

## THE IMPORTANCE OF ROTATION IN STELLAR EVOLUTION

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*Summary*

Eddington's assessment (1) of the rate of circulation in meridian planes in the Sun, produced by its rotation, is reviewed, using a first-order perturbation theory. The resulting velocity is found to be of the order of  $10^{-10}$  cm./sec., as compared with Eddington's figure of  $2 \times 10^{-4}$  cm./sec., a reduction of importance when considering the amount of mixing of material in the Sun during its lifetime.

The theory is applied to stars in general, and the equatorial rotation velocity necessary to provide a significant rate of stirring in a star is found in terms of its magnitude and effective temperature. These rotation rates are found to lie within the range actually observed for early-type stars, but are in excess of those for the later types.

In view of the widely held opinion that giants are stars with non-uniform composition, the distribution of rotation rates among the stars thus assumes an added interest. The rotation rate can similarly govern the possibility of local exhaustion of hydrogen in the core, with its consequent effects on evolution.

1. *Introduction.*—Circulation currents in the radiative envelopes of stars may have a considerable effect on their structure and evolution. In the absence of mixing currents the composition of the radiative envelope, apart from the possible accretion of interstellar material, must remain almost constant during the life of a star, while the molecular weight in the convective core increases slowly as the hydrogen there is transformed into helium. According to the theory of stellar structure the resultant discontinuity of molecular weight at the core surface would cause the star to expand. Although it is doubtful whether in light stars this increase in radius would be sufficient to produce giants, in the heavy stars where the polytropic index in the core is nearer its maximum value 3 it is possible that the effect might be sufficient to account for giants. The effect could not occur in well-stirred stars. Further, the work of Schönberg and Chandrasekhar (2), and its subsequent development by M. Hall Harrison (3), on the effects of a local exhaustion of hydrogen in the core, show that the evolution of a star in which there were no general mixing would differ significantly from a star in which stirring allowed hydrogen from all parts of the star to be consumed. Finally, the work of Hoyle and Lyttleton (4) and Hen and Schwarzschild (5) on the possible production of giant stars through accretion of interstellar material producing a non-uniformity in molecular weight in the outer part of the radiative envelope, shows that a star which has consumed hydrogen from its radiative envelope is more likely to become a giant than a star whose hydrogen content in the radiative envelope has not been affected. On this accretion theory, therefore, a well-stirred star stands more chance of becoming a giant than an unstirred one.

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In order to allow non-uniform composition to develop, the meridian-plane circulation velocity must not exceed a value of the order of magnitude of the radius of the star divided by the time-scale of its evolution. In the Sun, for example, taking a radius of  $7 \times 10^{10}$  cm., together with a time of  $3 \times 10^9$  years, this maximum velocity is seen to be of the order of  $7 \times 10^{-7}$  cm./sec. Thus Eddington's figure of  $2 \times 10^{-4}$  cm./sec. would preclude an appreciable difference in the compositions of the core and radiative envelope, while the figure of  $3 \times 10^{-10}$  cm./sec. derived in the present paper involves a negligible mixing rate.

A survey of the problem included in Wasiutynski's work (6) on stellar hydrodynamics shows that in the past great difficulty has been experienced in attempting to assess the rate of circulation produced by rotation. The main difficulty has been that, with an arbitrary distribution of angular velocity, the boundary conditions adopted for the flow cannot be satisfied simultaneously at the surface of the star and on the surface of the convective core. In consequence much of the work in this field has been restricted to general theorems, emphasis being laid mostly on the relationship between the circulation velocity, the thermal equilibrium and the variation of angular velocity throughout the star. Schwarzschild, for example, has shown (7) that thermal equilibrium could be maintained without meridian-plane circulation if the star possessed a certain special distribution of angular velocity. However, even if a star could develop such a differential angular velocity, it is difficult to see how it could maintain it against viscosity drag, and more especially, magnetic forces, without meridian-plane circulation, since the other forces are all directed in meridian planes. It will be seen later that the boundary conditions adopted were unnecessarily severe.

The first quantitative investigation of the effect of rotation was made by Eddington (8) in 1929. A few years previously, in 1924, H. von Zeipel had shown, in a famous theorem (8), that the radiative transfer of energy in a star is upset slightly by rotation and that thermal equilibrium would only be possible, in a rigid star, if the rate of generation of energy obeyed a special law. This law is clearly not satisfied in any real star; the first effect of radiative transfer alone attempting to maintain thermal equilibrium is therefore to produce a slight rise in temperature over some parts of any given equipotential surface and a slight fall over other parts. The resulting variation in pressure over the surface must set up convection currents in meridian planes. The circulation settles down to a steady state sufficient to maintain thermal equilibrium. An exception to this would be in a region where the molecular weight was already strongly non-uniform. In such a case circulation would only proceed for a short time, until the distortion of the surfaces of constant molecular weight prevented further motion. This effect is referred to in more detail later.

It was by equating the rate of transport of energy out of any region, by the circulation, to the amount by which radiative transfer alone fails to get rid of it, that Eddington derived an approximate value for the radial component of the circulation velocity, viz.:

$$v_r = q\bar{\epsilon}_1 / \{1/\gamma - 1/4(\gamma - 1)\}g, \quad (\text{I.I})$$

where  $\bar{\epsilon}_1$  is the mean rate of generation of energy per gram of material interior to the point in question,  $g$  is the local acceleration of gravity,  $\gamma$  is the ratio of the specific heats of the gas and  $q$  is a dimensionless factor depending on the distortion

of the equipotential surfaces by the rotation. In Eddington's treatment of the problem the factor  $q$  can only be obtained by assessment, and for some reason Eddington took a value which is far too high. This resulted in an estimate of  $2 \times 10^{-4}$  cm./sec. for the radial velocity. In a more accurate treatment the velocity distribution correct to the first order in the ratio of the centrifugal acceleration to acceleration of gravity can be found, using a first-order perturbation theory. The calculations made are used to form a relationship between the maximum equatorial rotation velocity which would allow a star to develop a non-uniform composition, and its position in the H-R diagram. The results show that the magnitude of that rotation which has a critical effect on the structure of a star is within the range actually observed.

2. *A generalized first-order perturbation theory.*—The first step is to derive an expression for the distortion of the equipotential surfaces due to the rotation. The following procedure is an extension of a method given by Chandrasekhar (9). The equations of hydrostatic equilibrium, neglecting viscosity and the inertia of the circulation itself, and assuming conditions to be symmetrical about an axis through the centre of the star\*, are

$$\left. \begin{aligned} \partial P' / \partial r &= \rho' \partial \phi' / \partial r + \rho' f_r, \\ \partial P' / \partial \theta &= \rho' \partial \phi' / \partial \theta + r \rho' f_\theta, \end{aligned} \right\} \quad (2.1)$$

where  $\rho' f_r$ ,  $\rho' f_\theta$  are the components of the disturbing force (e.g. the centrifugal force of rotation or magnetic forces),  $\phi'$  is the gravitational potential of the star itself,  $P'$ ,  $\rho'$  are the hydrostatic pressure and density of the material, and  $(r, \theta)$  are the distance from the centre and the angular distance from the axis of symmetry. Dashes are used to denote values in the perturbed star.

On eliminating  $P'$  from equations (2.1) we have

$$\frac{\partial \rho'}{\partial \theta} = -\frac{\chi}{g} \frac{d\rho}{dr} + \frac{1}{g} \left\{ \rho \frac{\partial f_r}{\partial \theta} - \frac{\partial}{\partial r} (r \rho f_\theta) \right\} \quad (2.2)$$

correct to the first order, where  $g$  is the local acceleration of gravity and  $\chi = \partial \phi' / \partial \theta$ . Poisson's equation,

$$\nabla^2 \phi' = -4\pi G \rho', \quad (2.3)$$

can be used to substitute for  $\partial \rho' / \partial \theta$  in (2.2), giving

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \chi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \{ \chi \sin \theta \} \right) \\ = + \frac{4\pi G}{g} \frac{d\rho}{dr} \chi + \frac{4\pi G}{g} \left[ \frac{\partial}{\partial r} (r \rho f_\theta) - \rho \frac{\partial f_r}{\partial \theta} \right]. \end{aligned} \quad (2.4)$$

The form of the left-hand side of (2.4) suggests expanding  $\chi$  in a series of associated Legendre polynomials  $P_n^1(\cos \theta)$ . If we therefore expand  $f_r$  and  $f_\theta$  in the forms

$$f_r = \sum_{s=0}^{\infty} a_s(r) P_s, \quad f_\theta = \sum_{s=1}^{\infty} b_s(r) P_s^{\dagger}, \quad (2.5)$$

\* It is the neglect of the inertia of the circulation which enables the assumption of axial symmetry to be made, for the remaining perturbing effect, that of rotation, for example, is axially symmetrical. In an exhaustive treatment it would have to be shown that no permanent asymmetrical state, with consequently large circulation currents, is possible. Such states would probably produce non-axially symmetrical effects at the surface of the star. No such effects are observed on the solar surface, however.

†  $f_\theta$  can always be expanded in this form, since it must vanish at  $\theta=0, \pi$  in order to avoid a singularity on the axis of symmetry.

while noting that  $\partial P_n / \partial \theta = -P_n^1$ , the general solution of (2.4) can be written

$$\chi = \sum_{s=1}^{\infty} c_s(r) P_s^1, \quad (2.6)$$

where

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dc_s}{dr} \right) - \left[ \frac{s(s+1)}{r^2} + \frac{4\pi G}{g} \frac{d\rho}{dr} \right] c_s = \frac{4\pi G}{g} \left[ \frac{d}{dr} (r\rho b_s) + \rho a_s \right]. \quad (2.7)$$

The boundary conditions for the  $c_s$  are obtained as follows. We have  $\chi \rightarrow 0$  as  $r \rightarrow \infty$ , hence

$$c_s \rightarrow 0 \text{ as } r \rightarrow \infty, \quad s \geq 1. \quad (2.8)$$

The inner boundary conditions may be obtained in terms of the temperature variation over the core surface. This variation of course depends on conditions within the core which are not dealt with in detail in this paper. If it is known that the turbulence in the core does not affect the equations of hydrostatic equilibrium appreciably, then the solutions of (2.7) can be extended right to the centre of the star. The boundary conditions in this case are

$$c_s = 0 \text{ at } r = 0, \quad s \geq 1. \quad (2.9)$$

The functions  $\partial \rho' / \partial \theta$  and  $\partial T' / \partial \theta$  can then be determined; the next step is to find the velocity components  $v_r$  and  $v_\theta$  of the meridian-plane circulation in terms of these. The actual circulation is governed by the equation of thermal equilibrium. This can be written in the entropy form

$$\rho' \mathbf{v} \cdot \text{grad } S' / \mu' = (\rho' \epsilon' - \text{div } \mathbf{H}) / T', \quad (2.10)$$

where  $S' = \frac{\Re}{\gamma' - 1} \log \left( \frac{P'}{\rho'^{\gamma'}} \right)$ , the entropy per g. mol. of the gas,

$\Re$  = gas constant,

$\gamma'$  = ratio of specific heats of the gas,

$\mu'$  = molecular weight of the gas,

$\mathbf{H}$  = flux of radiation, and

$\epsilon'$  = rate of production of nuclear energy per gram.

In this equation the radiation pressure has been neglected. The equation can be written correct to the first order as

$$\rho \frac{T v_r}{\mu} \frac{dS}{dr} = \rho' \epsilon' - \text{div } \mathbf{H} \quad (2.11)$$

since the left-hand side is already a first-order quantity. We can write, correct to the first order,

$$\frac{\partial}{\partial \theta} (\rho' \epsilon') = \rho \epsilon \left[ \frac{\partial}{\partial T} (\log \rho \epsilon) \frac{\partial T'}{\partial \theta} + \frac{\partial}{\partial \rho} (\log \rho \epsilon) \frac{\partial \rho'}{\partial \theta} \right], \quad (2.12)$$

all quantities on the right-hand side being known. Further

$$\frac{\partial}{\partial \theta} (\text{div } \mathbf{H}) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial H_r}{\partial \theta} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (H_\theta \sin \theta) \right\}, \quad (2.13)$$

in which, using the equation of radiative transfer, we have

$$\left. \begin{aligned} \frac{\partial H_r}{\partial \theta} &= -\frac{4}{3}ac \left[ \frac{\partial}{\partial \theta} \left( \frac{T'^3}{k'\rho'} \right) \frac{\partial T'}{\partial r} + \frac{T'^3}{k'\rho'} \frac{\partial}{\partial r} \left( \frac{\partial T'}{\partial \theta} \right) \right], \\ H_\theta &= -\frac{4}{3}ac \frac{T'^3}{k'\rho'} \cdot \frac{1}{r} \frac{\partial T'}{\partial \theta}, \end{aligned} \right\} \quad (2.14)$$

$k'$  being the opacity. (2.13) can therefore be written, correct to the first order, in the form

$$\frac{\partial}{\partial \theta} (\text{div } \mathbf{H}) = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 H \left\{ \frac{\partial}{\partial T} \left( \log \frac{T^3}{k\rho} \right) \frac{\partial T'}{\partial \theta} + \frac{\partial}{\partial \rho} \left( \log \frac{T^3}{k\rho} \right) \frac{\partial \rho'}{\partial \theta} + \frac{1}{dT/dr} \frac{\partial}{\partial r} \left( \frac{\partial T'}{\partial \theta} \right) \right\} \right] + \frac{H}{r^2 dT/dr} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\partial T'}{\partial \theta} \sin \theta \right) \right], \quad (2.15)$$

all quantities on the right-hand side again being known. By differentiating (2.11) with respect to  $\theta$  we can now write

$$\partial v_r / \partial \theta = Q(r, \theta),$$

where  $Q$  is a known function. Hence

$$v_r = \int_0^\theta Q \, d\theta + Q_1(r),$$

where  $Q_1(r)$  is a function which can be determined immediately from the condition that the total rate of outflow of material over any sphere must vanish. We thus have finally

$$v_r = \int_0^\theta Q \, d\theta - \frac{1}{2} \int_0^\pi \sin \theta_1 \int_0^{\theta_1} Q(r, \theta) \, d\theta \, d\theta_1. \quad (2.16)$$

The lines of flow can be found by introducing a stream function  $C$  given by

$$v_r = \frac{1}{\rho r^2 \sin \theta} \frac{\partial C}{\partial \theta}.$$

$C$  can then be expressed as

$$C = \rho r^2 \int_0^\theta v_r \sin \theta \, d\theta, \quad (2.17)$$

while

$$v_\theta = -\frac{1}{\rho r \sin \theta} \frac{\partial C}{\partial r}. \quad (2.18)$$

The method just given will be used in the next section to determine the meridian-plane circulation velocity in uniformly rotating stars. This provides the simplest example of its use, although the effect of a simple non-uniform distribution of angular velocity is not essentially more difficult to handle, and does not introduce any intrinsically new factor.

3. *Calculation for uniformly rotating stars.*—The discussion will be restricted to the point-convective model with uniform molecular weight, neglecting radiation pressure and the guillotine factor in the opacity law. The Cowling model (10) will be used; in this case the density  $\rho$  and the temperature  $T$  are expressed in units of their central values  $\rho_c$  and  $T_c$ , while  $z$  is the radial coordinate  $r$  in Emden units given by

$$z = r \{ 8\pi\mu G\rho_c / 5RT_c \}^{1/2}.$$

If the mass interior to any point, as expressed in units of  $4\pi\rho_c\{5\mathfrak{R}T_c/8\pi\mu G\rho_c\}^{3/2}$ , is denoted by  $m$ , the equation of hydrostatic equilibrium and the mass equation for the unperturbed model are given by

$$\left. \begin{aligned} dT/dz &= -5m/2(n+1)z^2, \\ dm/dz &= \rho z^2, \end{aligned} \right\} \quad (3.1)$$

where  $n$  is the effective polytropic index  $d \log \rho / d \log T$ . These equations will be used at several stages to simplify various expressions.

The disturbing functions  $f_r$  and  $f_\theta$  are given by

$$\left. \begin{aligned} f_r &= r\Omega^2 \sin^2 \theta, \\ f_\theta &= r\Omega^2 \sin \theta \cos \theta, \end{aligned} \right\} \quad (3.2)$$

where  $\Omega$  is the angular velocity of the star. Following the method of the previous section we then find

$$\chi = r^2\Omega^2(h-1) \sin \theta \cos \theta, \quad (3.3)$$

where  $h$  is given by

$$\frac{d^2h}{dz^2} + \frac{6}{z} \frac{dh}{dz} + \frac{5n\rho}{2(n+1)T} h = 0. \quad (3.4)$$

The boundary conditions for  $h$  are

$$\left. \begin{aligned} h &\text{ finite at } z=0, \\ h &\rightarrow 1 \text{ as } z \rightarrow \infty. \end{aligned} \right\} \quad (3.5)$$

In the core of the Sun the convective equilibrium has no effect on the hydrostatic equations, significant in this problem. The velocity  $u$  of the currents is of order  $2 \times 10^3$  cm./sec. according to Cowling's work (10). The Reynolds stresses  $\frac{1}{2}\rho u^2$  are therefore of order  $10^8$  dynes/cm.<sup>2</sup>. The gas pressure in the core is of order  $10^{17}$  dynes/cm.<sup>2</sup> and the effect of rotation, which can alter this value at any point by a factor of order  $10^{-5}$ , reduces this to a pressure change of order  $10^{12}$  dynes/cm.<sup>2</sup>. The Reynolds stresses are therefore negligible.

The equation of thermal equilibrium (2.11) reduces to

$$\rho_c g \rho \frac{n - \frac{3}{2}}{n+1} v_r = -\text{div } \mathbf{H}, \quad (3.6)$$

from which, following the method laid down in Section 2, we have finally

$$v_r = (1 - \frac{3}{2} \sin^2 \theta) p(z) LM^{-3} R^5 (\Omega/\Omega_\odot)^2 \text{ cm./sec.} \quad (3.7)$$

In this expression  $M$ ,  $L$  and  $R$  are the mass, luminosity and radius of the star in solar units, and  $\Omega_\odot$  is the Sun's angular velocity, taken as  $2.85 \times 10^{-6}$  radians/sec., while

$$p(z) = \frac{1.7 \times 10^{-12} z^5 (n+1)}{m^3 (n - \frac{3}{2})} \left[ h \left( 3 - \frac{\rho z^3}{m} \right) + z \frac{dh}{dz} \right]. \quad (3.8)$$

The values of  $v_r$  calculated from this expression are given for the Sun in Table I, using the Cowling model, and the resulting circulation system is illustrated in Fig. 1.

TABLE I

*Circulation velocity along rotation axis of Sun (units of  $10^{-10}$  cm./sec.)*

$z$	1.2	2.0	2.8	3.6	4.4	5.2	6.0	7.0
$[v_r]_{\theta=0}$	17.9	0.72	1.11	2.59	6.33	14.6	30.5	67.3

At the surface of the core the divisor  $n - \frac{3}{2}$  in the expression for  $v_r$  vanishes. The resulting singularity, however, is only a formal one, since the formula is only valid when  $v_r$  is small compared with the rotation velocity. The layer in which  $n - \frac{3}{2}$  is small enough to involve values of  $v_r$  of the order of magnitude of the rotation velocity is of course extremely thin, and is certainly thinner than the transition layer between the core and the radiative envelope. It will be noticed also that  $v_r$  does not vanish at the surface of the star. It has often been assumed in the past that  $v_r$  must vanish at the surface, in which case no steady state could exist in a star constrained to rotate uniformly. However, the actual physical requirement at the surface is that the radial flow of material should vanish. The required boundary condition is therefore simply that  $v_r$  is finite, and this is satisfied in this case.

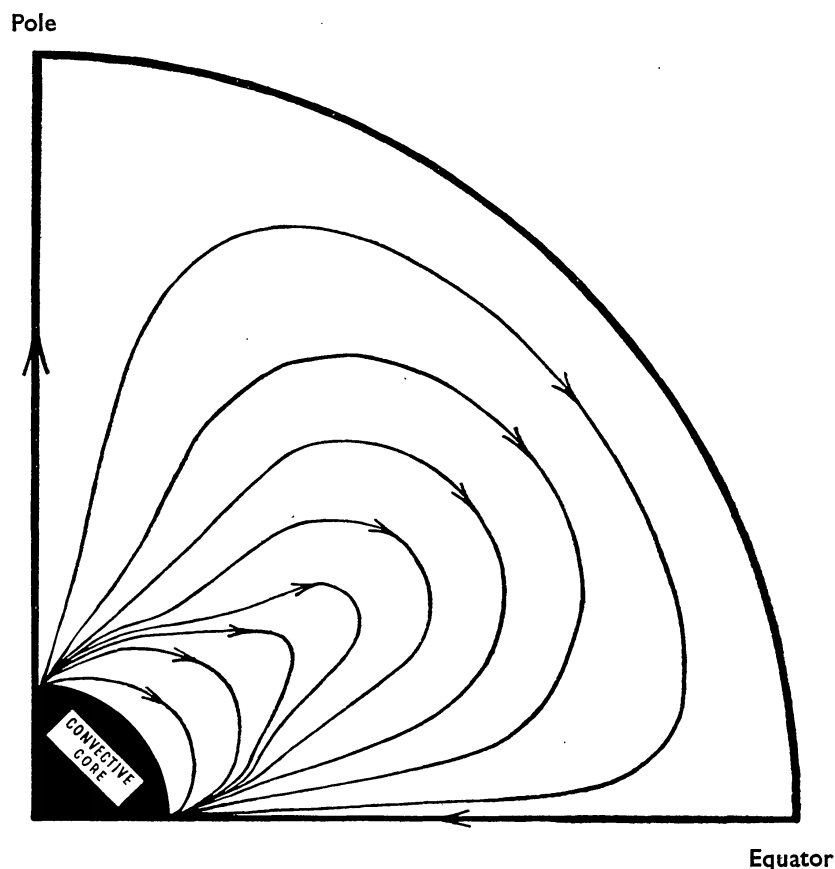


FIG. 1.—Circulation in a uniformly rotating star.

The total time taken for material to travel from the core of the Sun to the surface along the axis of rotation is given by

$$t_{\odot} = \int_{r_{\text{core}}}^R \frac{dr}{[v_r]_{\theta=0}} \sim 8 \times 10^{12} \text{ years.}$$

4. *The maximum rotation velocity allowing non-uniform molecular weight.*—The results obtained in the previous section for the Sun can be extended to give an approximate time for circulation for stars in general by adopting the Cowling model for these also. The approximation will therefore be good for light stars, but will fall off in accuracy with increasing stellar mass, as electron scattering becomes important. Equation (3.7) shows that

$$v_r \propto LM^{-3}R^5\Omega^2,$$

hence for stars in general the time taken for material to travel from the core to the surface along the axis of rotation is given by

$$t = L^{-1} M^3 R^{-4} (\Omega/\Omega_{\odot})^{-2} t_{\odot} \sim 8 \times 10^{12} L^{-1} M^3 R^{-4} (\Omega/\Omega_{\odot})^{-2} \text{ years.} \quad (4.1)$$

In discussing this expression it will be more convenient to use the equatorial rotation velocity  $v_0$  in place of the angular velocity  $\Omega$ . On taking the solar equatorial rotation velocity to be 2 km./sec. we have

$$v_0 = 2R(\Omega/\Omega_{\odot}) \text{ km./sec.};$$

the time  $t$  can then be written

$$t \sim 3.2 \times 10^{13} L^{-1} M^3 R^{-2} / v_0^2 \text{ years.} \quad (4.2)$$

The maximum rotation velocity which would allow the star to develop a non-uniform molecular weight is therefore given roughly by

$$v_{\max} \sqrt{10^{-9} t_1} \sim 180 L^{-1/2} M^{3/2} R^{-1} \text{ km./sec.,} \quad (4.3)$$

where  $t_1$  years is the time which the star would normally take to develop the non-uniformity. The factor  $10^9$  has been introduced as a suitable time scale for  $t_1$ . By expressing  $R$  in terms of  $L$  and of  $T_e$  the effective temperature, we have

$$v_{\max} \sqrt{10^{-9} t_1} \sim 180 L^{-1} M^{3/2} (T_e/T_{e\odot})^2 \text{ km./sec.,} \quad (4.4)$$

where  $T_{e\odot}$  is the Sun's effective temperature. This expression contains  $M$ , but since the range of mass is much less than the range of luminosity among stars, and since moreover there exists a rough mass-luminosity relationship, we can express  $M$  in terms of  $L$  from this relationship without undue loss of accuracy. The empirical mass-luminosity relation adopted for the purpose is given in Table II; here  $m$  denotes absolute bolometric magnitude, with  $m_{\odot} = +4^m.62$ .

TABLE II

*Mean mass-luminosity relation adopted*

$m$	-6	-4	-2	0	+2	+4	+5	+8
$\log_{10} M$	+1.35	+1.08	+0.80	+0.53	+0.28	+0.06	-0.17	-0.42

The results are given in graphical form in Fig. 2; here  $T_e$  has been reduced to spectral type, using the mean temperature scale for main sequence stars given by Kuiper (11). Stars brighter than  $0^m$  begin to depart from the model used as the effect of electron scattering increases. The  $v_{\max} \sqrt{10^{-9} t_1}$  curves may therefore lose accuracy above the zero magnitude line.

It must also be noted that if the star is expanding, the equation of thermal equilibrium in the form (2.10) does not hold. The effect of an expansion can be allowed for, however, by introducing an energy sink term on the right-hand side, equal to the rate at which the gravitational energy of the material at the point in question is increasing. This, like the normal energy production term and the  $\text{div } \mathbf{H}$  term, will be spherically symmetrical to the zero order, and of the same order of magnitude as  $H/r$ . It would therefore have no essentially new effect on the circulation.

In interpreting the results embodied in Fig. 2, consider for example stars of types around A0. Such stars would take times of order  $10^9$  years to change

composition appreciably, due to consumption of hydrogen, this figure decreasing with increasing luminosity from star to star. On taking  $t_1 = 10^9$  years, therefore, Fig. 2 shows that an A0-type star with rotation velocity greater than 50 km./sec. would normally maintain uniform composition. Such a star could not therefore develop an isothermal core in the early part of its life, as in Chandrasekhar's work (2). Neither could it become a giant as the result of a discontinuity of molecular weight at the core surface, even if that were otherwise possible. On the other hand, certain theories of structure in connection with accretion of interstellar material put forward by Hoyle and Lyttleton (4) and by Hen and Schwarzschild (5) show that it might temporarily become a giant even if its rotation velocity exceeded this limit. This could take place if the non-uniformity in the outer part of the radiative envelope, due to the addition of a layer of interstellar material on the surface, were built up more rapidly than the internal circulation currents could smooth it out. Once such a non-uniformity were established it would take a far larger rotation rate than normal to smooth it out again, since the forces tending to restore a distorted surface of constant molecular weight in a region of non-uniformity are comparable with the gravitational forces themselves.

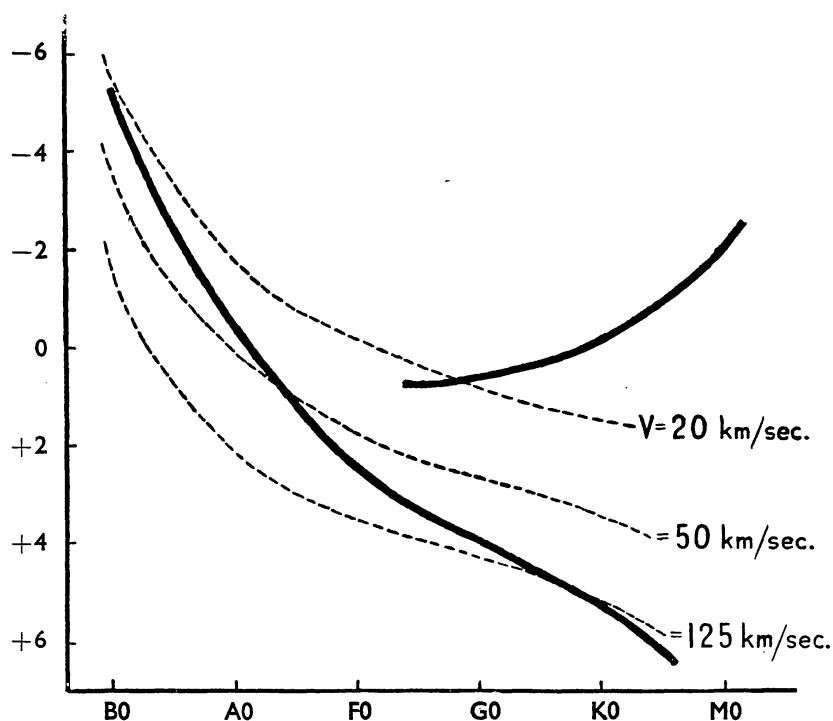


FIG. 2.—H-R diagram

$$V_{\max} = V \sqrt{10^9 / t_1}$$

Turning now to the actual observed rotation velocities, using Struve's 1945 survey (12), it is interesting to note that A- and B-type stars (which would be massive enough to form giants) do in fact rotate with velocities in the neighbourhood of 50 km./sec.

5. *The effects of magnetic fields and viscosity.*—The most important effect is that the angular velocity is constrained, by magnetic fields, to be constant along the lines of force, according to the Ferraro-Alfvén law of isorotation (13). In thus preventing any special distribution of angular velocity from being set up

by the meridian-plane circulation a magnetic field would play an important part in stabilizing the circulation against breaking up into cellules. A magnetic field also exerts a force in meridian planes and can contribute, along with the centrifugal force of rotation, to the net rate of circulation. The mechanical force exerted by a field of  $H$  gauss produced by electric currents in a region of dimensions of order  $l$  cm. is of order  $H^2/4\pi l$  dynes/cm.<sup>3</sup>. In the inner part of the radiative envelope of the Sun, for example, if  $H$  is taken as  $10^4$  gauss and  $l \sim 10^{10}$  cm. this force would be of order  $10^{-3}$  dynes/cm.<sup>3</sup>. This is negligible compared with the centrifugal force which is of order 1 dyne/cm.<sup>3</sup> in this region. The action of the field in impeding the circulation is also negligible. The author has shown (14) that the action of a magnetic field on material moving in it is like that of a semi-permeable membrane. A force  $\sigma H^2 v$  dynes/cm.<sup>3</sup> is exerted by the field opposing the motion of material moving across the lines of force, where  $\sigma$  e.m.u. is the conductivity of the material and  $v$  cm./sec. is the relative velocity of the material perpendicular to the lines of force. Taking  $\sigma \sim 10^{-4}$  e.m.u. together with  $v \sim 10^{-10}$  cm./sec. and  $H \sim 10^4$  gauss, this force is of order  $10^{-6}$  dynes/cm.<sup>3</sup>; this again is negligible as compared with the centrifugal force. The relative effect of the magnetic field may increase in the outer regions where the density is less, but Table I shows that the time of circulation is determined principally by conditions in the inner part of the radiative envelope where  $v$  is least. These rough considerations show that magnetic effects are not likely to contribute directly to the general mixing throughout the star, except possibly in individual giant stars in the unlikely event of their having low angular velocity accompanied by a strong magnetic field.

With circulation velocities as small as involved in the theory, viscosity plays a negligible part in the equations of motion in meridian planes. The only appreciable influence which viscosity effects could have is through the viscous production of heat affecting the thermal equilibrium. Here again the normal viscosity effect is negligible, although turbulent viscosity in non-uniformly rotating stars might give rise to an appreciable source of heat. The usual conditions for turbulent flow in a fluid cannot however apply in a radiative envelope owing to the high degree of thermal stability there. It is therefore doubtful whether differential rotation could produce turbulence.

6. *Conclusions.*—(i) Meridian-plane circulation currents in stars are much slower than has been supposed hitherto.

(ii) In fast-rotating stars, except possibly where no magnetic field is present, the meridian-plane circulation in radiative envelopes stirs material from the convective core throughout the whole star.

(iii) The resulting homogenizing effect on the chemical composition would be sufficient to prevent early A-type and late B-type stars with equatorial rotation velocities greater than 50 km./sec. from becoming giants as the result of inhomogeneity in the central regions. It would likewise prevent these stars from developing isothermal cores through local exhaustion of hydrogen. The corresponding figure is higher for the earlier B types.

(iv) The effects of accretion of small quantities of interstellar material would be more likely to form giants from highly rotating stars than from slowly rotating ones. Giants, once formed in this way, would remain as giants even if highly rotating, since a discontinuity in molecular weight, once formed, cannot easily be broken down by meridian-plane circulation.

(v) The observed rotation velocities of early-type stars are of the same order of magnitude as the critical values. This would produce bifurcations in the evolutionary tracks of stars of a given mass, according to rotation, if acted on by the processes mentioned in (iii) and (iv). Stars later than F5 have small rotation velocities, and are not subject to a significant mixing effect due to rotation.

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