from Brown's tables derived from occultations were given for occasional normals from 1681 to 1813 and for each year from 1820 to 1908. These proved to be a great improvement over the results of the meridian circle observations, especially before 1850.

The discussion of the residuals in the moon's longitude had led de Sitter[5] to a representation of the fluctuation curve requiring instantaneous changes of the earth's rate of rotation. De Sitter had used Newcomb's eclipse and occultation results up to 1835, and the Greenwich meridian observations of the moon after that date. Spencer Jones showed that the occultation results did not agree in detail with de Sitter's representation. He remarked that the latter could only be considered a rough approximation.

Spencer Jones's paper of 1939 established clearly that the fluctuations in the mean longitudes of the sun, Mercury and Venus correspond to fluctuations in time identical with those required to account for the observed fluctuations in the moon's mean longitude. Consequently, the fluctuations in the mean longitude of the sun or any planet may be obtained from those in the moon's mean longitude by multiplication by a factor equal to the ratio between the mean motion of the body concerned and the moon's mean motion. Previous to 1939 the observations had seemed to indicate that the fluctuations in the longitudes of the sun and inner planets were greater by about 20 per cent than would correspond to the ratio of their mean motions to that of the moon. A stumbling block had been the presence of systematic errors in the observed right ascensions of the sun in the nineteenth century. It was shown by Spencer Jones that excellent agreement could be obtained by deriving the sun's mean longitude from declination observations only.

With regard to the secular accelerations in the mean longitudes of the sun and moon, a series of papers by J. K. Fotheringham[6] had led to the conclusion that all classes of ancient observations including solar eclipses were satisfied by terms, in addition to gravitational terms,

in the moon's mean longitude, $+4''.7\ T^2$,
in the sun's mean longitude, $+1.5\ T^2$.

A rediscussion of essentially the same material by de Sitter led to the non-gravitational terms,

in the moon's mean longitude,
$(+5''.22 \pm ''.30)\ T^2$,

in the sun's mean longitude,

$(+1''80 \pm ''16)\ T^2$.

The coefficients are within the range indicated as possible by Fotheringham.

Spencer Jones in 1939 adopted de Sitter's value, $+5''22\ T^2$, in the moon's mean longitude, and found that the observations of the sun and planets during the past 250 years then required the term $(+1''23 \pm ''04)\ T^2$ in the sun's mean longitude. In the longitudes of Mercury and Venus the coefficients corresponded to that in the sun, increased in the ratio of their mean motions to that of the sun. The subsequent discussion of the observations of Mercury by G. M. Clemence[7] confirmed Spencer Jones's conclusions, both as to the ratio of the fluctuations in seconds of arc in the mean longitude of Mercury to those in the moon and as to the coefficient of $T^2$.

Spencer Jones concludes that the correction to Newcomb's tables of the sun may be written

$$\begin{aligned} \Delta L_\odot &= +1''00 + 2''97\ T' + 1''23\ T'^2 + .0748\ B, \\ &= +5''20 + 5''43\ T + 1''23\ T^2 + .0748\ B, \end{aligned} \qquad (1)$$

$B$ = observed mean longitude *minus* (Brown's tables $+\Sigma$),

$$\begin{aligned} \Sigma &= +4''65 + 12''96\ T' + 5''22\ T'^2 - 10''71 \sin(140^\circ\!0\ T' + 240^\circ\!7) \\ &= +22''83 + 23''40\ T + 5''22\ T^2 - 10''71 \sin(140^\circ\!0\ T + 20^\circ\!7). \end{aligned} \qquad (2)$$

Throughout this paper $T'$ will designate time counted in centuries from 1900, $T$ centuries from 2000, while $t$ will designate time counted in years from any epoch.

For Mercury and Venus the corrections to the tabular mean longitudes have the same form as that for the sun, with the coefficients of $T^2$ and $B$ multiplied by 4.152 for Mercury, 1.626 for Venus, the ratios of the mean motions of these planets to the sun's mean motion.

The lack of agreement between the $T^2$ term in the sun's longitude found from modern observations and that found from the discussion of ancient observations may be interpreted in various ways. As Spencer Jones remarks, on the basis of the modern observations alone the $T^2$ term in $\Sigma$ may be changed from

$$+5''22\ T^2 \quad \text{to} \quad (+5''22 + s)T^2,$$

provided that the corresponding changes are made:

$$B \quad \text{to} \quad B' = B - s\,T^2,$$

and in the correction to the sun's tabular longitude

$$+1''23\ T^2 \quad \text{to} \quad (+1''23 + .0748s)T^2.$$

Before concluding that the secular accelerations have changed since the date of the oldest recorded eclipses it may be well to attempt to satisfy the ancient observations by a solution in which $s$ is retained as the only unknown.

The term $1''23\ T^2$ in the sun's longitude arises from the secular decrease of the rate of rotation of the earth. Hence

$$\begin{aligned} \Delta t &= 24.349\ \Delta L_\odot \\ &= +24^s\!349 + 72^s\!3165\ T' + 29^s\!949\ T'^2 + 1.821\ B \\ &= +126^s\!6145 + 132^s\!2145\ T + 29^s\!949\ T^2 + 1.821\ B. \end{aligned} \qquad (3)$$

may be adopted as the difference ephemeris time *minus* mean solar time, so chosen that the correction to the tabular mean longitude of the sun, including the constant and the $T$ term, is directly proportional to $\Delta t$. This proposal was made by G. M. Clemence,[8] and adopted as a recommendation to the International Astronomical Union by the conference on the Fundamental Constants of Astronomy held in Paris in 1950.[9] It should be noted that this definition of $\Delta t$ is independent of the quantity $s$, and is therefore not affected by a possible modification of the coefficient of $T^2$ in the sun's longitude and the corresponding modification in the definition of $B$.

2. The character of the changes in the rate of rotation of the earth may be examined by forming the derivative of the fluctuation curve. For the purpose of a first orientation I derived an approximate derivative from differences between successive three-year means based on Spencer Jones's Occ.-Th.[10] The appearance of the derivative curve so obtained suggested that it might consist of straight line sections. However, since these values were "read off a smoothed curve, drawn to represent the directly observed values as closely as possible," the possibility could not be ruled out that this smooth curve had unknowingly been drawn of nearly parabolic sections, by which the straight-line character of the derivative might have been introduced artificially. In order to examine this point, a fluctuation curve free from any smoothing had

to be used. The first step was the derivation of a set of annual mean deviations of the observed mean longitude of the moon from Brown's tables from all readily available series of observations. A derivative curve was recently published by N. Stoyko,[11] but since the data on which the derivative is based are not given, Stoyko's diagram can be used for comparison only.

For the years 1820 to 1908 annual values of Occultations *minus* Tables may be obtained from Spencer Jones's Table III.[3] These data were used without change from 1820 to 1879. For the years 1880 to 1908, annual values from the Revision of Newcomb's Memoir (N) were combined with those of the Cape occultations (C).[12] A systematic correction −0″.64 was applied in order to reduce the Cape occultation residuals to the same basis as the revision of Newcomb's work. There is some duplication between the Cape material and the observations included in the "Revision." A discussion, the details of which need not be given, indicated that the best relative weights for combining the two series are weights proportional to the two-thirds power of the number of observations. These are the numbers given as Wt. in Table I.

For the years 1909 to 1922, the Cape results were taken with the correction −0″.64.

TABLE I. OCCULTATIONS *minus* TABLES, 1880–1908

| | N | Wt | C−″.64 | Wt | Occ. |
|---|---|---|---|---|---|
| 1880.5 | −2″.70 | 12 | −2″.80 | 11 | −2″.75 |
| 1 | −2.20 | 7 | −2.93 | 5 | −2.50 |
| 2 | −2.51 | 7 | −2.51 | 10 | −2.51 |
| 3 | −3.17 | 6 | −3.63 | 6 | −3.40 |
| 4 | −2.46 | 15 | −1.58 | 8 | −2.15 |
| 5 | −2.38 | 26 | −3.17 | 7 | −2.55 |
| 6 | −2.52 | 11 | −2.10 | 10 | −2.32 |
| 7 | −2.24 | 13 | −2.03 | 11 | −2.14 |
| 8 | −2.85 | 6 | −2.44 | 12 | −2.58 |
| 9 | −2.67 | 6 | −2.69 | 8 | −2.68 |
| 1890.5 | −2.64 | 8 | −2.33 | 8 | −2.49 |
| 1 | −2.76 | 10 | −2.86 | 10 | −2.81 |
| 2 | −2.84 | 9 | −3.12 | 6 | −2.95 |
| 3 | −2.53 | 9 | −2.95 | 8 | −2.73 |
| 4 | −3.11 | 14 | −3.45 | 6 | −3.21 |
| 5 | −2.72 | 36 | −2.94 | 5 | −2.75 |
| 6 | −2.20 | 34 | −2.91 | 18 | −2.45 |
| 7 | −3.20 | 10 | −2.71 | 28 | −2.84 |
| 8 | −2.78 | 12 | −2.58 | 27 | −2.64 |
| 9 | −2.53 | 8 | −2.06 | 23 | −2.18 |
| 1900.5 | −1.46 | 10 | −1.53 | 18 | −1.51 |
| 1 | −1.30 | 11 | −1.04 | 20 | −1.13 |
| 2 | + .05 | 20 | − .17 | 12 | − .03 |
| 3 | + .37 | 17 | + .64 | 6 | + .44 |
| 4 | + .81 | 13 | +1.39 | 6 | + .99 |
| 5 | +1.31 | 12 | +1.41 | 6 | +1.34 |
| 6 | +2.28 | 14 | +1.73 | 8 | +2.08 |
| 7 | +2.32 | 10 | +2.66 | 8 | +2.47 |
| 8 | +2.96 | 6 | +3.10 | 7 | +3.04 |

For the years 1923–1948 the occultation results furnished by the campaign started by Innes and Brown were used. From 1932 on, the results are based on stars in the new *Zodiacal Catalogue*, which is essentially on the same system as the $FK_3$. A comparison with the meridian circle observations in Greenwich 1932 to 1937 and in Washington 1932 to 1948 show that for this period the occultations and the meridian circle results are on essentially the same system, showing no important systematic differences. For the four years, 1932 to 1935, both the old annual values based on star positions not reduced to a common system, and revised annual values based on the new *Zodiacal Catalogue* are available. These are as follows.

| | Old Redn. | New Redn. | Corr. |
|---|---|---|---|
| 1932 | +5″.03 | +4″.16 | −0″.87 |
| 33 | +4.33 | +3.66 | −0.67 |
| 34 | +3.85 | +3.28 | −0.57 |
| 35 | +3.41 | +2.87 | −0.54 |

Table II gives the meridian circle results and the unrevised occultation results for the years 1923 to 1935. The column "Mer." is the weighted mean of the three preceding. The annual values of the Greenwich Limb (GL) and Washington (W) observations were given equal weight, the Greenwich Mösting A results (GMA) were used with one-third the weight of the other two. The difference Mer. *minus* Occ. was then represented by the formula

$$-1''.05+0''.064\ (t-1929.5).$$

The occultation residuals were corrected with this formula, and are given in the last column of Table II, except for the years 1932 to 1935, for which the results of the new reductions[13] are tabulated. The linear formula finds its explanation in the circumstance that for the early years

TABLE II. COMPARISON OF OCCULTATIONS WITH MERIDIAN OBSERVATIONS 1923–1935. OBSERVED MEAN LONGITUDES *minus* TABLES

| | GL | GMA | W | Mer. | Occ. | Corrected Occ. |
|---|---|---|---|---|---|---|
| 1923.5 | +6″.39 | +6″.87 | +6″.23 | +6″.39 | +7″.96 | +6″.53 |
| 24.5 | +6.61 | +6.54 | +6.26 | +6.45 | +7.72 | +6.35 |
| 25.5 | +6.01 | +6.11 | +6.54 | +6.25 | +7.42 | +6.11 |
| 26.5 | +5.80 | +5.95 | +6.26 | +6.02 | +7.19 | +5.95 |
| 27.5 | +5.60 | +5.64 | +5.67 | +5.64 | +6.85 | +5.67 |
| 28.5 | +4.91 | +5.28 | +5.28 | +5.12 | +6.25 | +5.14 |
| 29.5 | +4.60 | +5.29 | +5.05 | +4.89 | +5.90 | +4.85 |
| 30.5 | +4.48 | +4.80 | +4.63 | +4.59 | +5.71 | +4.72 |
| 31.5 | +4.08 | +4.27 | +4.49 | +4.28 | +5.20 | +4.28 |
| 32.5 | +4.01 | +3.96 | +4.02 | +4.01 | +5.03 | +4.16 |
| 33.5 | +3.61 | +3.50 | +3.45 | +3.53 | +4.33 | +3.66 |
| 34.5 | +3.38 | +2.73 | +3.26 | +3.24 | +3.85 | +3.28 |
| 1935.5 | +2.55 | +2.66 | +2.90 | +2.72 | +3.41 | +2.87 |

most of the star positions were taken from Hedrick's *Zodiacal Catalogue*. As the years progressed, more and more star positions based on more modern catalogues were used. The empirical correction is, of course, not entirely satisfactory; a new reduction of the occultations in the years 1923 to 1931 based on positions from the new *Zodiacal Catalogue* would strengthen this part of the fluctuation curve.

Previous to 1923 Newcomb's equinox was used, as this is the equinox used in Spencer Jones's revision. From 1923 on, the residuals such as those in Table II are on Newcomb's equinox *minus* 0″.6. This requires the change of the constant term in the correction to Brown's tables from 4″.65 to 4″.05, which leaves the fluctuation $B$ unaffected.[14]

For the years 1820 to 1850 the values of $-\Sigma$ and $B$ are listed in Table III. For 1851 to 1950 they are contained in Table V, along with data referring to the meridian observations.

TABLE III. ANNUAL MEANS FROM OCCULTATIONS, 1820–1850

| | $-\Sigma$ | $B$ | | $-\Sigma$ | $B$ |
|---|---|---|---|---|---|
| 1820.5 | +10″.63 | +10″.63 | 1835.5 | +6″.83 | +5″.42 |
| 21.5 | +10.41 | +10.21 | 36.5 | +6.54 | +5.19 |
| 22.5 | +10.19 | +10.47 | 37.5 | +6.24 | +4.99 |
| 23.5 | + 9.96 | + 9.30 | 38.5 | +5.94 | +4.52 |
| 24.5 | + 9.73 | + 9.03 | 39.5 | +5.64 | +4.65 |
| 1825.5 | + 9.49 | + 8.43 | 1840.5 | +5.33 | +4.18 |
| 26.5 | + 9.25 | + 9.14 | 41.5 | +5.02 | +4.77 |
| 27.5 | + 9.00 | + 7.74 | 42.5 | +4.70 | +3.55 |
| 28.5 | + 8.75 | + 8.43 | 43.5 | +4.39 | +5.16 |
| 29.5 | + 8.49 | + 7.00 | 44.5 | +4.07 | +3.72 |
| 1830.5 | + 8.22 | + 7.61 | 1845.5 | +3.74 | +3.59 |
| 31.5 | + 7.95 | + 6.63 | 46.5 | +3.42 | +4.70 |
| 32.5 | + 7.68 | + 7.21 | 47.5 | +3.09 | +3.60 |
| 33.5 | + 7.40 | + 7.01 | 48.5 | +2.76 | +4.65 |
| 34.5 | + 7.11 | + 5.46 | 49.5 | +2.43 | +3.13 |
| | | | 1850.5 | +2.09 | +3.72 |

Annual values of the observed corrections to Brown's tables for the Greenwich observations (G) for 1851.5 to 1922.5 are given in *Greenwich Observations* for the year 1920, page G vi. However, the column Br—H.N. on this page corresponds to a table in a publication by E. W. Brown which includes the equinox corrections.[15] Since these corrections were included in the preceding column, they should have been removed, as they were in a similar publication for Cowell periods.[16] Thus, for 1851.5 the difference Greenwich *minus* Brown's Tables becomes:

$$+2''.71+0''.33-0''.36 = +2''.68.$$

For the years 1866 to 1922 the values so obtained agree with those listed by H. R. Morgan.[17] Of the Washington observations (W), also listed by Morgan, there are two series, 1866 to 1891 and 1894 to the present. The residuals 1866 to 1922 have been taken from Morgan's publication.

For the observations since 1923 I have used the Greenwich results, reduced to Newcomb's equinox *minus* 0″.6, from the annual volumes of the *Greenwich Observations*. The limb and Mösting A residuals were averaged with weights 3 and 1, respectively. For Washington the results published in the *Astronomical Journal* are available. Beginning with the year 1933 they were corrected for limb irregularities.[18]

The differences between the results of the meridian observations and of the occultations are frequently of a systematic character. The same sign occasionally persists for many years. The discontinuity between the old and the new Washington series is also apparent. The agreement among Greenwich, Washington and occultations has been very much better since 1923 when Brown's tables were used for the ephemeris than before that date. This fact as well as other comparisons indicate that some of the inconsistencies in the meridian observations of the moon before 1923 may be ascribed to the use of Hansen's tables and the imperfect differential correction from Hansen to Brown. If the meridian results before 1923 were combined directly with the occultations, it is likely that systematic effects would be introduced greater than those present in the occultation results. In order to make use of the meridian results without this objection I proceeded as follows. Three-year means were formed for each of the three series of observations, and the differences Occ. *minus* W and Occ. *minus* G were represented by quadratic formulae. For W the observations 1866 to 1891 and 1894 to 1922 were treated independently. For the Greenwich observations, 1851 to 1887 and 1888 to 1922 were represented by two different quadratic formulae that were made to overlap for five years, 1886 to 1890. The corrections to the meridian series so obtained are listed in Table IV, columns 2 and 4. The corrected residuals Obs. *minus* Tables are given in columns 3 and 5. For 1892 and 1893 interpolated values for Washington are supplied and listed in parentheses. The principal purpose of these provisional corrections is the elimination of the discontinuities in the Washington observations between the end of the old series (1891) and the beginning of the new series (1894), and in both Greenwich and

Washington at the introduction of Brown's tables in 1923. No apparent discontinuity exists between the occultation residuals before 1923 and the corrected occultation residuals after 1923.

TABLE IV. PROVISIONAL CORRECTIONS TO GREENWICH AND WASHINGTON AND CORRECTED RESULTS

| | Correction to G | G | Correction to W | W |
|---|---|---|---|---|
| 1851.5 | − ″36 | +2″32 | | |
| 52.5 | − .37 | +1.91 | | |
| 53.5 | − .37 | +2.02 | | |
| 54.5 | − .37 | +2.47 | | |
| 55.5 | − .38 | +2.49 | | |
| 56.5 | − .38 | +2.72 | | |
| 57.5 | − .38 | +2.48 | | |
| 58.5 | − .38 | +2.40 | | |
| 59.5 | − .38 | +2.51 | | |
| 1860.5 | − .37 | +3.18 | | |
| 61.5 | − .37 | +2.59 | | |
| 62.5 | − .37 | +3.45 | | |
| 63.5 | − .36 | +3.67 | | |
| 64.5 | − .35 | +2.86 | | |
| 65.5 | − .35 | +2.55 | | |
| 66.5 | − .35 | +1.97 | +″77 | +2″91 |
| 67.5 | − .34 | +1.46 | +.59 | +1.98 |
| 68.5 | − .32 | +1.33 | +.42 | +1.58 |
| 69.5 | − .31 | + .94 | +.25 | +1.85 |
| 1870.5 | − .30 | + .71 | +.10 | + .17 |
| 71.5 | − .28 | − .18 | −.04 | + .08 |
| 72.5 | − .27 | − .26 | −.18 | − .32 |
| 73.5 | − .26 | −1.29 | −.30 | −1.49 |
| 74.5 | − .25 | −2.11 | −.41 | −1.79 |
| 75.5 | − .24 | −2.32 | −.52 | −1.95 |
| 76.5 | − .22 | −2.45 | −.61 | −2.49 |
| 77.5 | − .20 | −2.09 | −.69 | −1.90 |
| 78.5 | − .18 | −2.31 | −.76 | −2.46 |
| 79.5 | − .16 | −2.35 | −.82 | −1.99 |
| 1880.5 | − .14 | −2.06 | −.87 | −2.07 |
| 81.5 | − .13 | −1.86 | −.91 | −2.03 |
| 82.5 | − .10 | −2.03 | −.94 | −1.73 |
| 83.5 | − .08 | −2.29 | −.96 | −3.11 |
| 84.5 | − .05 | −2.48 | −.97 | −3.71 |
| 85.5 | − .03 | −2.60 | −.97 | −2.99 |
| 86.5 | − .01 | −2.85 | −.96 | −2.70 |
| 87.5 | + .02 | −2.40 | −.94 | −2.59 |
| 88.5 | + .06 | −2.77 | −.91 | −2.88 |
| 89.5 | + .08 | −2.28 | −.87 | −2.24 |
| 1890.5 | + ″11 | −1″97 | −″82 | −2″41 |
| 91.5 | + .13 | −2.91 | −.75 | −2.63 |
| 92.5 | + .14 | −3.34 | | (−2.70) |
| 93.5 | + .16 | −3.54 | | (−2.85) |
| 94.5 | + .17 | −3.06 | +.35 | −3.21 |
| 95.5 | + .18 | −2.79 | +.37 | −2.92 |
| 96.5 | + .17 | −3.09 | +.39 | −2.88 |
| 97.5 | + .17 | −2.77 | +.40 | −3.35 |
| 98.5 | + .17 | −1.97 | +.40 | −2.52 |
| 99.5 | + .15 | −1.81 | +.41 | −1.89 |
| 1900.5 | + .14 | −1.23 | +.41 | −1.66 |
| 01.5 | + .13 | −1.13 | +.41 | −1.01 |
| 02.5 | + .11 | − .07 | +.40 | − .05 |
| 03.5 | + .09 | + .49 | +.39 | + .59 |
| 04.5 | + .05 | + .83 | +.38 | +1.10 |
| 05.5 | + .02 | +1.90 | +.36 | +1.34 |
| 06.5 | − .01 | +2.41 | +.35 | +1.79 |
| 07.5 | − .05 | +2.76 | +.33 | + 2.41 |
| 08.5 | − .09 | +2.89 | +.30 | + 3.34 |
| 09.5 | − .13 | +2.94 | +.27 | +3.55 |
| 1910.5 | − .18 | +4.12 | +.24 | +4.34 |
| 11.5 | − .23 | +4.20 | +.21 | +5.42 |
| 12.5 | − .28 | +4.69 | +.17 | +4.42 |
| 13.5 | − .34 | +5.57 | +.13 | +5.62 |
| 14.5 | − .40 | +6.27 | +.08 | +6.19 |
| 15.5 | − .46 | +6.49 | +.04 | +6.39 |
| 16.5 | − .53 | +7.16 | −.01 | +7.24 |
| 17.5 | − .60 | +7.50 | −.07 | +7.67 |
| 18.5 | − .68 | +7.06 | −.12 | +7.06 |
| 19.5 | − .75 | +6.67 | −.18 | +6.61 |
| 1920.5 | − .83 | +7.13 | −.25 | +7.30 |
| 21.5 | − .90 | +7.37 | −.31 | +7.01 |
| 22.5 | −1.00 | +7.50 | −.38 | +7.22 |

The corresponding values of $B$ listed in Table V were then treated by obtaining parabolic representations by least-squares solutions for overlapping sequences of nine consecutive an-

TABLE V. ANNUAL VALUES OF $-\Sigma$, $B$, AND MID-POINT VALUES OF NINE-YEAR SOLUTIONS

| | $-\Sigma$ | $B_G$ | $B_W$ | $B_{Occ.}$ | $(B_9)_G$ | $(B_9)_W$ | $(B_9)_{Occ.}$ |
|---|---|---|---|---|---|---|---|
| 1851.5 | + 1″75 | + 4″07 | | + 3″70 | | | |
| 52.5 | + 1.41 | + 3.32 | | + 3.24 | | | |
| 53.5 | + 1.07 | + 3.09 | | + 3.17 | | | |
| 54.5 | + 0.73 | + 3.20 | | + 3.09 | | | |
| 55.5 | + 0.39 | + 2.88 | | + 2.92 | + 2″82 | | + 2″77 |
| 56.5 | + 0.04 | + 2.76 | | + 2.19 | + 2.53 | | + 2.56 |
| 57.5 | − 0.31 | + 2.17 | | + 2.37 | + 2.29 | | + 2.40 |
| 58.5 | − 0.65 | + 1.75 | | + 2.54 | + 1.89 | | + 2.28 |
| 59.5 | − 1.00 | + 1.51 | | + 1.79 | + 1.61 | | + 2.12 |
| 1860.5 | − 1.35 | + 1.83 | | + 2.09 | + 1.51 | | + 1.77 |
| 61.5 | − 1.70 | + .89 | | + 1.51 | + 1.42 | | + 1.26 |
| 62.5 | − 2.05 | + 1.40 | | + .55 | + 1.23 | | + .62 |
| 63.5 | − 2.40 | + 1.27 | | − .29 | + .83 | | + .23 |
| 64.5 | − 2.75 | + .11 | | − .17 | + .18 | | − .26 |
| 65.5 | − 3.10 | − .55 | | − .87 | − .48 | | − .78 |
| 66.5 | − 3.45 | − 1.48 | − ″54 | − .47 | − 1.40 | | − 1.13 |
| 67.5 | − 3.80 | − 2.34 | − 1.82 | − 1.86 | − 2.16 | | − 1.48 |
| 68.5 | − 4.15 | − 2.82 | − 2.57 | − 2.74 | − 2.89 | | − 2.21 |
| 69.5 | − 4.49 | − 3.55 | − 2.64 | − 2.42 | − 3.57 | | − 3.08 |

TABLE V.—*Continued*

| | $-\Sigma$ | $\Gamma_G$ | $B_W$ | $B_{Occ.}$ | $(B_9)_G$ | $(B_9)_W$ | $(B_9)_{Occ.}$ |
|---|---|---|---|---|---|---|---|
| 1870.5 | − 4″.84 | − 4″.13 | − 4″.67 | − 3″.73 | − 4″.28 | − 4″.21 | − 4″.14 |
| 71.5 | − 5.19 | − 5.37 | − 5.11 | − 5.58 | − 5.16 | − 5.16 | − 5.08 |
| 72.5 | − 5.53 | − 5.79 | − 5.85 | − 6.35 | − 6.16 | − 6.14 | − 6.15 |
| 73.5 | − 5.87 | − 7.16 | − 7.36 | − 7.07 | − 7.18 | − 7.20 | − 7.15 |
| 74.5 | − 6.21 | − 8.32 | − 8.00 | − 7.93 | − 8.04 | − 7.85 | − 8.03 |
| 75.5 | − 6.55 | − 8.87 | − 8.50 | − 8.59 | − 8.71 | − 8.62 | − 8.64 |
| 76.5 | − 6.89 | − 9.34 | − 9.38 | − 8.96 | − 9.30 | − 9.14 | − 9.10 |
| 77.5 | − 7.23 | − 9.32 | − 9.13 | −10.32 | − 9.63 | − 9.45 | − 9.56 |
| 78.5 | − 7.56 | − 9.87 | −10.02 | − 9.52 | − 9.83 | − 9.79 | − 9.93 |
| 79.5 | − 7.89 | −10.24 | − 9.88 | − 9.78 | −10.03 | − 9.92 | −10.20 |
| 1880.5 | − 8.22 | −10.28 | −10.29 | −10.97 | −10.28 | −10.14 | −10.71 |
| 81.5 | − 8.55 | −10.41 | −10.58 | −11.05 | −10.63 | −10.74 | −11.05 |
| 82.5 | − 8.87 | −10.90 | −10.60 | −11.38 | −10.96 | −11.32 | −11.60 |
| 83.5 | − 9.19 | −11.48 | −12.30 | −12.59 | −11.45 | −11.98 | −11.94 |
| 84.5 | − 9.51 | −11.99 | −13.22 | −11.66 | −11.93 | −12.46 | −12.06 |
| 85.5 | − 9.82 | −12.42 | −12.81 | −12.37 | −12.44 | −12.94 | −12.29 |
| 86.5 | −10.14 | −12.99 | −12.84 | −12.46 | −12.83 | −13.19 | −12.54 |
| 87.5 | −10.44 | −12.84 | −13.03 | −12.58 | −12.99 | −13.14 | −12.74 |
| 88.5 | −10.75 | −13.50 | −13.63 | −13.33 | −13.17 | −13.22 | −13.21 |
| 89.5 | −11.05 | −13.33 | −13.29 | −13.73 | −13.45 | −13.54 | −13.60 |
| 1890.5 | −11.35 | −13.32 | −13.76 | −13.84 | −13.92 | −13.87 | −13.99 |
| 91.5 | −11.64 | −14.55 | −14.27 | −14.45 | −14.49 | −14.29 | −14.45 |
| 92.5 | −11.94 | −15.28 | −14.79 | −14.89 | −14.92 | −14.71 | −14.82 |
| 93.5 | −12.22 | −15.76 | −15.19 | −14.95 | −15.43 | −15.15 | −15.10 |
| 94.5 | −12.51 | −15.57 | −15.72 | −15.72 | −15.84 | −15.59 | −15.40 |
| 95.5 | −12.79 | −15.58 | −15.71 | −15.54 | −15.90 | −15.94 | −15.64 |
| 96.5 | −13.06 | −16.15 | −15.94 | −15.51 | −15.91 | −16.12 | −15.87 |
| 97.5 | −13.33 | −16.10 | −16.68 | −16.17 | −15.84 | −16.20 | −16.02 |
| 98.5 | −13.60 | −15.57 | −16.12 | −16.24 | −15.86 | −16.18 | −16.06 |
| 99.5 | −13.86 | −15.67 | −15.75 | −16.04 | −15.71 | −16.02 | −16.00 |
| 1900.5 | −14.12 | −15.35 | −15.78 | −15.63 | −15.40 | −15.66 | −15.74 |
| 01.5 | −14.37 | −15.50 | −15.38 | −15.50 | −15.18 | −15.15 | −15.28 |
| 02.5 | −14.62 | −14.60 | −14.67 | −14.65 | −14.89 | −14.79 | −14.88 |
| 03.5 | −14.86 | −14.37 | −14.27 | −14.42 | −14.44 | −14.45 | −14.48 |
| 04.5 | −15.10 | −14.27 | −14.00 | −14.11 | −13.99 | −14.14 | −14.14 |
| 05.5 | −15.33 | −13.43 | −13.99 | −13.99 | −13.53 | −13.84 | −13.86 |
| 06.5 | −15.56 | −13.15 | −13.77 | −13.48 | −13.39 | −13.61 | −13.57 |
| 07.5 | −15.78 | −13.02 | −13.37 | −13.31 | −13.15 | −13.36 | −13.24 |
| 08.5 | −16.00 | −13.11 | −12.66 | −12.92 | −12.96 | −12.81 | −12.83 |
| 09.5 | −16.22 | −13.28 | −12.67 | −12.24 | −12.90 | −12.45 | −12.47 |
| 1910.5 | −16.43 | −12.31 | −12.09 | −12.19 | −12.71 | −12.15 | −12.07 |
| 11.5 | −16.64 | −12.44 | −11.22 | −11.58 | −12.35 | −11.88 | −11.65 |
| 12.5 | −16.83 | −12.14 | −12.41 | −11.63 | −11.94 | −11.71 | −11.41 |
| 13.5 | −17.03 | −11.46 | −11.41 | −10.91 | −11.50 | −11.44 | −11.29 |
| 14.5 | −17.22 | −10.95 | −11.03 | −11.03 | −11.10 | −11.10 | −10.96 |
| 15.5 | −17.40 | −10.91 | −11.01 | −11.27 | −10.67 | −10.78 | −10.98 |
| 16.5 | −17.58 | −10.42 | −10.34 | −10.84 | −10.56 | −10.53 | −10.85 |
| 17.5 | −17.75 | −10.25 | −10.08 | −10.30 | −10.65 | −10.58 | −10.86 |
| 18.5 | −17.92 | −10.86 | −10.86 | −11.58 | −10.80 | −10.73 | −10.83 |
| 19.5 | −18.08 | −11.41 | −11.47 | −10.36 | −10.90 | −10.88 | −10.86 |
| 1920.5 | −18.24 | −11.11 | −10.94 | −11.16 | −11.10 | −11.18 | −11.02 |
| 21.5 | −18.39 | −11.02 | −11.38 | −11.22 | −11.18 | −11.44 | −11.19 |
| 22.5 | −18.54 | −11.04 | −11.32 | −11.30 | −11.24 | −11.50 | −11.27 |
| 23.5 | −18.08 | −11.57 | −11.85 | −11.55 | −11.41 | −11.57 | −11.64 |
| 24.5 | −18.21 | −11.62 | −11.95 | −11.86 | −11.74 | −11.81 | −11.82 |
| 25.5 | −18.34 | −12.30 | −11.80 | −12.23 | −12.18 | −12.06 | −12.17 |
| 26.5 | −18.47 | −12.63 | −12.21 | −12.52 | −12.66 | −12.40 | −12.60 |
| 27.5 | −18.59 | −12.98 | −12.92 | −12.92 | −13.09 | −12.80 | −13.01 |
| 28.5 | −18.70 | −13.70 | −13.42 | −13.56 | −13.59 | −13.27 | −13.45 |
| 29.5 | −18.81 | −14.04 | −13.76 | −13.96 | −13.99 | −13.77 | −13.86 |

TABLE V.—*Continued*

| | $-\Sigma$ | $B_G$ | $B_W$ | $B_{Occ.}$ | $(B_9)_G$ | $(B_9)_W$ | $(B_9)_{Occ.}$ |
|---|---|---|---|---|---|---|---|
| 1930.5 | −18″.91 | −14″.35 | −14″.28 | −14″.19 | −14″.42 | −14″.24 | −14″.28 |
| 31.5 | −19.01 | −14.88 | −14.52 | −14.73 | −14.79 | −14.67 | −14.67 |
| 32.5 | −19.10 | −15.10 | −15.08 | −14.94 | −15.17 | −15.10 | −15.04 |
| 33.5 | −19.19 | −15.61 | −15.74 | −15.53 | −15.65 | −15.57 | −15.49 |
| 34.5 | −19.27 | −16.05 | −16.01 | −15.99 | | −16.05 | −16.02 |
| 35.5 | −19.35 | −16.77 | −16.45 | −16.48 | | −16.54 | −16.51 |
| 36.5 | −19.42 | −17.26 | −17.05 | −17.09 | | −17.03 | −17.05 |
| 37.5 | −19.49 | −17.67 | −17.54 | −17.64 | | −17.53 | −17.58 |
| 38.5 | −19.55 | | −17.95 | −17.89 | | −18.00 | −18.02 |
| 39.5 | −19.61 | | −18.55 | −18.55 | | −18.43 | −18.45 |
| 1940.5 | −19.66 | | −18.83 | −18.88 | | −18.81 | −18.70 |
| 41.5 | −19.71 | | −18.98 | −18.94 | | −19.12 | −19.07 |
| 42.5 | −19.75 | | −19.48 | −19.51 | | −19.41 | −19.39 |
| 43.5 | −19.79 | | −19.73 | −19.53 | | −19.68 | −19.68 |
| 44.5 | −19.83 | | −19.85 | −19.94 | | −19.98 | −19.98 |
| 45.5 | −19.86 | | −20.31 | −20.41 | | −20.27 | |
| 46.5 | −19.89 | | −20.59 | −20.65 | | −20.51 | |
| 47.5 | −19.91 | | −20.74 | −20.75 | | | |
| 48.5 | −19.93 | | −21.02 | −21.08 | | | |
| 49.5 | −19.95 | | −21.26 | | | | |
| 1950.5 | −19.96 | | −21.54 | | | | |

nual values. The data listed as $B_9$ in the last three columns of Table V for W, G and Occ. are mid-point values given by the nine-year parabolic solutions. If it is assumed that the parabola absorbs the change in systematic error, then $B-B_9$ for each of the three series will be comparable and may be treated as the deviation of the moon's mean longitude from the nine-year solution. The final values for $B$ were obtained with the formula

$$B_{\text{adopted}} = (B_9)_{\text{Occ.}} + \alpha(B-B_9)_{\text{G}} + \beta(B-B_9)_{\text{W}} + \gamma(B-B_9)_{\text{Occ.}}$$

The factors $\alpha$, $\beta$, $\gamma$ are given by the relative weights in Table VI with the obvious condition,

TABLE VI. RELATIVE WEIGHTS FOR COMBINING ANNUAL MEANS

| | G | W | Occ. |
|---|---|---|---|
| 1820–1854 | 0 | 0 | 1 |
| 1855–1869 | 5 | 0 | 2 |
| 1870–1879 | 5 | 3 | 2 |
| 1880–1922 | 5 | 3 | 5 |

$\alpha+\beta+\gamma=1$. The relative weights were assigned in accordance with the standard deviations given in Table VII. They were derived from a discussion of the differences $B-B_9$ for each of the three series.

TABLE VII. MEAN ERRORS OF ANNUAL MEANS, IN SECONDS OF TIME

| | G | W | Occ. | $F$ | $F-F_Q$ |
|---|---|---|---|---|---|
| 1820–1854 | — | — | $^{s}$95 | $^{s}$95 | $^{s}$87 |
| 1855–1879 | $^{s}$46 | $^{s}$51 | .78 | .33 | .35 |
| 1880–1922 | .51 | .66 | .53 | .40 | .40 |
| 1923–1950 | .17 | .20 | .21 | .16 | .17 |

This method was used for the observations previous to 1923. After 1923 the annual values listed are obtained as the straight average of Greenwich, Washington and Occultations.

The value of $B_9$ was found as the constant term, $a$, in the solution by least squares of nine equations with equal weight,

$$a + tb + t^2c = B(t), \quad t = -4 \text{ to } +4.$$

The value of $a$ obtained from the solution may be expressed explicitly by

$$231\,B_9 = +59\,B(0) + 54\{B(+1)+B(-1)\} + 39\{B(+2)+B(-2)\} + 14\{B(+3)+B(-3)\} - 21\{B(+4)+B(-4)\}.$$

From this expression it is easily derived that the mean error of any of the values of $B$ equals 1.159 times the root mean square value of $B(0)-B_9$ from a large number of such solutions. This factor has been applied in obtaining the mean errors given in Table VII. Throughout this article the mean error will be used to indicate the uncertainty of a result.

The essential feature of the procedure is the adoption of the occultation series as essentially free from systematic errors. The choice of a nine-point formula was made because the use of a period of about this length tends to reduce systematic effects in annual means arising from limb errors and related causes.

The adoption of the occultation data as standard is not an arbitrary step. It was shown by Spencer Jones that for the observations before

TABLE VIIIa. ADOPTED VALUES OF THE FLUCTUATION AND RELATED DATA 1820–1950

| Year | $B$ | $F$ | $\dot{F}_9$ | $F_Q$ | $F-F_Q$ | $\dot{F}_Q$ | $\Delta t$ | $\Delta t_Q$ | | $\frac{d}{dt}\Delta t_Q$ | $10^8 \frac{\Delta\nu}{\nu}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1820.5 | +10″.63 | +19$^s$.36 | | +19$^s$.53 | - $^s$.17 | - $^s$.793 | + 5$^s$.15 | + 5$^s$.32 | +$^d$.000062 | - $^s$.546 | +1.73 |
| 21.5 | +10.21 | +18.60 | | +18.76 | - .16 | - .754 | + 4.64 | + 4.80 | + 56 | - .501 | +1.59 |
| 22.5 | +10.47 | +19.07 | | +18.03 | +1.04 | - .716 | + 5.36 | + 4.32 | + 50 | - .457 | +1.45 |
| 23.5 | + 9.30 | +16.94 | | +17.33 | - .39 | - .678 | + 3.49 | + 3.88 | + 45 | - .413 | +1.31 |
| 24.5 | + 9.03 | +16.45 | - $^s$.600 | +16.67 | - .22 | - .639 | + 3.27 | + 3.49 | + 40 | - .368 | +1.17 |
| 25.5 | + 8.43 | +15.35 | - .667 | +16.05 | - .70 | - .601 | + 2.45 | + 3.15 | + 36 | - .324 | +1.03 |
| 26.5 | + 9.14 | +16.65 | - .613 | +15.47 | +1.18 | - .562 | + 4.03 | + 2.85 | + 33 | - .279 | + .88 |
| 27.5 | + 7.74 | +14.10 | - .563 | +14.93 | - .83 | - .524 | + 1.76 | + 2.59 | + 30 | - .235 | + .74 |
| 28.5 | + 8.43 | +15.35 | - .500 | +14.42 | + .93 | - .486 | + 3.30 | + 2.37 | + 27 | - .191 | + .61 |
| 29.5 | + 7.00 | +12.75 | - .441 | +13.95 | -1.20 | - .471 | + 1.00 | + 2.20 | + 25 | - .170 | + .54 |
| 1830.5 | + 7.61 | +13.86 | - .598 | +13.45 | + .41 | - .522 | + 2.42 | + 2.01 | +.000023 | - .215 | + .68 |
| 31.5 | + 6.63 | +12.08 | - .564 | +12.91 | - .83 | - .574 | + 0.94 | + 1.77 | + 20 | - .261 | + .83 |
| 32.5 | + 7.21 | +13.13 | - .657 | +12.31 | + .82 | - .625 | + 2.31 | + 1.49 | + 17 | - .306 | + .97 |
| 33.5 | + 7.01 | +12.77 | - .590 | +11.66 | +1.11 | - .676 | + 2.27 | + 1.16 | + 13 | - .351 | +1.11 |
| 34.5 | + 5.46 | + 9.95 | - .695 | +10.96 | -1.01 | - .728 | - 0.22 | + 0.79 | + 9 | - .397 | +1.26 |
| 35.5 | + 5.42 | + 9.87 | - .616 | +10.20 | - .33 | - .779 | + 0.03 | + 0.36 | + 4 | - .442 | +1.40 |
| 36.5 | + 5.19 | + 9.45 | - .653 | + 9.46 | - .01 | - .670 | - 0.05 | - 0.04 | 0 | - .327 | +1.04 |
| 37.5 | + 4.99 | + 9.09 | - .456 | + 8.87 | + .22 | - .515 | - 0.06 | - 0.28 | - 3 | - .166 | + .53 |
| 38.5 | + 4.52 | + 8.23 | - .363 | + 8.43 | - .20 | - .360 | - 0.57 | - 0.37 | - 4 | - .005 | + .02 |
| 39.5 | + 4.65 | + 8.47 | - .205 | + 8.15 | + .32 | - .205 | + 0.03 | - 0.29 | - 3 | + .156 | - .49 |
| 1840.5 | + 4.18 | + 7.61 | - .218 | + 8.02 | - .41 | - .050 | - 0.47 | - 0.06 | -.000001 | + .317 | -1.00 |
| 41.5 | + 4.77 | + 8.69 | - .231 | + 7.96 | + .73 | - .063 | + 0.98 | + 0.25 | + 3 | + .310 | - .98 |
| 42.5 | + 3.55 | + 6.47 | - .091 | + 7.89 | -1.42 | - .082 | - 0.86 | + 0.56 | + 6 | + .297 | - .94 |
| 43.5 | + 5.16 | + 9.40 | - .148 | + 7.80 | +1.60 | - .100 | + 2.45 | + 0.85 | + 10 | + .285 | - .90 |
| 44.5 | + 3.72 | + 6.78 | - .027 | + 7.69 | - .91 | - .119 | + 0.22 | + 1.13 | + 13 | + .272 | - .86 |
| 45.5 | + 3.59 | + 6.54 | - .162 | + 7.56 | -1.02 | - .138 | + 0.37 | + 1.39 | + 16 | + .259 | - .82 |
| 46.5 | + 4.70 | + 8.56 | - .108 | + 7.42 | +1.14 | - .156 | + 2.79 | + 1.65 | + 19 | + .247 | - .78 |
| 47.5 | + 3.60 | + 6.56 | - .207 | + 7.25 | - .69 | - .175 | + 1.20 | + 1.89 | + 22 | + .234 | - .74 |
| 48.5 | + 4.65 | + 8.47 | - .122 | + 7.07 | +1.40 | - .194 | + 3.52 | + 2.12 | + 25 | + .221 | - .70 |
| 49.5 | + 3.13 | + 5.70 | - .206 | + 6.86 | -1.16 | - .212 | + 1.17 | + 2.33 | + 27 | + .209 | - .66 |
| 1850.5 | + 3.72 | + 6.78 | - .303 | + 6.64 | + .14 | - .231 | + 2.67 | + 2.53 | +.000029 | + .196 | - .62 |
| 51.5 | + 3.70 | + 6.74 | - .246 | + 6.40 | + .34 | - .250 | + 3.06 | + 2.72 | + 31 | + .183 | - .58 |
| 52.5 | + 3.24 | + 5.90 | - .325 | + 6.14 | - .24 | - .268 | + 2.66 | + 2.90 | + 34 | + .171 | - .54 |
| 53.5 | + 3.17 | + 5.77 | - .255 | + 5.86 | - .09 | - .287 | + 2.97 | + 3.06 | + 35 | + .158 | - .50 |
| 54.5 | + 3.09 | + 5.63 | - .353 | + 5.57 | + .06 | - .306 | + 3.28 | + 3.22 | + 37 | + .145 | - .46 |
| 55.5 | + 2.86 | + 5.21 | - .368 | + 5.25 | - .04 | - .324 | + 3.31 | + 3.35 | + 39 | + .133 | - .42 |
| 56.5 | + 2.62 | + 4.77 | - .318 | + 4.90 | - .13 | - .385 | + 3.33 | + 3.46 | + 40 | + .078 | - .25 |
| 57.5 | + 2.31 | + 4.21 | - .429 | + 4.48 | - .27 | - .448 | + 3.23 | + 3.50 | + 41 | + .021 | - .07 |
| 58.5 | + 2.25 | + 4.10 | - .505 | + 4.01 | + .09 | - .510 | + 3.60 | + 3.51 | + 41 | - .035 | + .11 |
| 59.5 | + 1.95 | + 3.55 | - .557 | + 3.46 | + .09 | - .573 | + 3.52 | + 3.43 | + 40 | - .092 | + .29 |
| 1860.5 | + 2.09 | + 3.81 | - .649 | + 2.86 | + .95 | - .636 | + 4.27 | + 3.32 | +.000038 | - .149 | + .47 |
| 61.5 | + .95 | + 1.73 | - .751 | + 2.19 | - .46 | - .699 | + 2.68 | + 3.14 | + 36 | - .206 | + .65 |
| 62.5 | + .72 | + 1.31 | - .811 | + 1.46 | - .15 | - .762 | + 2.75 | + 2.90 | + 34 | - .263 | + .83 |
| 63.5 | + .40 | + .73 | - .868 | + .67 | + .06 | - .825 | + 2.67 | + 2.61 | + 30 | - .320 | +1.01 |
| 64.5 | - .28 | - .51 | - .921 | - .19 | - .32 | - .888 | + 1.94 | + 2.26 | + 26 | - .377 | +1.19 |
| 65.5 | - .86 | - 1.57 | - .890 | - 1.11 | - .46 | - .951 | + 1.39 | + 1.85 | + 21 | - .434 | +1.38 |
| 66.5 | - 1.00 | - 1.82 | -1.034 | - 2.09 | + .27 | -1.013 | + 1.66 | + 1.39 | + 16 | - .490 | +1.55 |
| 67.5 | - 1.72 | - 3.13 | -1.201 | - 3.14 | + .01 | -1.092 | + 0.88 | + 0.87 | + 10 | - .564 | +1.79 |
| 68.5 | - 2.31 | - 4.21 | -1.312 | - 4.34 | + .13 | -1.312 | + 0.33 | + 0.20 | + 2 | - .778 | +2.47 |
| 69.5 | - 2.88 | - 5.25 | -1.485 | - 5.76 | + .51 | -1.531 | - 0.17 | - 0.68 | - 8 | - .991 | +3.14 |
| 1870.5 | - 4.12 | - 7.50 | -1.660 | - 7.40 | - .10 | -1.751 | - 1.88 | - 1.78 | -.000021 | -1.205 | +3.82 |
| 71.5 | - 5.27 | - 9.60 | -1.695 | - 9.26 | - .34 | -1.971 | - 3.43 | - 3.09 | - 36 | -1.419 | +4.50 |
| 72.5 | - 5.92 | -10.78 | -1.665 | -11.21 | + .43 | -1.846 | - 4.05 | - 4.48 | - 52 | -1.288 | +4.08 |
| 73.5 | - 7.17 | -13.06 | -1.532 | -12.94 | - .12 | -1.607 | - 5.77 | - 5.65 | - 65 | -1.043 | +3.31 |
| 74.5 | - 8.19 | -14.92 | -1.332 | -14.43 | - .49 | -1.367 | - 7.06 | - 6.57 | - 76 | - .797 | +2.53 |
| 75.5 | - 8.67 | -15.79 | -1.134 | -15.67 | - .12 | -1.127 | - 7.36 | - 7.24 | - 84 | - .551 | +1.75 |
| 76.5 | - 9.16 | -16.68 | -1.004 | -16.68 | .00 | - .888 | - 7.67 | - 7.67 | - 89 | - .306 | + .97 |
| 77.5 | - 9.46 | -17.23 | - .817 | -17.46 | + .23 | - .728 | - 7.64 | - 7.87 | - 91 | - .140 | + .44 |
| 78.5 | - 9.94 | -18.11 | - .711 | -18.18 | + .07 | - .702 | - 7.93 | - 8.00 | - 93 | - .108 | + .34 |
| 79.5 | -10.21 | -18.60 | - .753 | -18.87 | + .27 | - .676 | - 7.82 | - 8.09 | - 94 | - .076 | + .24 |
| 1880.5 | -10.84 | -19.74 | - .721 | -19.53 | - .21 | - .650 | - 8.35 | - 8.14 | -.000094 | - .044 | + .14 |
| 81.5 | -10.93 | -19.91 | - .678 | -20.17 | + .26 | - .624 | - 7.91 | - 8.17 | - 95 | - .012 | + .04 |
| 82.5 | -11.33 | -20.64 | - .615 | -20.78 | + .14 | - .598 | - 8.03 | - 8.17 | - 95 | + .020 | - .06 |
| 83.5 | -12.28 | -22.37 | - .547 | -21.37 | -1.00 | - .572 | - 9.14 | - 8.14 | - 94 | + .052 | - .16 |
| 84.5 | -12.10 | -22.04 | - .544 | -21.93 | - .11 | - .546 | - 8.18 | - 8.07 | - 93 | + .084 | - .27 |
| 85.5 | -12.28 | -22.37 | - .547 | -22.46 | + .09 | - .520 | - 7.88 | - 7.97 | - 92 | + .116 | - .37 |
| 86.5 | -12.49 | -22.75 | - .496 | -22.97 | + .22 | - .495 | - 7.62 | - 7.84 | - 91 | + .147 | - .47 |
| 87.5 | -12.60 | -22.95 | - .517 | -23.45 | + .50 | - .507 | - 7.17 | - 7.67 | - 89 | + .141 | - .45 |
| 88.5 | -13.49 | -24.57 | - .652 | -24.01 | - .56 | - .606 | - 8.14 | - 7.58 | - 88 | + .048 | - .15 |
| 89.5 | -13.55 | -24.68 | - .700 | -24.67 | - .01 | - .705 | - 7.59 | - 7.58 | - 88 | - .045 | + .14 |

TABLE VIIIa.—*Continued*

| Year | $B$ | $F$ | $\dot{F}_9$ | $F_Q$ | $F-F_Q$ | $\dot{F}_Q$ | $\Delta t$ | $\Delta t_Q$ | | $\frac{d}{dt}\Delta t_Q$ | $10^8\frac{\Delta\nu}{\nu}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1890.5 | -13″.68 | -24″.92 | - $^{s}$.715 | -25″.42 | + ″.50 | - $^{s}$.803 | - 7$^{s}$.17 | - 7$^{s}$.67 | -$^{d}$.000089 | - $^{s}$.137 | + .43 |
| 91.5 | -14.47 | -26.36 | - .660 | -26.27 | - .09 | - .902 | - 7.94 | - 7.85 | - 91 | - .230 | + .73 |
| 92.5 | -15.00 | -27.32 | - .579 | -27.13 | - .19 | - .783 | - 8.23 | - 8.04 | - 93 | - .105 | + .33 |
| 93.5 | -15.18 | -27.65 | - .598 | -27.84 | + .19 | - .630 | - 7.88 | - 8.07 | - 93 | + .054 | - .17 |
| 94.5 | -15.45 | -28.14 | - .502 | -28.39 | + .25 | - .477 | - 7.68 | - 7.93 | - 92 | + .213 | - .67 |
| 95.5 | -15.43 | -28.10 | - .347 | -28.79 | + .69 | - .324 | - 6.94 | - 7.63 | - 88 | + .372 | -1.18 |
| 96.5 | -15.78 | -28.74 | - .214 | -29.04 | + .30 | - .171 | - 6.89 | - 7.19 | - 83 | + .531 | -1.68 |
| 97.5 | -16.29 | -29.67 | - .105 | -29.13 | - .54 | - .018 | - 7.11 | - 6.57 | - 76 | + .690 | -2.19 |
| 98.5 | -16.00 | -29.14 | + .097 | -29.07 | - .07 | + .135 | - 5.87 | - 5.80 | - 67 | + .849 | -2.69 |
| 99.5 | -15.94 | -29.03 | + .280 | -28.86 | - .17 | + .288 | - 5.04 | - 4.87 | - 56 | +1.008 | -3.19 |
| 1900.5 | -15.71 | -28.61 | + .457 | -28.50 | - .11 | + .441 | - 3.90 | - 3.79 | -.000044 | +1.168 | -3.70 |
| 01.5 | -15.54 | -28.31 | + .579 | -27.98 | - .33 | + .594 | - 2.87 | - 2.54 | - 29 | +1.327 | -4.21 |
| 02.5 | -14.69 | -26.76 | + .618 | -27.31 | + .55 | + .746 | - 0.58 | - 1.13 | - 13 | +1.484 | -4.70 |
| 03.5 | -14.39 | -26.21 | + .647 | -26.57 | + .36 | + .725 | + 0.71 | + 0.35 | + 4 | +1.470 | -4.66 |
| 04.5 | -14.20 | -25.86 | + .641 | -25.86 | .00 | + .705 | + 1.80 | + 1.80 | + 21 | +1.455 | -4.61 |
| 05.5 | -13.91 | -25.34 | + .616 | -25.16 | - .18 | + .685 | + 3.08 | + 3.26 | + 38 | +1.441 | -4.57 |
| 06.5 | -13.48 | -24.55 | + .597 | -24.49 | - .06 | + .664 | + 4.63 | + 4.69 | + 54 | +1.426 | -4.52 |
| 07.5 | -13.22 | -24.08 | + .653 | -23.83 | - .25 | + .644 | + 5.86 | + 6.11 | + 71 | +1.412 | -4.47 |
| 08.5 | -12.90 | -23.50 | + .630 | -23.20 | - .30 | + .623 | + 7.21 | + 7.51 | + 87 | +1.397 | -4.43 |
| 09.5 | -12.58 | -22.91 | + .631 | -22.59 | - .32 | + .603 | + 8.58 | + 8.90 | + 103 | +1.383 | -4.38 |
| 1910.5 | -11.95 | -21.77 | + .607 | -21.99 | + .22 | + .582 | +10.50 | +10.28 | +.000119 | +1.368 | -4.34 |
| 11.5 | -11.51 | -20.96 | + .518 | -21.42 | + .46 | + .557 | +12.10 | +11.64 | + 135 | +1.349 | -4.27 |
| 12.5 | -11.73 | -21.37 | + .458 | -20.91 | - .46 | + .469 | +12.49 | +12.95 | + 150 | +1.267 | -4.02 |
| 13.5 | -11.12 | -20.25 | + .418 | -20.48 | + .23 | + .382 | +14.41 | +14.18 | + 164 | +1.186 | -3.76 |
| 14.5 | -10.91 | -19.87 | + .252 | -20.15 | + .28 | + .295 | +15.59 | +15.31 | + 177 | +1.105 | -3.50 |
| 15.5 | -11.24 | -20.47 | + .161 | -19.89 | - .58 | + .207 | +15.81 | +16.39 | + 190 | +1.023 | -3.24 |
| 16.5 | -10.75 | -19.58 | + .107 | -19.73 | + .15 | + .120 | +17.52 | +17.37 | + 201 | + .942 | -2.99 |
| 17.5 | -10.38 | -18.91 | - .010 | -19.65 | + .74 | + .033 | +19.01 | +18.27 | + 211 | + .861 | -2.73 |
| 18.5 | -11.18 | -20.36 | - .056 | -19.67 | - .69 | - .055 | +18.39 | +19.08 | + 221 | + .779 | -2.47 |
| 19.5 | -11.00 | -20.04 | - .129 | -19.76 | - .28 | - .142 | +19.55 | +19.83 | + 230 | + .698 | -2.21 |
| 1920.5 | -11.02 | -20.07 | - .248 | -19.95 | - .12 | - .229 | +20.36 | +20.48 | +.000237 | + .616 | -1.95 |
| 21.5 | -11.13 | -20.27 | - .312 | -20.22 | - .05 | - .317 | +21.01 | +21.06 | + 244 | + .535 | -1.70 |
| 22.5 | -11.16 | -20.33 | - .320 | -20.58 | + .25 | - .404 | +21.81 | +21.56 | + 250 | + .454 | -1.44 |
| 23.5 | -11.66 | -21.24 | - .445 | -21.03 | - .21 | - .492 | +21.76 | +21.97 | + 254 | + .372 | -1.18 |
| 24.5 | -11.81 | -21.51 | - .566 | -21.57 | + .06 | - .579 | +22.35 | +22.29 | + 258 | + .291 | - .92 |
| 25.5 | -12.11 | -22.06 | - .654 | -22.19 | + .13 | - .666 | +22.68 | +22.55 | + 261 | + .210 | - .67 |
| 26.5 | -12.45 | -22.68 | - .714 | -22.90 | + .22 | - .754 | +22.94 | +22.72 | + 263 | + .128 | - .41 |
| 27.5 | -12.94 | -23.57 | - .737 | -23.68 | + .11 | - .789 | +22.93 | +22.82 | + 264 | + .099 | - .31 |
| 28.5 | -13.56 | -24.70 | - .768 | -24.47 | - .23 | - .779 | +22.69 | +22.92 | + 265 | + .115 | - .36 |
| 29.5 | -13.92 | -25.35 | - .792 | -25.24 | - .11 | - .769 | +22.94 | +23.05 | + 267 | + .131 | - .42 |
| 1930.5 | -14.27 | -25.99 | - .792 | -26.01 | + .02 | - .759 | +23.20 | +23.18 | +.000268 | + .147 | - .47 |
| 31.5 | -14.71 | -26.79 | - .792 | -26.76 | - .03 | - .749 | +23.31 | +23.34 | + 270 | + .162 | - .51 |
| 32.5 | -15.04 | -27.39 | - .809 | -27.52 | + .13 | - .789 | +23.63 | +23.50 | + 272 | + .129 | - .41 |
| 33.5 | -15.63 | -28.47 | - .853 | -28.34 | - .13 | - .851 | +23.47 | +23.60 | + 273 | + .073 | - .23 |
| 34.5 | -16.02 | -29.18 | - .864 | -29.22 | + .04 | - .913 | +23.68 | +23.64 | + 274 | + .017 | - .05 |
| 35.5 | -16.57 | -30.18 | - .884 | -30.17 | - .01 | - .975 | +23.62 | +23.63 | + 273 | - .039 | + .12 |
| 36.5 | -17.13 | -31.20 | - .877 | -31.15 | - .05 | - .950 | +23.53 | +23.58 | + 273 | - .008 | + .03 |
| 37.5 | -17.62 | -32.09 | - .807 | -32.05 | - .04 | - .863 | +23.59 | +23.63 | + 273 | + .085 | - .27 |
| 38.5 | -17.92 | -32.64 | - .774 | -32.87 | + .23 | - .776 | +23.99 | +23.76 | + 275 | + .178 | - .56 |
| 39.5 | -18.55 | -33.79 | - .697 | -33.60 | - .19 | - .688 | +23.80 | +23.99 | + 278 | + .272 | - .86 |
| 1940.5 | -18.86 | -34.35 | - .628 | -34.25 | - .10 | - .601 | +24.20 | +24.30 | +.000281 | + .365 | -1.16 |
| 41.5 | -18.96 | -34.53 | - .598 | -34.81 | + .28 | - .538 | +24.99 | +24.71 | + 286 | + .434 | -1.38 |
| 42.5 | -19.50 | -35.52 | - .576 | -35.34 | - .18 | - .527 | +24.97 | +25.15 | + 291 | + .450 | -1.43 |
| 43.5 | -19.63 | -35.75 | - .524 | -35.86 | + .11 | - .517 | +25.72 | +25.61 | + 296 | + .467 | -1.48 |
| 44.5 | -19.90 | -36.25 | - .519 | -36.38 | + .13 | - .507 | +26.21 | +26.08 | + 302 | + .483 | -1.53 |
| 45.5 | -20.36 | -37.08 | - .510 | -36.88 | - .20 | - .496 | +26.37 | +26.57 | + 308 | + .500 | -1.58 |
| 46.5 | -20.62 | -37.56 | - .477 | -37.37 | - .19 | - .486 | +26.89 | +27.08 | + 313 | + .516 | -1.64 |
| 47.5 | -20.74 | -37.78 | | -37.85 | + .07 | - .475 | +27.68 | +27.61 | + 320 | + .532 | -1.69 |
| 48.5 | -21.05 | -38.34 | | -38.32 | - .02 | - .465 | +28.13 | +28.15 | + 326 | + .549 | -1.74 |
| 49.5 | -21.26 | -38.72 | | -38.78 | + .06 | - .454 | +28.76 | +28.70 | + 332 | + .565 | -1.79 |
| 1950.5 | -21.54 | -39.23 | | -39.23 | .00 | - .444 | +29.28 | +29.28 | +.000339 | + .582 | -1.84 |

1830 the differences between the occultations and the Greenwich observations run closely parallel to the differences between the sun's tabular longitude and the Greenwich observed longitude. Thus the Greenwich observations of both sun and moon appear to have been affected by similar errors. It is thus established that it is extremely unlikely that the large discordances between the meridian and occultation results before 1830 can be due to the occultations.[19]

TABLE VIIIb. ADOPTED VALUES OF THE FLUCTUATION AND RELATED DATA 1621–1812

| | $B$ | $F$ | $\Delta t$ |
|---|---|---|---|
| 1621 | $+23''$ | $+42^{s}$ | $+98^{s}$ |
| 1635 | $-3$ | $-5$ | $+38$ |
| 1639 | $-29$ | $-53$ | $-13$ |
| 1645 | $-12$ | $-22$ | $+13$ |
| 1653 | $-21$ | $-38$ | $-10$ |
| 1662 | $-15$ | $-27$ | $-5$ |
| 1681 | $-12''.7$ | $-23^{s}.1$ | $-13^{s}.5$ |
| 1710 | $-3.9$ | $-7.1$ | $-12.0$ |
| 1727 | $+1.9$ | $+3.5$ | $-7.6$ |
| 1738 | $+6.2$ | $+11.3$ | $-2.9$ |
| 1747 | $+8.7$ | $+15.8$ | $-0.4$ |
| 1760.9 | $+11.2$ | $+20.4$ | $+2.1$ |
| 1774.1 | $+14.0$ | $+25.5$ | $+6.6$ |
| 1785.1 | $+15.1$ | $+27.5$ | $+8.3$ |
| 1792.6 | $+14.4$ | $+26.2$ | $+7.4$ |
| 1801.8 | $+12.9$ | $+23.5$ | $+5.7$ |
| 1811.9 | $+11.4$ | $+20.8$ | $+4.7$ |

TABLE VIIIc. REVISED VALUES OBTAINED BY COMBINING OCCULTATIONS AND GREENWICH MERIDIAN RESULTS, 1824 TO 1854

| Year | $B$ | $F$ | $F-F_Q$ |
|---|---|---|---|
| 1824.5 | $+9''.20$ | $+16^{s}.76$ | $+{}^{s}.09$ |
| 25.5 | $+8.85$ | $+16.12$ | $+.07$ |
| 26.5 | $+8.59$ | $+15.65$ | $+.18$ |
| 27.5 | $+7.96$ | $+14.50$ | $-.43$ |
| 28.5 | $+7.68$ | $+13.99$ | $-.43$ |
| 29.5 | $+7.87$ | $+14.33$ | $+.38$ |
| 1830.5 | $+7.59$ | $+13.82$ | $+.37$ |
| 31.5 | $+6.91$ | $+12.59$ | $-.32$ |
| 32.5 | $+6.57$ | $+11.97$ | $-.34$ |
| 33.5 | $+6.60$ | $+12.02$ | $+.36$ |
| 34.5 | $+5.99$ | $+10.91$ | $-.05$ |
| 35.5 | $+5.65$ | $+10.29$ | $+.09$ |
| 36.5 | $+5.13$ | $+9.34$ | $-.12$ |
| 37.5 | $+4.77$ | $+8.69$ | $-.18$ |
| 38.5 | $+4.55$ | $+8.29$ | $-.14$ |
| 39.5 | $+4.74$ | $+8.63$ | $+.48$ |
| 1840.5 | $+4.31$ | $+7.85$ | $-.17$ |
| 41.5 | $+4.43$ | $+8.07$ | $+.11$ |
| 42.5 | $+3.96$ | $+7.21$ | $-.68$ |
| 43.5 | $+4.53$ | $+8.25$ | $+.45$ |
| 44.5 | $+4.10$ | $+7.47$ | $-.22$ |
| 45.5 | $+4.28$ | $+7.80$ | $+.24$ |
| 46.5 | $+3.87$ | $+7.05$ | $-.37$ |
| 47.5 | $+3.68$ | $+6.70$ | $-.55$ |
| 48.5 | $+4.47$ | $+8.14$ | $+1.07$ |
| 49.5 | $+3.68$ | $+6.70$ | $-.16$ |
| 1850.5 | $+3.33$ | $+6.07$ | $-.57$ |
| 51.5 | $+3.80$ | $+6.92$ | $+.52$ |
| 52.5 | $+3.21$ | $+5.85$ | $-.29$ |
| 53.5 | $+3.09$ | $+5.63$ | $-.23$ |
| 54.5 | $+3.06$ | $+5.57$ | $.00$ |

After 1830 the systematic differences between occultations and Greenwich become much smaller. I have assumed that these smaller discordances also may be ascribed to the meridian results. This appears justified by the excellent accordance between the Cape occultation results in the years 1880–1908 and Spencer Jones's revision of Newcomb's occultations during these same years. This is shown graphically in Figure 2 to which further reference is made later in this section.

For the years 1750 to 1850 Greenwich meridian results for Cowell periods are available. Before 1820, the few occultation normals do not permit a reduction of the Greenwich observations to the occultation system by a method comparable to that used for the data after 1850. In view of the large systematic differences between the Greenwich results and the occultations and the difficulties involved in dealing with Cowell periods for the Greenwich results and annual values for the occultations I also abandoned an attempt to use the Greenwich observations in the years 1820 to 1850.

*Note added in proof*. After the manuscript had been sent to the printer I returned to this question, and found that the annual values of $F$ before 1850 can be improved considerably by using the Greenwich residuals for Cowell periods.[16] By linear interpolation annual values for the years 1820.5 to 1858.5 are obtained. This is the same procedure as was used by E. W. Brown for the Greenwich observations after 1850.[19a] These annual values are combined with the occultation residuals with the formula

$$B_{\text{adopted}} = (B_9)_{\text{Occ}} + .7(B - B_9)_{\text{G}} + .3(B - B_9)_{\text{Occ}},$$

the relative weights having been obtained from the root mean square values of $B - B_9$ for the two series. The relevant data are given in Table VIIIc. The circumstance that $F_Q$ could be left unchanged shows that no significant systematic change was introduced. The standard deviation of $F - F_Q$ for the 31 years listed in Table VIIIc is reduced to ${}^{s}.38$ as compared with ${}^{s}.83$ for the corresponding data in Table VIIIa.

The adopted values of $B$ are listed in Table VIIIa. The values of $F$ in this table were obtained by multiplying $B$ by the factor 1.821; hence $F$ is the fluctuation in the earth's rotation in seconds of time. The values of $\dot{F}_9$ were obtained from $F$ by the solution of $b$ from the nine point formula.

$$\begin{aligned} 60\,\dot{F}_9 = {} & \{F(+1) - F(-1)\} \\ & +2\{F(+2) - F(-2)\} \\ & +3\{F(+3) - F(-3)\} \\ & +4\{F(+4) - F(-4)\}. \end{aligned}$$

These derivatives are plotted in Figure 1b.

Let $\mu$ be the mean error of any of the annual values of $F$, assumed to be independent. The

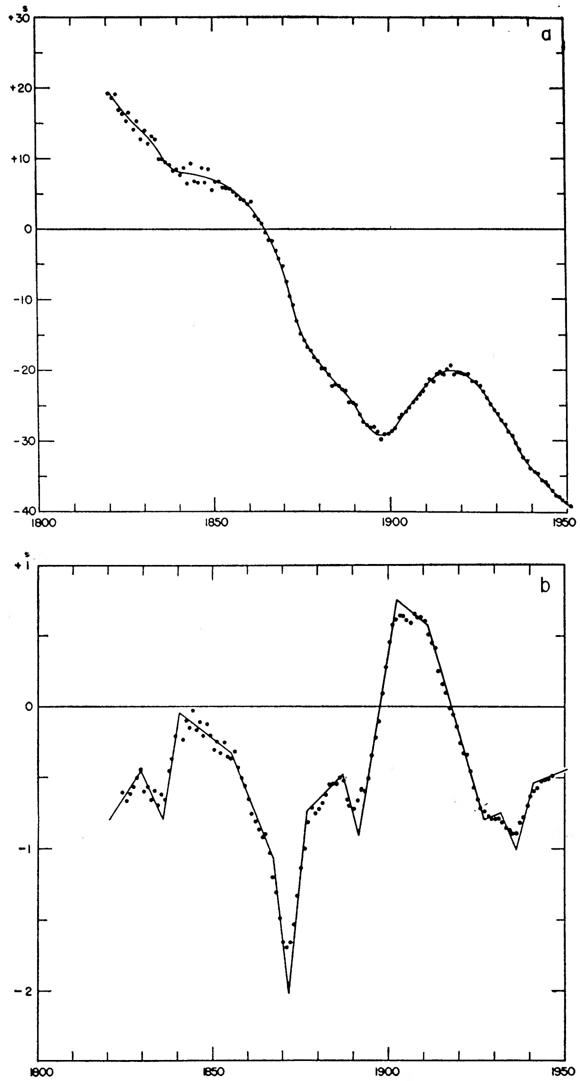


Figure 1a. Dots, the annual values of the fluctuation $F$ expressed in seconds of time; curve, a representation $F_Q$ by arcs of parabolas. Figure 1b. Dots, the derivative of $F$ obtained by a nine-point formula; straight lines, the derivative of $F_Q$.

mean error of $\dot{F}_9$ obtained with the formula given is then $.129\ \mu$.

A modified formula, giving independent values of $\dot{F}$ with an interval of 5 years, is

$$\dot{F} = .08\{F(+4) - F(-4)\} + .06\{F(+3) - F(-3)\},$$

the mean error of $\dot{F}$ being $.141\ \mu$.

For the years 1851 to 1922 the mean error of an annual value of $F$ may be taken to be $\mu = {}^{s}.40$. The mean error of any of the derivatives plotted is then ${}^{s}.052$, but successive values are, of course, not independent. The reality of the large range in $\dot{F}$ over this interval in years can be questioned only if it is supposed that the annual values of the occultation results are affected by large systematic errors that would have to change progressively over a sequence of years by amounts large compared with the mean error $\mu$ in order to simulate the change in the derivative.

The reductions of the occultations were made with star positions referred to a well-defined system. Occultations so reduced in recent years have been demonstrated to yield annual means essentially free from systematic errors if the observations are well distributed over the year. This condition is not satisfied for every single year. The effect of non-uniform distribution over any one year merely tends to increase the mean error of the annual values if the distributions in successive years are independent. I have examined this point with some care and cannot find any cause for the presence of systematic errors large compared with the accidental errors.

Figure 1b suggests that the derivative of the fluctuation curve may consist of a series of straight lines. However, this curve is affected by systematic deviations from the true derivative of the fluctuation curve on account of the rounding effect that must be present near the junctions of the straight line sections. A better approximation to the derivative was obtained by representing the fluctuation curve by parabolic formulae over intervals for which the derivative curve was apparently nearly straight. The difficulty of properly fitting such parabolic arcs together is best illustrated by the formulae. Let for an interval $t_0$ to $t_1$ the parabolic representation be

$$\text{(I)} \qquad a_0 + b_0\,(t-t_0) + c_0\,(t-t_0)^2,$$

and from $t_1$ to $t_2$

$$\text{(II)} \qquad a_1 + b_1\,(t-t_1) + c_1\,(t-t_1)^2,$$

The derivatives will agree for a time $t'$ if

$$b_0 + 2c_0\,(t'-t_0) = b_1 + 2c_1\,(t'-t_1).$$

The time $t'$ was obtained from this equation, but $t'$ substituted into (I) and (II) will as a rule give different values. If the two expressions did agree for $t = t'$, the difference between the two formulae for a time $t > t'$ would be

$$\text{(II)} - \text{(I)} = (c_1 - c_0)\,(t-t')^2.$$

In a second approximation, after provisional

TABLE IX. COEFFICIENTS FOR REPRESENTING PARABOLIC ARCS IN $F_Q$ AND $\Delta t_Q$

| | Representation of $F_Q$ | | | Representation of $\Delta t_Q$ | | |
|---|---|---|---|---|---|---|
| $t$ | $a$ | $b$ | $c$ | $a$ | $b$ | $c$ |
| 1820.5 | +19ˢ.5343 | −ˢ.79254 | +ˢ.01917 | + 5ˢ.3202 | −ˢ.54556 | +ˢ.022165 |
| 1829.241 | +14.0714 | − .45741 | −.02570 | + 2.2450 | − .15808 | −.022705 |
| 1835.725 | +10.0251 | − .79069 | +.07757 | + 0.2655 | − .45252 | +.080565 |
| 1840.532 | + 8.0167 | − .04493 | −.00933 | − 0.0482 | + .32203 | −.006335 |
| 1855.555 | + 5.2360 | − .32526 | −.03144 | + 3.3599 | + .13169 | −.028445 |
| 1867.402 | − 3.0299 | −1.07020 | −.10991 | + 0.9279 | − .54229 | −.106915 |
| 1871.750 | − 9.7610 | −2.02597 | +.11980 | − 3.4513 | −1.47202 | +.122795 |
| 1877.126 | −17.1902 | − .73788 | +.01298 | − 7.8159 | − .15173 | +.015975 |
| 1887.188 | −23.3007 | − .47667 | −.04929 | − 7.7253 | + .16975 | −.046295 |
| 1891.636 | −26.3961 | − .91516 | +.07652 | − 7.8862 | − .24209 | +.079515 |
| 1902.491 | −27.3138 | + .74609 | −.01023 | − 1.1448 | +1.48418 | −.007235 |
| 1911.425 | −21.4647 | + .56330 | −.04368 | +11.5374 | +1.35490 | −.040685 |
| 1926.964 | −23.2586 | − .79419 | +.00493 | +22.7673 | + .09049 | +.007925 |
| 1931.816 | −26.9959 | − .74635 | −.03097 | +23.3929 | + .16739 | −.027975 |
| 1935.922 | −30.5826 | −1.00067 | +.04367 | +23.6086 | − .06234 | +.046665 |
| 1941.183 | −34.6384 | − .54118 | +.00522 | +24.5722 | + .42867 | +.008215 |
| 1950.5 | −39.2274 | − .44391 | | +29.2792 | + .58174 | |

values of the coefficients $c$ and the times of discontinuity $t'$ were available, this difference with the sign reversed was applied usually to five annual values preceding the beginning and following the end of each section. New solutions were then made with these overlaps. In about one-third of the total number of sections a third approximation was required. The result of these calculations is the function $F_Q$, consisting of sixteen parabolic sections between 1820.5 and 1950.5 This function, the differences $F-F_Q$, and the derivative $\dot{F}_Q$ are tabulated in Table VIIIa. The curve in Figure 1a consisting of arcs of parabolas, represents $F_Q$.

Table IX gives the details of the function $F_Q$. The final calculations were made to five decimal places in $b$ and $c$, and the times of the discontinuities in $c$ are given to three decimals of a year. This excess of two decimal places was required in order to achieve numerical consistency. The adequacy of the representation of the fluctuation curve by the parabolic arcs is demonstrated by the fact that the values of the mean error of $F$ obtained from the differences $F-F_Q$, shown in the last column of Table VIIIa, are nearly identical with the mean errors obtained previously from $F-F_9$. In the calculation of these mean errors I made allowance for the fact that in the solutions a total of 18 constants had been obtained from 131 annual values of $F$.

While the differences $F-F_Q$ on the whole show a satisfactory distribution, there are a few sections which show systematic trends. This is especially the case between 1892 and 1901 and again between 1902 and 1910. Comparison with Stoyko's derivative curve shows good agreement in the principal features. The nine-point formula used here has introduced greater smoothing than in Stoyko's curve. The irregularity noticeable in his curve at about 1927 may be due to his use of the uncorrected occultation data 1923–1931.

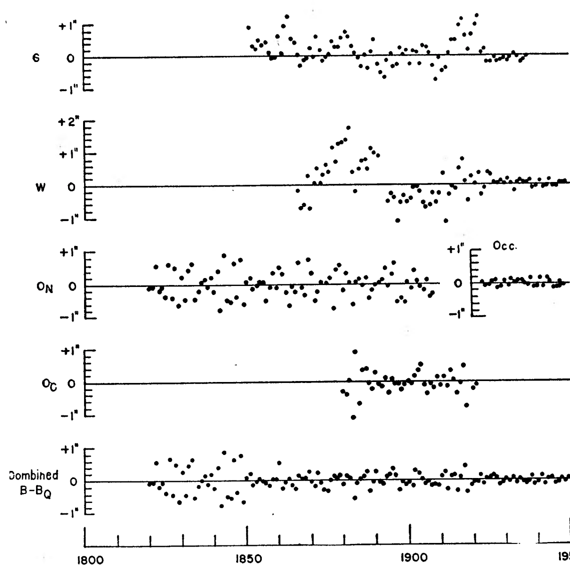


Figure 2. Individual series of observations compared with computed values based on a representation of the fluctuation curve by arcs of parabolas. G, Greenwich uncorrected; W, Washington uncorrected; $O_N$, Spencer Jones's revision of Newcomb's Occultation Memoir; $O_C$, Cape occultations, Occ., occultations after 1923; $B$-$B_Q$, residuals of adopted annual means.

Further improvement of the representation could be obtained by increasing the number of parabolic sections. However, this would probably place undue confidence in the accuracy of the annual means. How well the curve $F_Q$ represents the annual means is illustrated in Figure 2, giving $B-B_Q = .5490(F-F_Q)$. The upper curves in this figure give the residuals for the uncorrected annual values G, W, and Occ. The good agreement between the Cape occultation and the revision of Newcomb's occultations for the interval

1880 to 1908 proves that for this interval two essentially independent series of occultations agree systematically, except for the constant $-0''.64$ applied to the Cape results. These plots further show that using G and W in the manner explained has appreciably reduced the standard deviation of the annual means after 1850 without apparently affecting the results in a systematic manner.

Beginning with 1923 the annual means have so much smaller mean errors that the derivative becomes better defined, and the shorter intervals for the parabolic arcs may be introduced with more confidence. The last four columns of Table VIIIa give the values of $\Delta t$, defined by (3), that correspond to the adopted values of $B$ or of $F = 1.821B$, $\Delta t_Q$ obtained by replacing $F$ by $F_Q$ in this expression, the derivative $d\Delta t_Q/dt$, and $10^8\Delta\nu/\nu$, to be defined below. The coefficients used in the representation of $\Delta t_Q$ are listed in the last three columns of Table IX.

For dates before 1820, Table VIIIb gives $B$, $F$ and $\Delta t$. The sources for these data are discussed in the next section.

The amplitude of the oscillation of $\Delta t$ during the 250 years, 1650 to 1900, is much smaller than the amplitude of $F$. Since $\Delta t$ is directly proportional to the observed deviations of the sun's mean longitude from Newcomb's tables, the small amplitude of this curve shows how well Newcomb succeeded in adjusting the tabular mean longitude of the sun to the observations during this period.

The functions $F_Q$ and $\Delta t_Q$ are presented only as convenient representations of $F$ and $\Delta t$, respectively. They represent the annual values derived from observations so well that in many applications it may be preferable to use $\Delta t_Q$, which may be interpolated for any date, rather than $\Delta t$ for the reduction of observations since 1820.

If $\Delta t_Q$ is considered a suitable representation of the correction to time measured by the rotation of the earth, then the variable mean solar second of time measured by the rotation of the earth equals

$$1 + 3.169 \times 10^{-8} \frac{d\Delta t_Q}{dt}$$

seconds of ephemeris time, measured by the revolution of the earth around the sun. The factor $3.169 \times 10^{-8}$ is the reciprocal of the number of seconds in the year. To the precision needed no distinction need be made between different kinds of year.

A frequency measured with the variable mean solar second as unit of time requires to be multiplied by the factor

$$1 + \frac{\Delta\nu}{\nu} = 1 - 3.169 \times 10^{-8} \frac{d\Delta t_Q}{dt}$$

in order to be converted to frequency per second of ephemeris time which may be regarded as an invariable unit of time.

The last four columns of Table VIIIa are not essential for the discussion in the following sections. These data are given because they are not readily available elsewhere; their compilation here may serve a useful purpose.

3. There is a degree of arbitrariness in the representation of the derivative of the fluctuation curve by a series of straight lines. Obviously, by increasing the number of straight line sections, improvement of the representation of the annual values can be obtained. I have not attempted by statistical discussion of the data to answer the question whether or not all of the discontinuities introduced are fully justified. At this stage of the discussion the important aspect of this representation of the derivative of the fluctuation curve is that the straight-line character appears to be a significant feature the understanding of which should advance the interpretation of the fluctuation phenomenon.

It would appear that sudden large changes must occur at the times of discontinuity in the derivative curve. I held this view when I wrote an article on the accurate measurement of time in the summer of 1950.[20] Further contemplation of the problem led me to undertake experiments with accumulations of random numbers with mean value zero and either a uniform distribu-

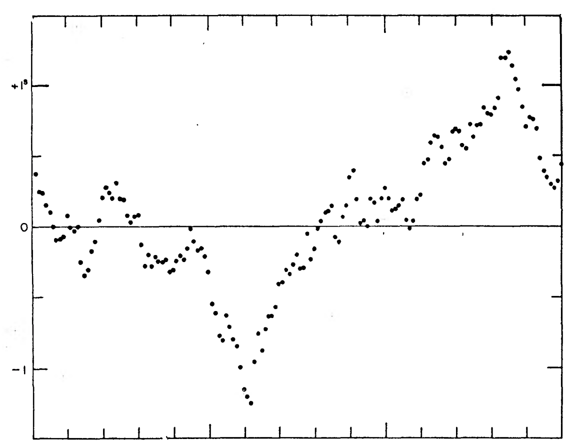


Figure 3. Accumulation of random numbers with normal distribution, mean value zero, and standard deviation 0.10.

tion between $-50$ and $+50$ or with a normal distribution with a prescribed standard deviation. It was found that such accumulations produce curves that resemble the derivative of the fluctuation curve. See Figure 3. This led to the hypothesis that the fluctuations in time as measured by the earth's rotation are caused by cumulative random changes in the angular velocity of the earth's rotation on its axis.

This is not the first time that this explanation has been advanced. About seventeen years ago, Dr. T. E. Sterne, now at Aberdeen Proving Ground, then on the Harvard Observatory staff, was studying cumulative changes in periods of variable stars. During a visit to New Haven he brought up the question whether the fluctuations in the earth's rotation might be explained as the accumulation of random changes. As I remember our discussion,. we concluded that the evidence strongly favored large abrupt changes.

In very recent years Sir Harold Spencer Jones has remarked that the fluctuations in the earth's rotation resemble observed errors in pendulum clocks that were shown to be affected by frequent small erratic changes. In 1949 he expressed himself in the Arthur Lecture at the Smithsonian Institution:[21] "It may prove, however, that the earth itself is rather like a pendulum in its behavior and that its rate of rotation is liable to frequent and small irregular changes, so that we can at present merely observe their integrated effect."

The remainder of this paper will be concerned principally with an examination of the consequences that may be deduced from the hypothesis and further confrontation with observational evidence.

In order to study various aspects of problems that arose in this discussion, my associate, Dr. A. J. J. van Woerkom, undertook some extensive experiments with accumulations of random numbers. I am permitted to use some of his results in this paper. A more detailed account of his experiments is presented in a separate article in the *Astronomical Journal*.

4. The assumption that the fluctuation curve is the result of random cumulative changes in the earth's rate of rotation requires that values of $F$ with an interval of a year can be represented as in the table below:

$$\begin{array}{lll}
F(0) & & \\
 & \Delta_0 & \\
F(1) & & \delta_1 \\
 & \Delta_0 + \delta_1 & \\
F(2) & & \delta_2 \\
 & \Delta_0 + \delta_1 + \delta_2 & \\
F(3) & & \delta_3 \\
 & \Delta_0 + \delta_1 + \delta_2 + \delta_3 & \\
F(4) & &
\end{array}$$

The second differences $\delta$ are supposed to have a mean value zero and a root mean square value $\sigma$ to be determined from the observational data. Taken with the opposite sign they measure the cumulative effects in consecutive years in time measured by the earth's rotation due to changes in the length of the day. The $\delta$'s may themselves be the accumulations of numerous smaller random changes with average intervals much smaller than a year. The astronomical evidence throws no further light on this, though perhaps something may be gained by an analysis of residuals in the moon's mean longitude taken by lunations. Since

$$F(n) - F(0) - n\Delta_0 = (n - 1)\delta_1 + (n - 2)\delta_2 + \cdots + \delta_{n-1},$$

the mean value of the square of the right hand member of this expression equals

$$\sigma_n{}^2 = \sum_{k=1}^{k=n-1} k^2\sigma^2 = \left(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}\right)\sigma^2.$$

Hence, for large values of $n$,

$$\sigma_n = 3^{-\frac{1}{2}}\, n^{\frac{3}{2}}\sigma. \tag{4}$$

Put

$$\Delta_n = F(n + 1) - F(n) = \Delta_0 + \sum_{k=1}^{k=n} \delta_k.$$

The root mean square value of $\Delta_n - \Delta_0$ is $n^{\frac{1}{2}}\sigma$.

Consequently, the mean value of the amplitude of the fluctuation curve may be expected to increase proportionally as the power $\frac{3}{2}$ of the time counted from any epoch for which the curve has the initial values $F(0) = 0$, $\Delta_0 = 0$.

To test the hypothesis, a solution of $F$ was made in the form

$$x + Ty + T^2z = F, \tag{5}$$

in which the mean errors assigned to the right hand members were taken to be proportional to $(-T)^{\frac{3}{2}}$, $T$ counted in centuries from 2000 A.D. The choice of zero epoch is arbitrary. For practical numerical reasons it is not advisable to take it too near a date for which an equation is available.

TABLE X. EQUATIONS USED FOR OBTAINING THE SECULAR ACCELERATION IN THE MOON'S MEAN LONGITUDE

| Year | $T$ | $F$ | $a$ | $b$ | $c$ | $l$ | $v^*$ | $F^*$ |
|---|---|---|---|---|---|---|---|---|
| −683 | −26.83 | −661$^s$ | .00719 | −.1929 | 5.176 | −4$^s$.8 | +16$^s$.0 | +2327$^s$ |
| −380 | −23.80 | −2051 | .00861 | −.2049 | 4.877 | −17.7 | +1.0 | +202 |
| −189 | −21.89 | −1472 | .00977 | −.2139 | 4.682 | −14.4 | +2.9 | +370 |
| +135 | −18.65 | −1339 | .01242 | −.2313 | 4.314 | −16.6 | −1.8 | −100 |
| 850 | −11.50 | −517 | .0256 | −.2944 | 3.386 | −13.2 | −5.4 | −192 |
| 927 | −10.73 | −297 | .0284 | −.3047 | 3.269 | −8.4 | −1.5 | −37 |
| 986 | −10.14 | −617 | .0310 | −.3143 | 3.187 | −19.1 | −12.9 | −401 |
| 1621 | −3.79 | +42 | .136 | −.5154 | 1.953 | +5.7 | +2.7 | +22 |
| 1635 | −3.65 | −5 | .143 | −.5220 | 1.905 | −0.7 | −3.8 | −25 |
| 1639 | −3.61 | −53 | .146 | −.5271 | 1.903 | −7.7 | −10.9 | −73 |
| 1645 | −3.55 | −22 | .149 | −.5290 | 1.878 | −3.3 | −6.6 | −42 |
| 1653 | −3.47 | −38 | .155 | −.5378 | 1.866 | −5.9 | −9.3 | −58 |
| 1662 | −3.38 | −27 | .161 | −.5442 | 1.839 | −4.3 | −7.7 | −47 |
| 1681 | −3.19 | −23.1 | .176 | −.5614 | 1.791 | −4.1 | −7.7 | −42.1 |
| 1710 | −2.90 | −7.1 | .202 | −.5858 | 1.699 | −1.4 | −5.1 | −24.1 |
| 1727 | −2.73 | +3.5 | .222 | −.6061 | 1.655 | +0.8 | −2.8 | −11.9 |
| 1738 | −2.62 | +11.3 | .236 | −.6183 | 1.620 | +2.7 | −0.9 | −2.9 |
| 1747 | −2.53 | +15.8 | .249 | −.6300 | 1.594 | +3.9 | +0.4 | +2.7 |
| 1760.9 | −2.391 | +20.4 | .270 | −.6456 | 1.544 | +5.5 | +2.2 | +9.1 |
| 1774.1 | −2.259 | +25.5 | .295 | −.6664 | 1.505 | +5.0 | +2.1 | +16.2 |
| 1785.5 | −2.145 | +27.5 | .318 | −.6821 | 1.463 | +8.7 | +6.1 | +20.1 |
| 1792.6 | −2.074 | +26.2 | .335 | −.6948 | 1.441 | +8.8 | +6.5 | +20.0 |
| 1801.8 | −1.982 | +23.5 | .358 | −.7096 | 1.406 | +8.4 | +6.6 | +19.0 |
| 1811.9 | −1.881 | +20.8 | .388 | −.7298 | 1.373 | +8.1 | +6.9 | +18.2 |
| 1825 | −1.750 | +15.9 | .432 | −.7560 | 1.323 | +6.9 | +6.8 | +16.0 |
| 1835 | −1.650 | +9.9 | .472 | −.7788 | 1.285 | +4.7 | +5.6 | +12.2 |
| 1845 | −1.550 | +6.7 | .518 | −.8029 | 1.244 | +3.5 | +5.7 | +11.3 |
| 1855 | −1.450 | +5.4 | .573 | −.8308 | 1.205 | +3.1 | +7.0 | +12.4 |
| 1865 | −1.350 | −1.0 | .637 | −.8600 | 1.161 | −0.6 | +5.3 | +8.5 |
| 1875 | −1.250 | −15.4 | .715 | −.8938 | 1.117 | −11.0 | −2.4 | −3.2 |
| 1885 | −1.150 | −22.2 | .811 | −.9326 | 1.072 | −18.0 | −6.0 | −7.3 |
| 1895 | −1.050 | −28.1 | .929 | −.9754 | 1.024 | −26.1 | −9.7 | −10.3 |
| 1905 | −.950 | −25.6 | 1.080 | −1.0260 | .975 | −27.6 | −5.3 | −4.9 |
| 1915 | −.850 | −20.2 | 1.276 | −1.0846 | .922 | −25.8 | +4.8 | +3.6 |
| 1925 | −.750 | −21.8 | 1.540 | −1.1550 | .866 | −33.6 | +7.9 | +5.2 |
| 1935 | −.650 | −29.7 | 1.908 | −1.2402 | .806 | −56.7 | +1.1 | +0.6 |
| 1945 | −.550 | −36.7 | 2.452 | −1.3486 | .742 | −90.0 | −7.3 | −3.0 |

The solution involves the evaluation of the coefficient of $T^2$. Since the observations since the middle of the seventeenth century do not give a determinate solution it is necessary to include ancient observations.

The data used are collected in Table X. For the years 1681 to 1820 Spencer Jones's results are immediately available. His six values 1800.4 to 1813.5 were combined into two normals, 1801.8 and 1811.9. From 1825 to 1945, values with an interval of ten years were taken from Table VIIIa. I used the mean of 1824.5 and 1825.5, 1834.5 and 1835.5, etc. For the older data I decided to use the same material as was used by Newcomb in his discussion of the secular acceleration.[2] Since only an exploratory discussion was contemplated, the practical advantage that Newcomb's data were available in a form ready for use was a major consideration.

The first six dates refer to observations of eclipses. Newcomb[1] gives the Greenwich mean time indicated by Ptolemy or the Arabian observers and the tabular time of geometric phase. The former is the observed astronomical time, the latter the ephemeris time at which, according to the tables of sun and moon employed, the geometrical condition of the eclipse is satisfied. Let the value of the difference between the longitudes of moon and sun, computed for astronomical time, be

$$D_c = (\lambda_{☾} - \lambda_{\odot})_c,$$

and for the geometric phase

$$D_g = (\lambda_{☾} - \lambda_{\odot})_g.$$

Then evidently

$$\delta t = \frac{D_g - D_c}{\lambda_{☾}' - \lambda_{\odot}'},$$

$\lambda'$ designating the motion in longitude per unit of time, and $\delta t$ the difference ephemeris time *minus* astronomical time.

Use of different tables for moon and sun changes $D_c$ to $D_c + \Delta D_c$ but leaves $D_g$ unchanged. Hence $\delta t$ changes to $\delta t + \delta' t$, with

$$\delta' t = -\frac{\Delta D_c}{\lambda_{☾}' - \lambda_{\odot}'}.$$

The differences in mean longitude to be introduced for the moon are, if $T''$ is expressed in centuries from 1800.0,

$$\begin{aligned} H_N - H &= -1''.14 - 29''.17T'' \\ &\qquad -3''.76T''^2 - ''.0067T''^3, \\ B - H_N &= +2.25 + 1.36T'' - 2''.42T''^2, \\ B_{SJ} - B &= -3.09 + 2.52T'' + 5.22T''^2, \\ B_{SJ} - H &= -1.98 - 25.29T'' \\ &\qquad -0.96T''^2 - .0067T''^3. \end{aligned}$$

in which H = Hansen's tables, $H_N$ = Hansen with Newcomb's corrections, B = Brown's tables, $B_{SJ}$ = Brown with Spencer Jones's corrections. For the observations before 1600 it is unnecessary to apply the difference between the empirical terms in Hansen's and Brown's tables.

For the sun, Newcomb used Hansen's tables. The difference between Newcomb's own tables and Hansen's for the older observations is insignificant. Hence the correction to be applied is

$$N_{SJ} - N = -''.74 + ''.51T'' + 1''.23T''^2.$$

N = Newcomb's tables; $N_{SJ}$ = Newcomb with Spencer Jones's corrections. The differences in true longitude are obtained from those in mean longitude by the factors given by Newcomb. Thus, finally

$$\Delta D_c = \frac{\lambda_{☾}'}{n_{☾}}(B_{SJ} - H) - \frac{\lambda_{\odot}'}{n_{\odot}}(N_{SJ} - N).$$

The value of $\delta' t$ added to Newcomb's (Obs. − H) $/n_{☾}$ gives the value of $F$ that corresponds to the system adopted for the modern observations of the moon. The residuals for the individual observations were combined with the same weights as those given by Newcomb. It is of interest to note that the principal contribution to the correction to Newcomb's residuals is due to the introduction of the term $+1''.23T^2$ in the sun's mean longitude.

The six residuals for 1621 to 1662 depend in part on occultations, in part on eclipses. The effect of the correction to the sun's mean longitude amounts to $3''$ at most at these epochs and may be ignored. The residuals obtained agree with those given by de Sitter[5] as they should.

The equations actually used were

$$ax + by + cz = l, \tag{6}$$

obtained by multiplying (5) by $(-T)^{-\frac{3}{2}}$. Hence

$$a = (-T)^{-\frac{3}{2}},\ b = (-T)^{-\frac{1}{2}},\quad c = (-T)^{+\frac{1}{2}},\ l = (-T)^{-\frac{3}{2}} F.$$

The normal equations are:

$$\begin{aligned} 19.265x - 16.896y + 19.580z &= -470^s.2, \\ +19.580 - 37.000 &= +325.4, \\ +186.544 &= -574.8, \end{aligned}$$

with the solution

$$\begin{aligned} x &= -54^s.84 \pm .544\mu^*, \\ y &= -41.03 \pm .645\mu^*, \\ z &= -5.46 \pm .109\mu^*, \end{aligned} \tag{7}$$

The substitution gives the residuals $v^*$ in equations (6), which correspond to residual fluctuations $F^*$. The residuals $v^*$ give for the mean error of a right hand member of the $l$ equations

$$\mu^* = \pm 6^s.8.$$

The substitution was actually made with numerical values slightly different from (7). These give:

$$F^* = F + \underset{\pm 3.70}{54^s.54} + \underset{\pm 4.39}{40^s.91}\, T + \underset{\pm .74}{5^s.46}\, T^2, \tag{8.1}$$

$$B^* = B + \underset{\pm 2.03}{29''.94} + \underset{\pm 2.41}{22''.46}\, T + \underset{\pm .41}{3''.00}\, T^2. \tag{8.2}$$

The correction $\Sigma^*$ to Brown's tables that would correspond to $B^*$ is

$$\Sigma^* = \Sigma + B - B^*,$$

$$\Sigma^* = -7''.11 + 0''.94\, T + 2''.22\, T^2 - 10''.71 \sin(140^\circ.0\, T + 20^\circ.7). \tag{9}$$

Finally

$$\Delta L_{\odot} = +2''.96 + 3''.75\, T + 1''.01\, T^2 + .0748\, B^*, \tag{10}$$

$$\Delta t = +72^s.073 + 91^s.3088\, T + 24^s.592\, T^2 + F^*. \tag{11}$$

In the expression for $\Delta L_{\odot}$, $+.0748B^*$ may be replaced by $+.04107\ F^*$. The expression for $\Delta t$ was obtained by multiplying $\Delta L_{\odot}$ by 24.349. The mean errors, based upon $\mu^* = \pm 6^s.8$, have only the formal significance of indicating the accuracy with which the past record of the earth's rotation may be represented by a quadratic formula. They do not furnish a reliable estimate of the accuracy with which the future of the fluctuation can be predicted. The problem is treated in Section 6.

5. According to this solution the quadratic term in the moon's mean longitude ascribed to

tidal friction is $2''.22\ T^2$ as compared with $5''.22\ T^2$ previously used. This large change can be accounted for by an examination of Newcomb's solution. Newcomb introduced five unknowns, three of which correspond to the unknowns $x$, $y$, $z$ of the solution presented here; the remaining two unknowns were introduced to evaluate the amplitude and phase of a sine term with period 275 years. A provisional discussion had yielded this period as the best suited to represent the fluctuation curve during the modern period, 1621 to 1908. If Newcomb's equations are solved without the two unknowns connected with the 275-year periodic term, a change from $-0''.47$ to $-3''.30$ is obtained for the correction to the coefficient of the $T^2$ term in the moon's mean longitude. The difference is very nearly the same as the difference between $+5''.22$ and $+2''.22$, and indicates that it is the omission of the two unknowns introduced by Newcomb to represent the fluctuation by an empirical sine term rather than the new weighting of the equations that is the principal cause of the large change in the solution.

Brown followed Newcomb's attempt to represent the modern data by a sine term. He modified Newcomb's empirical term to one with a smaller coefficient and a period of 257 years. The constant and linear term in the moon's mean longitude were so adjusted that Brown's new tables, without $T^2$ term to be ascribed to tidal friction, yield a system of deviations of the moon from the tables that agrees as closely to Newcomb's system for the modern period as the modified $T^2$ term permitted. A similar adjustment was made by Fotheringham when he obtained his coefficient of $T^2$ in the moon's mean longitude. A further modification was made by de Sitter, while Spencer Jones adopted de Sitter's solution.

Hence the solution of the secular acceleration in the moon's mean longitude was made with the prescribed condition that the last three centuries be represented by an empirical sine term. If the fluctuations are due to the accumulation of random changes in the angular speed of the earth's rotation, the empirical term has no real significance, and this feature of the solution must be abandoned.

Nevertheless, the new value for the coefficient of $T^2$ in the moon's mean longitude must be regarded as provisional. A new solution, to be based on all reliable observations, as well distributed in time as possible, is desirable.

6. An important further aspect of the problem must now be considered: the random process may produce a spurious quadratic term. An evaluation of its magnitude is necessary in order to obtain a valid estimate of the full uncertainty of the coefficient of the $T^2$ term that must be ascribed to tidal friction.

The problem may be stated as follows. The coefficient $z$ of $T^2$ obtained in the solution in Section 4 may be considered to be made up of two parts, $z = z_r + z_a$, $z_r$ being due to the tidal retardation of the earth's rotation, $z_a$ the spurious effect introduced by the random process. Only the sum of the two parts is obtained from observation. It is possible, however, to give the root mean square value, $\sigma_z$, of $z_a$, and hence

$$z_r = z \pm \sigma_z.$$

While the numerical value of $z_r$ is the same as that of $z$ found in (7), its true mean error is quite different from the formal mean error given there.

In order to solve this and related problems, Dr. van Woerkom and I constructed fifteen artificial fluctuation curves with initial values $F(0) = 0$, $\Delta = 0$ from sequences of random $\delta$'s with mean value zero and a normal distribution with standard deviation $\sigma$. Equations were then solved with identical left hand members as the equations in Table X but with the right hand members taken from the artificial fluctuation curves for the appropriate values of $T$ in terms of 100 steps, only the sign of $T$ being reversed. If $\mu$ is the root mean square value of the accumulation after 100 steps,

$$\mu = 577\ \sigma.$$

The arithmetic means of the fifteen independent values of $x$, $y$, $z$ obtained are

$$\frac{x}{\mu} = 1.08, \qquad \frac{y}{\mu} = 1.73, \qquad \frac{z}{\mu} = .198.$$

The scatter among the individual values of $x$, $y$, $z$ is large. For $z/\mu$ it amounts to a standard deviation $\pm .138$ from the arithmetic mean.

The mean values of the coefficients $x$ and $y$ are found to be of the same order of magnitude as the mean value of the accumulation after 100 steps. This can be verified by considering the fact that $x$ and $y$ for the individual sequences are principally determined by the equations for the smaller values of $T$. It is now possible to evaluate the order of magnitude of the mean value of $z/\mu$ as follows. Let the equation for the largest value, $T_e$, of $T$ be written

$$z\ T_e^{\frac{1}{2}} = l - x\ T_e^{-\frac{3}{2}} - y\ T_e^{-\frac{1}{2}},$$

in which $l = T_e^{-\frac{3}{2}}F(T_e)$. The mean values of the three terms in the right hand member are of the orders, $\mu$, $\mu T_e^{-\frac{3}{2}}$, $\mu T_e^{-\frac{1}{2}}$, respectively. Hence, if $T_e$ is large compared with unity, the mean value of $l$ is large compared with the other two terms, and it follows that the mean value of $z/\mu$ is approximately $T_e^{-\frac{1}{2}}$. Since in the equations used, $T_e = 26.83$, $T_e^{-\frac{1}{2}} = .193$, this estimate is in excellent agreement with the result obtained numerically.

Substitution of the solutions yields residuals $F^*$ for the 37 equations for each of the 15 artificial fluctuation curves, and $v^*$ for the residuals of the corresponding $l$-equations.

Let $\mu^*$ designate the standard deviation derived from the residuals $v^*$ obtained by substitution of the quadratic solution. The ratios $\mu/\mu^*$ obtained for the fifteen solutions separately gave a large scattering. The arithmetic mean of the fifteen determinations and the standard deviation of the individual ratios from this mean may be expressed by

$$\frac{\mu}{\mu^*} = 5.1 \pm 2.9.$$

This may be applied to the results from the fluctuation curve in the earth's rotation. The only mean error known is that after the quadratic solution,

$$\mu^* = 6^{\rm s}8,$$

for the mean accumulation after one century. With the factor $5.1 \pm 2.9$, it follows that

$$\mu = 35^{\rm s} \pm 20^{\rm s},$$
$$\sigma = {}^{\rm s}061 \pm {}^{\rm s}035,$$

the latter being the apparent standard deviation in the length of the year caused by the varying length of the day. The average length of the day varies from year to year by random amounts with standard deviation

$$\sigma_d = {}^{\rm s}00017 \pm {}^{\rm s}00010.$$

The uncertainty in $z$ due to the random process is found to be

$$\sigma_z = .198\mu = 7^{\rm s}0 \pm 4^{\rm s}0.$$

7. If values of the derivative of the fluctuation curve were available, the unknowns $y$ and $z$ could be obtained from the equation

$$y + 2Tz = 100\dot{F}, \qquad (12)$$

$\dot{F}$ being the derivative with the year as unit of time. If the observational error in $\dot{F}$ is ignored, if $\dot{F}$ is assumed to be the accumulation of annual random $\delta$'s with standard deviation $\sigma$ and mean value zero, and if the zero epoch is again taken at 2000 A.D., the mean error to be assigned to $\dot{F}$ is $(2000 - t)^{\frac{1}{2}}\sigma$. Thus the equations (12) multiplied by factors $(-T)^{\frac{1}{2}}$ will have mean errors of the right hand members all equal to $1000\ \sigma$. These equations may be written

$$a'y + b'z = l'.$$
$$a' = (-T)^{-\frac{1}{2}},\ b' = -2(-T)^{\frac{1}{2}},\ l' = 100(-T)^{-\frac{1}{2}}\dot{F}.$$

For the years after 1820 the values of $\dot{F}$ to be used may, for example, be those designated $\dot{F}_9$ or $\dot{F}_Q$ in Table VIII. In Table XI the latter are chosen with an interval of 5 years. The numerical result would have been changed very little if derivatives otherwise determined had been employed.

An interval of 130 years is, of course, hope-

TABLE XI. SOLUTION FROM THE DERIVATIVE OF THE FLUCTUATION

| $T$ | $100\dot{F}$ | $a'$ | $b'$ | $l'$ | $v'^*$ |
|---|---|---|---|---|---|
| −25.32 | −458ˢ7 | .199 | −10.06 | − 91ˢ1 | −114ˢ2 |
| −22.84 | +303.1 | .209 | − 9.56 | + 63.4 | + 42.3 |
| −20.27 | + 41.0 | .222 | − 9.00 | + 9.1 | − 9.7 |
| −14.58 | +101.6 | .262 | − 7.64 | + 26.6 | + 13.7 |
| − 7.00 | + 69.0 | .378 | − 5.29 | + 26.1 | + 25.2 |
| − 3.350 | + 18.4 | .546 | − 3.66 | + 10.1 | + 21.3 |
| − 3.045 | + 55.2 | .573 | − 3.49 | + 31.6 | + 44.4 |
| − 2.815 | + 62.4 | .596 | − 3.36 | + 37.1 | + 51.4 |
| − 2.675 | + 70.9 | .611 | − 3.27 | + 43.3 | + 58.4 |
| − 2.575 | + 50.0 | .623 | − 3.21 | + 31.2 | + 47.0 |
| − 2.460 | + 33.1 | .638 | − 3.14 | + 21.1 | + 37.7 |
| − 2.325 | + 38.6 | .656 | − 3.05 | + 25.3 | + 42.9 |
| − 2.202 | + 17.5 | .674 | − 2.97 | + 11.8 | + 30.5 |
| − 2.110 | − 18.3 | .688 | − 2.91 | − 12.6 | + 6.9 |
| − 2.028 | − 29.3 | .702 | − 2.85 | − 20.6 | − 0.5 |
| − 1.932 | − 26.7 | .719 | − 2.78 | − 19.2 | + 2.0 |
| − 1.828 | − 26.4 | .740 | − 2.70 | − 19.5 | + 2.7 |
| − 1.75 | − 62.0 | .756 | − 2.65 | − 46.9 | − 23.8 |
| − 1.70 | − 49.6 | .767 | − 2.61 | − 38.0 | − 14.4 |
| − 1.65 | − 75.4 | .778 | − 2.57 | − 58.7 | − 34.5 |
| − 1.60 | − 12.8 | .791 | − 2.53 | − 10.1 | + 14.7 |
| − 1.55 | − 12.8 | .803 | − 2.49 | − 10.3 | + 15.2 |
| − 1.50 | − 22.2 | .816 | − 2.45 | − 18.1 | + 8.1 |
| − 1.45 | − 31.5 | .830 | − 2.41 | − 26.1 | + 0.7 |
| − 1.40 | − 60.4 | .845 | − 2.37 | − 51.0 | − 23.4 |
| − 1.35 | − 92.0 | .861 | − 2.32 | − 79.2 | − 50.8 |
| − 1.30 | −164.1 | .877 | − 2.28 | −143.9 | −114.7 |
| − 1.25 | −124.7 | .894 | − 2.24 | −111.5 | − 81.5 |
| − 1.20 | − 66.3 | .913 | − 2.19 | − 60.5 | − 29.5 |
| − 1.15 | − 53.3 | .933 | − 2.14 | − 49.7 | − 17.8 |
| − 1.10 | − 75.4 | .953 | − 2.10 | − 71.9 | − 39.0 |
| − 1.05 | − 40.0 | .976 | − 2.05 | − 39.0 | − 5.0 |
| − 1.00 | + 36.4 | 1.000 | − 2.00 | + 36.4 | + 71.6 |
| − .95 | + 69.5 | 1.026 | − 1.95 | + 71.3 | +107.7 |
| − .90 | + 59.2 | 1.054 | − 1.90 | + 62.4 | +100.1 |
| − .85 | + 25.1 | 1.085 | − 1.84 | + 27.2 | + 66.4 |
| − .80 | − 18.6 | 1.118 | − 1.79 | − 20.8 | + 19.9 |
| − .75 | − 62.3 | 1.155 | − 1.73 | − 72.0 | − 29.6 |
| − .70 | − 76.4 | 1.195 | − 1.67 | − 91.3 | − 47.0 |
| − .65 | − 94.4 | 1.240 | − 1.61 | −117.1 | − 70.8 |
| − .60 | − 64.4 | 1.291 | − 1.55 | − 83.1 | − 34.5 |
| − .55 | − 50.1 | 1.348 | − 1.48 | − 67.5 | − 16.3 |

lessly inadequate for an evaluation of the secular acceleration. In order to obtain a determinate solution, values of $\dot{F}$ for earlier epochs must be used in addition. These were obtained by the simple expedient of taking the average value of $\dot{F}$ over an interval $t_a$ to $t_b$,

$$\dot{F} = \frac{F(t_b) - F(t_a)}{t_b - t_a},$$

for the date $(t_b + t_a)/2$. If $F$ has a continuous derivative, the derivative must have had this average value somewhere in the interval. The arbitrariness of assigning this value to the average date is justified if a large number of dates is used.

The last three dates of the first group in Table X and all of the dates of the second group are too close together, considering their observational uncertainty, to be used individually. They were combined with approximately the relative weights used by Newcomb. Relative weights 1, 2, 4 for the Arabian eclipses, and 1, 4, 8, 6, 6, 12 for the six dates 1621 to 1662 yielded:

$$950, \qquad F = -511^s \pm 155^s,$$
$$1649, \qquad F = -\ 29^s \pm\ 6^s.4.$$

The remaining values of $F$ from $-683$ to $1811.9$ were used from Table X, to which I added

$$1822.5, \qquad F = +17^s.0.$$

The resulting values of $100\ \dot{F}$ for the average date of each interval are given in Table XI, together with the adopted values for the dates since 1825. The normal equations are

$$29.76y - 84\quad z = -\ 970^s.5,$$
$$+592.4\quad = +1634.2,$$

with the solution

$$y = -41.38 \pm .41\mu^* \quad z = -3.11 \pm .092\mu^*.$$

The residuals yield

$$1000\sigma^* = 50^s.0, \qquad \mu^* = 28^s.9.$$

Corresponding solutions were made from the fifteen artificial fluctuation sequences. The average of fifteen determinations gave:

$$\frac{y}{\mu} = .599, \frac{z}{\mu} = .216, \frac{\mu}{\mu^*} = 1.49 \pm .47.$$

Thus:

$$\mu\ = 42^s \pm 14^s,$$
$$\sigma\ = {}^s.074 + {}^s.025,$$
$$\sigma_d = {}^s.00020 \pm {}^s.00007,$$
$$\sigma_z = 9^s.1 \pm 3^s.0.$$

These results are in good agreement with those obtained by the parabolic solution from the values of $F$.

Again, an approximate estimate of $\sigma_z$ may be obtained by considering the equation for the largest value $T_e$ of $T$. This may be written

$$2z(T_e)^{\frac{1}{2}} = l' - y\ T_e^{-\frac{1}{2}}.$$

The mean value of $l'$ for a sequence built up by the accumulation of random $\delta$'s is $1000\sigma$; $y$ is of the same order, but on account of the factor $T_e^{-\frac{1}{2}} \simeq .2$, the second term in the right hand number is small compared with the mean value of $l'$. Thus the approximate value of $\sigma_z$ is

$$\sigma_z = \frac{1000\sigma}{2\ T_e^{\frac{1}{2}}} = \frac{1.73\mu}{10.4} = .167\mu.$$

This estimate is in fair agreement with $\sigma_z = .216\mu$ found numerically.

In the discussion so far, the effect on the secular acceleration coefficient by the observational uncertainty in the residuals has been ignored. This may be examined by expressing $z$ explicitly in terms of the individual values of $F$ used in the solution. The parabolic solution for $z$ from $F$ gives

$$\begin{aligned} z &= +.03137\ [al] + .04956\ [bl] + .07190\ [cl] \\ &= \Sigma\{+.03137\ (-T)^{-3} - .04956\ (-T)^{-2} \\ &\qquad\qquad + .07190\ (-T)^{-1}\}\ F(T) \\ &= \Sigma\ \gamma_j\ F_j \end{aligned}$$

if $F_j$ designates the individual values of $F$ used in the solution.

The linear solution from $\dot{F}$ gives

$$\begin{aligned} z &= +.00794\ [a'l'] + .00281\ [b'l'] \\ &= \Sigma\{+.00794\ (-T)^{-1} + .00563\} \times 100\dot{F}(T) \\ &= \Sigma\ \gamma_j'\ F_j. \end{aligned}$$

For the dates before 1825, $\dot{F}(T)$ may be expressed in terms of the $F_j$ on which they depend. The results are given in Table XII. The factors for the individual $F$'s are given for the dates up to 1662. For the later observations the observational mean errors are so small that the addition to the observational uncertainty is negligible.

A summary of the two evaluations of $z$ is:

| | $F$ solution | $\dot{F}$ solution |
|---|---|---|
| $z$ | $-5^s.46$ | $-3^s.10$ |
| Accumulation, m.e., | $\pm 7.0$ | $\pm 9.1$ |
| Observational, m.e., | $\pm 0.6$ | $\pm 1.5$ |
| Total, m.e., | $\pm 7.0$ | $\pm 9.2$ |

The formal mean errors, $\pm 0^s.7$ and $\pm 2^s.7$, respectively, for the two solutions have been ig-

TABLE XII. DEPENDENCE OF $z$ ON THE INDIVIDUAL RIGHT-HAND MEMBERS

| | $F$ | | Solution from $F$: $\gamma$ | contr. to $z$ | | Solution from $\dot{F}$: $\gamma'$ | contr. to $z$ | |
|---|---|---|---|---|---|---|---|---|
| − 683 | − 661$^{s}$ | ±648$^{s}$ | +.000376 | − $^{s}$.249 | ±$^{s}$.244 | +.00176 | −1$^{s}$.163 | ±1$^{s}$.140 |
| − 300 | −2051 | 486 | + 415 | − .851 | .202 | + 101 | −2.072 | .491 |
| − 189 | −1472 | 648 | + 443 | − .652 | .287 | − 115 | +1.693 | .745 |
| + 185 | −1339 | 486 | + 499 | − .668 | .243 | − 99 | +1.326 | .481 |
| 850 | − 517 | 389 | + 681 | − .352 | .265 | | | |
| 927 | − 297 | 275 | + 704 | − .209 | .194 | | | |
| 986 | − 617 | 211 | + 721 | − .445 | .152 | | | |
| 1621 | + 42 | 38 | + 265 | + .011 | .010 | | | |
| 1635 | − 5 | 19 | + 184 | − .001 | .003 | | | |
| 1639 | − 53 | 14 | + 160 | − .008 | .002 | | | |
| 1645 | − 22 | 16 | + 120 | − .003 | .002 | | | |
| 1653 | − 38 | 16 | + 64 | − .002 | .001 | | | |
| 1662 | − 27 | 11 | − 6 | .000 | .000 | | | |
| 950 | − 511 | 155 | | | | +.00001 | − .004 | .002 |
| 1649 | − 29 | 6.4 | | | | + 953 | − .276 | .061 |
| − 683 to 1662 | | | | $-3^{s}.429$ | ±$^{s}$.61 | | − $^{s}$.497 | ±1$^{s}$.53 |
| 1681 to 1822.5 | | | | − .333 | | | − .463 | |
| 1825 to 1945 | | | | −1.703 | | | −2.145 | |
| | | | | $z = -5^{s}.46$ | ±$^{s}$.61 | | $z = -3^{s}.10$ | ±1$^{s}$.53 |

nored in forming the total mean error. By comparing the actual solution with a solution from artificially constructed fluctuation sequences, the entire uncertainty $\sigma_z$ in $z$, due to the random process, has been included. This procedure also justifies using data so close together in time that their independence with regard to the accumulation process may be questioned. The evaluation of $\sigma_z$ from artificially constructed sequences which are used for dates corresponding to those present in the actual solution removes any objection that might otherwise exist.

Of the two solutions for $z$ I am strongly inclined to prefer the one derived by the parabolic solution from the fluctuation curve, rather than that by the linear solution from the derivatives. The latter depends less directly upon the data furnished by the observations; moreover, the coefficients $\gamma'$ are as a rule larger than the coefficients $\gamma$. Of course the two results may not be combined as if they had been derived from independent data.

One reason for presenting the solution from the derivatives was to demonstrate that this solution does not yield a significantly lower uncertainty in the coefficient $z$. Both solutions yield an uncertainty proportional to the reciprocal of the square root of the interval of time covered by the observations.

8. The evidence presented in favor of the hypothesis that the variations in the rate of rotation of the earth are the cumulative effect of random changes with mean value zero, superposed on a secular decrease by tidal friction, is the following.

(a) The observed data agree with an increase in amplitude of the fluctuation curve proportional to the three-halves power of the time, as shown by the residuals $v^*$ in Table X.

(b) The values for $\sigma$ and $\sigma_z$ obtained in the two solutions are in general agreement, which could hardly be expected if the basic assumption were not well founded.

(c) The derivative of the fluctuation curve for the last 130 years has an appearance that resembles plots obtained by the accumulation of random numbers. Proper allowance should be made for the rounding due to the use of a nine-point formula and for the presence of observational errors.

This evidence is indirect. Dr. van Woerkom and I have made an attempt to obtain more direct information from a statistical analysis of the observed data since 1820. He intends to report on this in a paper dealing with his experiments with sequences of random numbers. It may suffice to say that the results indicate that the simple random process used as a working hypothesis appears to account for the principal features of the fluctuation curve. However, certain statistical properties of the data indicate that some superimposed effects that follow different laws are present.

9. In my paper presented to the National Academy of Sciences in November, 1951,[22] I used essentially the same data as presented in detail in this article. At that time I accepted the explanation that the changes in the rate of rotation of the earth are caused by changes in the

moment of inertia, $C$, about the axis of rotation. The standard deviation $\sigma = \pm^{s}.07$ in the apparent length of the year corresponds to variations in $C$ such that the standard deviation of $\delta C/C$ from year to year amounts to $2.2 \times 10^{-9}$.

An attractive feature of this interpretation is that the random process may produce a secular term in $C$. It was shown by H. C. Urey[23] that the introduction of $dC/dt$ into the equations relating the coefficients of the $T^2$ terms in the mean longitudes of the sun and moon removes the apparent inconsistency that has made it difficult to interpret the results.

The solution

$$z = -5^{s}.46 \pm 7^{s}.0 \quad \text{(m.e.)}$$

corresponds to

$$s = -3''.00 \pm 3''.8.$$

Thus the coefficients of $T^2$ in the mean longitudes become, according to the relations of Section 1,

for the moon, $+2''.2 \pm 3''.8$,
for the sun, $+1''.01 \pm 0''.28$.

The latter result corresponds to a secular change in the length of the day per century,

$$+^{s}.00135 \pm {}^{s}.00038.$$

Jeffreys[24] gave the theoretical relation

$$\frac{\nu}{\nu_1} = \frac{.378\,N + N_1}{N + N_1}\frac{n}{n_1},$$

$-N$ and $-N_1$ being the retarding couples acting on the earth due to the lunar and solar tides, respectively, $n$ and $n_1$ the mean motions, and $\nu$ and $\nu_1$ the coefficients of the $T^2$ terms in the mean longitudes of the moon and the sun, respectively. The coefficient .378 is the numerical value of $\kappa - 3/\kappa$, $\kappa$ being the ratio between the orbital angular momentum of the earth–moon system and the rotational angular momentum of the earth. The present value of $\kappa$ is 4.82.

With $n_1/n = .0748$, the expression for $\nu/\nu_1$ may be written

$$\frac{\nu}{\nu_1} = 5.05 + \frac{8.32\,N_1}{N + N_1}.$$

Thus the minimum value of $\nu/\nu_1$ is 5.05 if $N_1$ is negligibly small compared with $N$. On two different assumptions Jeffreys obtained $N/N_1 = 5.1$ and 3.4, whence $N/(N + N_1) = .164$ and .227, respectively. These give 6.4 and 6.9 for the corresponding values for $\nu/\nu_1$. With $+5''.22$ and $+1''.23$ for the coefficients of $T^2$, $\nu/\nu_1 = 4.24$ is below the theoretical minimum. The values found in the discussion presented here, $+2''.2$ and $1''.01$ with $\nu/\nu_1 = 2.18$, emphasize the contradiction.

Urey proposes to modify Jeffreys' formula by introducing as an additional unknown the secular change in the moment of inertia about the axis of rotation. The formula may then be written

$$\frac{\nu}{\nu_1} = \frac{.378 + \dfrac{N_1}{N} + \dfrac{\Omega}{N}\dfrac{dC}{dt}}{1 + \dfrac{N_1}{N} + \dfrac{\Omega}{N}\dfrac{dC}{dt}}\cdot\frac{n}{n_1}.$$

If numerical values are introduced, including $\nu = 2.22 + \alpha$, $\nu_1 = 1.01 + .0748\alpha$, the result becomes

$$\frac{\Omega}{N}\frac{dC}{dt} = -\frac{N_1}{N} - .255 + .055\alpha.$$

Substitution of the result of the solution of this paper, $\alpha = 0 \pm 3.8$, gives

$$N/N_1 = \quad 5.1, \qquad 3.4,$$

$$\frac{\Omega}{N}\frac{dC}{dt} = -0.45 \pm 0.21, \quad -0.55 \pm 0.21.$$

In order to make $dC/dt = 0$, the values of $\alpha$ required are $+8.2$ and $+10.0$, respectively, for the two values of $N/N_1$. These correspond to improbably large coefficients of $T^2$ in the mean longitudes of the moon and sun.

10. Postulating random cumulative changes in the moment of inertia $C$ raises various problems of a geophysical nature. Among these is the question, what corresponding effect in the variation of latitude may be predicted. If the changes in $C$ are caused by displacements of masses distributed over all longitudes and latitudes, the moments of inertia $A$ and $B$ and the products of inertia $D$, $E$ and $F$ will be affected by changes of the same order of magnitude. In a coordinate system the $z$-axis of which coincides with the axis of figure, $D$ and $E$ are zero. Hence changes $\delta D$, $\delta E$ correspond to a motion of the pole of figure. This displacement is of the order

$$\delta\psi = \frac{C}{C - A}\cdot\frac{\delta C}{C}.$$

Thus if $\delta C/C$ represents a change from year to year with standard deviation $2.2 \times 10^{-9}$, the corresponding standard deviation in $\delta\psi$ is $\pm''.14$. A numerical factor, depending on the distribution of the disturbances and on the directions of the